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EXISTENCE RESULTS FOR A CLASS OF NONLINEAR PARABOLIC EQUATIONS IN ORLICZ SPACES

HICHAM REDWANE

Faculté des Sciences Juridiques, Économiques et Sociales. Université Hassan 1, B.P. 784. Settat. Morocco

Abstract. An existence result of a renormalized solution for a class of nonlinear parabolic equations in Orlicz spaces is proved. No growth assumption is made on the nonlinearities.

1. Introduction

In this paper we consider the following problem:

$$\frac{\partial b(x,u)}{\partial t} - div\Big(a(x,t,u,\nabla u)+\Phi(u)\Big) = f \quad \text{in } \Omega\times(0,T), \tag{1.1}$$

$$b(x,u)(t=0) = b(x,u_0) \quad \text{in } \Omega, \tag{1.2}$$

$$u = 0 \quad \text{on } \partial\Omega\times(0,T), \tag{1.3}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ and $T>0$, $Q=\Omega\times(0,T)$. Let b be a Carathéodory function (see assumptions (3.1)-(3.2) of Section 3), the data f and $b(x,u_0)$ in $L^1(Q)$ and $L^1(\Omega)$ respectively, $Au = -div\Big(a(x,t,u,\nabla u)\Big)$ is a Leray-Lions operator defined on $W_0^{1,x}L_M(\Omega)$, M is an appropriate N-function and which grows like $\bar{M}^{-1}M(\beta_K^4|\nabla u|)$ with respect to ∇u, but which is not restricted by any growth condition with respect to u (see assumptions (3.3)-(3.6)). The function Φ is just assumed to be continuous on $\mathbb{R}$.

Under these assumptions, the above problem does not admit, in general, a weak solution since the fields $a(x,t,u,\nabla u)$ and $\Phi(u)$ do not belong in $(L^1_{loc}(Q))^N$ in general. To overcome this difficulty we use in this paper the framework of renormalized solutions. This notion was introduced by Lions and DiPerna [31] for the study of Boltzmann equation (see also [27], [11], [29], [28], [2]).

A large number of papers was devoted to the study the existence of renormalized solution of parabolic problems under various assumptions and in different contexts: for a review on classical results see [7], [30], [9], [8], [4], [5], [34], [12], [13], [14].

The existence and uniqueness of renormalized solution of (1.1)-(1.3) has been proved in H. Redwane [34, 35] in the case where $Au = -div\Big(a(x,t,u,\nabla u)\Big)$ is a Leray-Lions operator defined on $L^p(0,T;W_0^{1,p}(\Omega))$, the existence of renormalized solution in Orlicz spaces has been proved in E. Azroul, H. Redwane and M.

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Rhoudaf [32] in the case where $b(x,u)=b(u)$ and where the growth of $a(x,t,u,\nabla u)$ is controlled with respect to u. Note that here we extend the results in [34, 32] in three different directions: we assume $b(x,u)$ depend on x , and the growth of $a(x,t,u,\nabla u)$ is not controlled with respect to u and we prove the existence in Orlicz spaces.

The paper is organized as follows. In section 2 we give some preliminaries and gives the definition of N-function and the Orlicz-Sobolev space. Section 3 is devoted to specifying the assumptions on b, a, Φ, f and $b(x,u_0)$. In Section 4 we give the definition of a renormalized solution of (1.1)-(1.3). In Section 5 we establish (Theorem 5.1) the existence of such a solution.

2. Preliminaries

Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function, i.e., M is continuous, convex, with $M(t)>0$ for $t>0$, $\frac{M(t)}{t}\to 0$ as $t\to 0$ and $\frac{M(t)}{t}\to\infty$ as $t\to\infty$. Equivalently, M admits the representation : $M(t)=\int_0^t a(s)\,ds$ where $a : \mathbb{R}^+\to\mathbb{R}^+$ is non-decreasing, right continuous, with $a(0)=0$, $a(t)>0$ for $t>0$ and $a(t)\to\infty$ as $t\to\infty$. The N-function $\overline{M}$ conjugate to M is defined by $\overline{M}(t)=\int_0^t \overline{a}(s)\,ds$, where $\overline{a} : \mathbb{R}^+\to\mathbb{R}^+$ is given by $\overline{a}(t)=\sup\{s : a(s)\le t\}$.

The N-function M is said to satisfy the Δ_2 condition if, for some $k>0$,

$$M(2t)\le k\,M(t) \quad \text{for all } t\ge 0. \tag{2.1}$$

When this inequality holds only for $t\ge t_0>0$, M is said to satisfy the Δ_2-condition near infinity.

Let P and Q be two N-functions. $P\ll Q$ means that P grows essentially less rapidly than Q ; i.e., for each $\varepsilon>0$,

$$\frac{P(t)}{Q(\varepsilon\,t)}\to 0 \quad \text{as } t\to\infty. \tag{2.2}$$

This is the case if and only if,

$$\frac{Q^{-1}(t)}{P^{-1}(t)}\to 0 \quad \text{as } t\to\infty. \tag{2.3}$$

We will extend these N-functions into even functions on all $\mathbb{R}$. Let Ω be an open subset of $\mathbb{R}^N$. The Orlicz class $\mathcal{L}_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real-valued measurable functions u on Ω such that :

$$\int_\Omega M(u(x))dx<+\infty \quad (\text{resp. } \int_\Omega M(\frac{u(x)}{\lambda})dx<+\infty \text{ for some } \lambda>0). \tag{2.4}$$

Note that $L_M(\Omega)$ is a Banach space under the norm

$$\|u\|_{M,\Omega}=\inf\Big\{\lambda>0 : \int_\Omega M(\frac{u(x)}{\lambda})dx\le 1\Big\} \tag{2.5}$$

and $\mathcal{L}_M(\Omega)$ is a convex subset of $L_M(\Omega)$. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$. The equality $E_M(\Omega)=L_M(\Omega)$ holds if and only if M satisfies the Δ_2-condition, for all t or for t large according to whether Ω has infinite measure or not.

The dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_\Omega u(x)v(x)dx$, and the dual norm on $L_{\overline{M}}(\Omega)$ is equivalent to $\|.\|_{\overline{M},\Omega}$. The space $L_M(\Omega)$ is reflexive if and only if M and $\overline{M}$ satisfy the Δ_2 condition, for all t or for t large, according to whether Ω has infinite measure or not.

We now turn to the Orlicz-Sobolev space. $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) is the space of all functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp. $E_M(\Omega)$). This is a Banach space under the norm

$$\|u\|_{1,M,\Omega} = \sum_{|\alpha|\leq 1} \|\nabla^\alpha u\|_{M,\Omega}. \tag{2.6}$$

Thus $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by ΠL_M, we will use the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$. We say that u_n converges to u for the modular convergence in $W^1L_M(\Omega)$ if for some $\lambda > 0$, $\displaystyle\int_\Omega M\Big(\frac{\nabla^\alpha u_n - \nabla^\alpha u}{\lambda}\Big)dx \to 0$ for all $|\alpha| \leq 1$. This implies convergence for $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. If M satisfies the Δ_2 condition on $\mathbb{R}^+$(near infinity only when Ω has finite measure), then modular convergence coincides with norm convergence.

Let $W^{-1}L_{\overline{M}}(\Omega)$ (resp. $W^{-1}E_{\overline{M}}(\Omega)$) denote the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}(\Omega)$ (resp. $E_{\overline{M}}(\Omega)$). It is a Banach space under the usual quotient norm.

If the open set Ω has the segment property, then the space $\mathcal{D}(\Omega)$ is dense in $W_0^1L_M(\Omega)$ for the modular convergence and for the topology $\sigma(\Pi L_M, \Pi L_{\overline{M}})$ (cf. [21]). Consequently, the action of a distribution in $W^{-1}L_{\overline{M}}(\Omega)$ on an element of $W_0^1L_M(\Omega)$ is well defined. For more details see [1], [23].

For $K > 0$, we define the truncation at height K, $T_K : \mathbb{R} \to \mathbb{R}$ by

$$T_K(s) = \min(K, \max(s, -K)). \tag{2.7}$$

The following abstract lemmas will be applied to the truncation operators.

Lemma 2.1. [21] *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly lipschitzian, with $F(0) = 0$. Let M be an N-function and let $u \in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$).*

Then $F(u) \in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Moreover, if the set of discontinuity points D of F' is finite, then

$$\frac{\partial}{\partial x_i}F(u) = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & a.e. \text{ in } \{x \in \Omega : u(x) \notin D\} \\ 0 & a.e. \text{ in } \{x \in \Omega : u(x) \in D\} \end{cases}$$

Lemma 2.2. [21] *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly lipschitzian, with $F(0) = 0$. We suppose that the set of discontinuity points of F' is finite. Let M be an N-function, then the mapping $F : W^1L_M(\Omega) \to W^1L_M(\Omega)$ is sequentially continuous with respect to the weak* topology $\sigma(\Pi L_M, \Pi E_{\overline{M}})$.*

Let Ω be a bounded open subset of $\mathbb{R}^N$, $T > 0$ and set $Q = \Omega \times (0, T)$. M be an N-function. For each $\alpha \in \mathbb{N}^N$, denote by ∇_x^α the distributional derivative on Q of order α with respect to the variable $x \in \mathbb{N}^N$. The inhomogeneous Orlicz-Sobolev spaces are defined as follows,

$$\text{(2.8)} \qquad \begin{array}{l} W^{1,x}L_M(Q) = \{u \in L_M(Q) : \nabla_x^\alpha u \in L_M(Q)\ \forall\ |\alpha| \le 1\} \\ \text{and}\ \ W^{1,x}E_M(Q) = \{u \in E_M(Q) : \nabla_x^\alpha u \in E_M(Q)\ \forall\ |\alpha| \le 1\} \end{array}$$

The last space is a subspace of the first one, and both are Banach spaces under the norm,

$$\text{(2.9)} \qquad \|u\| = \sum_{|\alpha|\le 1} \|\nabla_x^\alpha u\|_{M,Q}.$$

We can easily show that they form a complementary system when Ω satisfies the segment property. These spaces are considered as subspaces of the product space $\Pi L_M(Q)$ which have as many copies as there is α-order derivatives, $|\alpha| \le 1$. We shall also consider the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. If $u \in W^{1,x}L_M(Q)$ then the function : $t \longmapsto u(t) = u(t,.)$ is defined on $(0,T)$ with values in $W^1 L_M(\Omega)$. If, further, $u \in W^{1,x}E_M(Q)$ then the concerned function is a $W^1E_M(\Omega)$-valued and is strongly measurable. Furthermore the following imbedding holds: $W^{1,x}E_M(Q) \subset L^1(0,T; W^1E_M(\Omega))$. The space $W^{1,x}L_M(Q)$ is not in general separable, if $u \in W^{1,x}L_M(Q)$, we can not conclude that the function $u(t)$ is measurable on $(0,T)$. However, the scalar function $t \mapsto \|u(t)\|_{M,\Omega}$ is in $L^1(0,T)$. The space $W_0^{1,x}E_M(Q)$ is defined as the (norm) closure in $W^{1,x}E_M(Q)$ of $\mathcal{D}(Q)$. We can easily show as in [22] that when Ω has the segment property, then each element u of the closure of $\mathcal{D}(Q)$ with respect of the weak * topology $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ is a limit, in $W^{1,x}L_M(Q)$, of some subsequence $(u_i) \subset \mathcal{D}(Q)$ for the modular convergence; i.e., there exists $\lambda > 0$ such that for all $|\alpha| \le 1$,

$$\text{(2.10)} \qquad \int_Q M\Big(\frac{\nabla_x^\alpha u_i - \nabla_x^\alpha u}{\lambda}\Big)\, dx\, dt \to 0 \text{ as } i \to \infty.$$

This implies that (u_i) converges to u in $W^{1,x}L_M(Q)$ for the weak topology $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. Consequently,

$$\text{(2.11)} \qquad \overline{\mathcal{D}(Q)}^{\sigma(\Pi L_M, \Pi E_{\overline{M}})} = \overline{\mathcal{D}(Q)}^{\sigma(\Pi L_M, \Pi L_{\overline{M}})}.$$

This space will be denoted by $W_0^{1,x}L_M(Q)$. Furthermore, $W_0^{1,x}E_M(Q) = W_0^{1,x}L_M(Q) \cap \Pi E_M$. Poincaré's inequality also holds in $W_0^{1,x}L_M(Q)$, i.e., there is a constant $C > 0$ such that for all $u \in W_0^{1,x}L_M(Q)$ one has,

$$\text{(2.12)} \qquad \sum_{|\alpha|\le 1} \|\nabla_x^\alpha u\|_{M,Q} \le C \sum_{|\alpha|=1} \|\nabla_x^\alpha u\|_{M,Q}.$$

Thus both sides of the last inequality are equivalent norms on $W_0^{1,x}L_M(Q)$. We have then the following complementary system

$$\text{(2.13)} \qquad \begin{pmatrix} W_0^{1,x}L_M(Q) & F \\ W_0^{1,x}E_M(Q) & F_0 \end{pmatrix}$$

F being the dual space of $W_0^{1,x}E_M(Q)$. It is also, except for an isomorphism, the quotient of $\Pi L_{\overline{M}}$ by the polar set $W_0^{1,x}E_M(Q)^\perp$, and will be denoted by $F =$

$W^{-1,x}L_{\overline{M}}(Q)$ and it is shown that,

$$W^{-1,x}L_{\overline{M}}(Q) = \Big\{ f = \sum_{|\alpha|\le 1} \nabla_x^\alpha f_\alpha : f_\alpha \in L_{\overline{M}}(Q) \Big\}. \tag{2.14}$$

This space will be equipped with the usual quotient norm

$$\|f\| = \inf \sum_{|\alpha|\le 1} \|f_\alpha\|_{\overline{M},Q} \tag{2.15}$$

where the infimum is taken on all possible decompositions

$$f = \sum_{|\alpha|\le 1} \nabla_x^\alpha f_\alpha, \quad f_\alpha \in L_{\overline{M}}(Q). \tag{2.16}$$

The space F_0 is then given by,

$$F_0 = \Big\{ f = \sum_{|\alpha|\le 1} \nabla_x^\alpha f_\alpha : f_\alpha \in E_{\overline{M}}(Q) \Big\} \tag{2.17}$$

and is denoted by $F_0 = W^{-1,x}E_{\overline{M}}(Q)$.

Remark 2.3. We can easily check, using lemma 2.1, that each uniformly lipschitzian mapping F, with $F(0) = 0$, acts in inhomogeneous Orlicz-Sobolev spaces of order 1 : $W^{1,x}L_M(Q)$ and $W_0^{1,x}L_M(Q)$.

3. Assumptions and statement of main results

Throughout this paper, we assume that the following assumptions hold true: Ω is a bounded open set on $\mathbb{R}^N$ $(N \ge 2)$, $T > 0$ is given and we set $Q = \Omega \times (0,T)$. Let M and P be two N-function such that $P \ll M$.

(3.1) $\qquad b : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that,

for every $x \in \Omega$: $b(x,s)$ is a strictly increasing C^1-function, with $b(x,0) = 0$. For any $K > 0$, there exists $\lambda_K > 0$, a function A_K in $L^\infty(\Omega)$ and a function B_K in $L_M(\Omega)$ such that

$$\lambda_K \le \frac{\partial b(x,s)}{\partial s} \le A_K(x) \quad \text{and} \quad \Big|\nabla_x\Big(\frac{\partial b(x,s)}{\partial s}\Big)\Big| \le B_K(x), \tag{3.2}$$

for almost every $x \in \Omega$, for every s such that $|s| \le K$.

Consider a second order partial differential operator $A : D(A) \subset W^{1,x}L_M(Q) \to W^{-1,x}L_{\overline{M}}(Q)$ in divergence form,

$$A(u) = -\operatorname{div}\Big(a(x,t,u,\nabla u)\Big)$$

where

(3.3) $\qquad a : \Omega \times (0,T) \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying

for any $K > 0$, there exist $\beta_K^i > 0$ (for $i = 1,2,3,4$) and a function $C_K \in E_{\bar{M}}(Q)$ such that:

$$|a(x,t,s,\xi)| \le C_K(x,t) + \beta_K^1 \bar{M}^{-1} P(\beta_K^2 |s|) + \beta_K^3 \bar{M}^{-1} M(\beta_K^4 |\xi|) \tag{3.4}$$

for almost every $(x,t)\in Q$ and for every $|s|\leq K$ and for every $\xi\in\mathbb{R}^N$.

$$\Big[a(x,t,s,\xi)-a(x,t,s,\xi^*)\Big]\Big[\xi-\xi^*\Big]>0 \tag{3.5}$$

$$a(x,t,s,\xi)\xi\geq\alpha M(|\xi|) \tag{3.6}$$

for almost every $(x,t)\in Q$, for every $s\in\mathbb{R}$ and for every $\xi\neq\xi^*\in\mathbb{R}^N$, where $\alpha>0$ is a given real number.

$$\Phi\ :\ \mathbb{R}\to\mathbb{R}^N \text{ is a continuous function} \tag{3.7}$$

$$f \text{ is an element of } L^1(Q). \tag{3.8}$$

$$u_0 \text{ is an element of } L^1(\Omega) \text{ such that } b(x,u_0)\in L^1(\Omega). \tag{3.9}$$

Remark 3.1. As already mentioned in the introduction, problem (1.1)-(1.3) does not admit a weak solution under assumptions (3.1)-(3.9) (even when $b(x,u)=u$) since the growths of $a(x,t,u,Du)$ and $\Phi(u)$ are not controlled with respect to u (so that these fields are not in general defined as distributions, even when u belongs to $W_0^{1,x}L_M(Q)$.

4. Definition of a renormalized solution

The definition of a renormalized solution for problem (1.1)-(1.3) can be stated as follows.

Definition 4.1. A measurable function u defined on Q is a renormalized solution of Problem (1.1)-(1.3) if

$$T_K(u)\in W_0^{1,x}L_M(Q)\ \ \forall K\geq 0 \text{ and } b(x,u)\in L^\infty(0,T;L^1(\Omega)), \tag{4.1}$$

$$\int_{\{(t,x)\in Q\ ;\ m\leq|u(x,t)|\leq m+1\}} a(x,t,u,\nabla u)\nabla u\,dx\,dt\ \longrightarrow 0 \quad \text{as } m\to+\infty\ ; \tag{4.2}$$

and if, for every function S in $W^{2,\infty}(\mathbb{R})$, which is piecewise C^1 and such that S' has a compact support, we have

$$\begin{aligned}\frac{\partial B_S(x,u)}{\partial t}&-div\Big(S'(u)a(x,t,u,\nabla u)\Big)+S''(u)a(x,t,u,\nabla u)\nabla u\\ &-div\Big(S'(u)\Phi(u)\Big)+S''(u)\Phi(u)\nabla u=fS'(u)\ \text{ in } D'(Q),\end{aligned} \tag{4.3}$$

and

$$B_S(x,u)(t=0)=B_S(x,u_0) \text{ in } \Omega, \tag{4.4}$$

where $B_S(x,z)=\displaystyle\int_0^z\frac{\partial b(x,r)}{\partial r}S'(r)\,dr$.

The following remarks are concerned with a few comments on definition 4.1.

Remark 4.2. Equation (4.3) is formally obtained through pointwise multiplication of equation (1.1) by $S'(u)$. Note that due to (4.1) each term in (4.3) has a meaning in $L^1(Q) + W^{-1,x}L_{\overline{M}}(Q)$.

Indeed, if K is such that $suppS' \subset [-K, K]$, the following identifications are made in (4.3).

$\star$ $B_S(x, u) \in L^\infty(Q)$, because $|B_S(x, u)| \leq K\|A_K\|_{L^\infty(\Omega)}\|S'\|_{L^\infty(\mathbb{R})}$.

$\star$ $S'(u)a(x, t, u, \nabla u)$ identifies with $S'(u)a\Big(x, t, T_K(u), \nabla T_K(u)\Big)$ a.e. in Q. Since indeed $|T_K(u)| \leq K$ a.e. in Q. Since $S'(u) \in L^\infty(Q)$ and with (3.4), (4.1) we obtain that

$$S(u)a\Big(x, t, T_K(u), \nabla T_K(u)\Big) \in (L_{\overline{M}}(Q))^N.$$

$\star$ $S'(u)a(x, t, u, \nabla u)\nabla u$ identifies with $S'(u)a\Big(x, t, T_K(u), \nabla T_K(u)\Big)\nabla T_K(u)$ and in view of (3.2) and (4.1) one has

$$S'(u)a\Big(x, t, T_K(u), \nabla T_K(u)\Big)\nabla T_K(u) \in L^1(Q).$$

$\star$ $S'(u)\Phi(u)$ and $S''(u)\Phi(u)\nabla u$ respectively identify with $S'(u)\Phi(T_K(u))$ and $S''(u)\Phi(T_K(u))\nabla T_K(u)$. Due to the properties of S and (3.7), the functions S', S'' and $\Phi \circ T_K$ are bounded on $\mathbb{R}$ so that (4.1) implies that $S'(u)\Phi(T_K(u)) \in (L^\infty(Q))^N$, and $S''(u)\Phi(T_K(u))\nabla T_K(u) \in (L_M(Q))^N$.

The above considerations show that equation (4.3) takes place in $D'(Q)$ and that

$$\frac{\partial B_S(x, u)}{\partial t} \text{ belongs to } W^{-1,x}L_{\overline{M}}(Q) + L^1(Q). \tag{4.5}$$

Due to the properties of S and (3.2), we have

$$\Big|\nabla B_S(x, u)\Big| \leq \|A_K\|_{L^\infty(\Omega)}|\nabla T_K(u)|\|S'\|_{L^\infty(\Omega)} + K\|S'\|_{L^\infty(\Omega)}B_K(x) \tag{4.6}$$

and

$$B_S(x, u) \text{ belongs to } W_0^{1,x}L_M(Q). \tag{4.7}$$

Moreover (4.5) and (4.7) implies that $B_S(x, u)$ belongs to $C^0([0, T]; L^1(\Omega))$ (for a proof of this trace result see [30]), so that the initial condition (4.4) makes sense.

Remark 4.3. For every $S \in W^{2,\infty}(\mathbb{R})$, nondecreasing function such that supp$S' \subset [-K, K]$ and (3.2), we have

$$\lambda_K|S(r) - S(r')| \leq \Big|B_S(x, r) - B_S(x, r')\Big| \leq \|A_K\|_{L^\infty(\Omega)}|S(r) - S(r')| \tag{4.8}$$

for almost every $x \in \Omega$ and for every $r,\ r' \in \mathbb{R}$.

5. Existence result

This section is devoted to establish the following existence theorem.

Theorem 5.1. *Under assumption (3.1)-(3.9) there exists at at least a renormalized solution of Problem (1.1)-(1.3).*

Proof. The proof is divided into 5 steps. □

⋆ **Step 1**. For $n \in \mathbb{N}^*$, let us define the following approximations of the data:

$$b_n(x,r) = b(x,T_n(r)) + \frac{1}{n}\, r \quad \text{a.e. in } \Omega,\ \forall s \in \mathbb{R}, \tag{5.1}$$

$$a_n(x,t,r,\xi) = a(x,t,T_n(r),\xi) \quad \text{a.e. in } Q,\ \forall s \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N, \tag{5.2}$$

$$\Phi_n \text{ is a Lipschitz continuous bounded function from } \mathbb{R} \text{ into } \mathbb{R}^N, \tag{5.3}$$

such that Φ_n uniformly converges to Φ on any compact subset of $\mathbb{R}$ as n tends to $+\infty$.

$$f_n \in C_0^\infty(Q)\ :\ \|f_n\|_{L^1} \le \|f\|_{L^1} \text{ and } f_n \longrightarrow f \text{ in } L^1(Q) \text{ as } n \text{ tends to } +\infty, \tag{5.4}$$

(5.5)
$$u_{0n} \in C_0^\infty(\Omega)\ :\ \|b_n(x,u_{0n})\|_{L^1} \le \|b(x,u_0)\|_{L^1} \text{ and } b_n(x,u_{0n}) \longrightarrow b(x,u_0) \text{ in } L^1(\Omega)$$

as n tends to $+\infty$.

Let us now consider the following regularized problem:

$$\frac{\partial b_n(x,u_n)}{\partial t} - div\Big(a_n(x,t,u_n,\nabla u_n) + \Phi_n(u_n)\Big) = f_n \ \text{ in } Q, \tag{5.6}$$

$$u_n = 0 \ \text{ on } (0,T)\times\partial\Omega, \tag{5.7}$$

$$b_n(x,u_n)(t=0) = b_n(x,u_{0n}) \ \text{ in } \Omega. \tag{5.8}$$

As a consequence, proving existence of a weak solution $u_n \in W_0^{1,x}L_M(Q)$ of (5.6)-(5.8) is an easy task (see e.g. [25], [33]).

⋆ **Step 2**. The estimates derived in this step rely on usual techniques for problems of the type (5.6)-(5.8).

Proposition 5.2. *Assume that (3.1)-(3.9) hold true and let u_n be a solution of the approximate problem* (5.6) – (5.8). *Then for all $K,\ n > 0$, we have*

$$\|T_K(u_n)\|_{W_0^{1,x}L_M(Q)} \le K\Big(\|f\|_{L^1(Q)} + \|b(x,u_0)\|_{L^1(\Omega)}\Big) \equiv CK, \tag{5.9}$$

where C is a constant independent of n.

$$\int_\Omega B_K^n(x,u_n)(\tau)\, dx \le K(\|f\|_{L^1(Q)} + \|b(x,u_0)\|_{L^1(\Omega)}) \equiv CK, \tag{5.10}$$

for almost any τ in $(0,T)$, and where $B_K^n(x,r) = \displaystyle\int_0^r T_K(s)\frac{\partial b_n(x,s)}{\partial s}\, ds.$

$$\lim_{K\to\infty} meas\Big\{(x,t)\in Q:\ |u_n| > K\Big\} = 0 \ \textit{ uniformly with respect to } \ n. \tag{5.11}$$

Proof. We take $T_K(u_n)\chi_{(0,\tau)}$ as test function in (5.6), we get for every $\tau \in (0,T)$

(5.12)
$$\langle \frac{\partial b_n(x,u_n)}{\partial t}, T_K(u_n)\chi_{(0,\tau)}\rangle + \int_{Q_\tau} a_n(x,t,T_K(u_n),\nabla T_K(u_n))\nabla T_K(u_n)\, dx\, dt$$
$$+ \int_{Q_\tau} \Phi_n(u_n)\nabla T_K(u_n)\, dx\, dt = \int_{Q_\tau} f_n T_K(u_n)\, dx\, dt,$$

which implies that,

(5.13)
$$\int_\Omega B_K^n(x,u_n)(\tau)\,dx + \int_{Q_\tau} a_n(x,t,T_K(u_n),\nabla T_K(u_n))\nabla T_K(u_n)\,dx\,dt + \int_{Q_\tau} \Phi_n(u_n)\nabla T_K(u_n)\,dx\,dt = \int_{Q_\tau} f_n T_K(u_n)\,dx\,dt + \int_\Omega B_K^n(x,u_{0n})\,dx$$

where, $B_K^n(x,r)=\displaystyle\int_0^r T_K(s)\frac{\partial b_n(x,s)}{\partial s}\,ds.$

The Lipschitz character of Φ_n, Stokes formula together with the boundary condition (5.7), make it possible to obtain

$$\int_{Q_\tau} \Phi_n(u_n)\nabla T_K(u_n)\,dx\,dt = 0. \tag{5.14}$$

Due to the definition of B_K^n we have,

$$0 \le \int_\Omega B_K^n(x,u_{0n})\,dx \le K\int_\Omega |b_n(x,u_{0n})|\,dx \le K\|b(x,u_0)\|_{L^1(\Omega)}. \tag{5.15}$$

By using (5.14), (5.15) and the fact that $B_K^n(x,u_n)\ge 0$, permit to deduce from (5.13) that

(5.16)
$$\int_Q a_n(x,t,T_K(u_n),\nabla T_K(u_n))\nabla T_K(u_n)\,dx\,dt \le K(\|f_n\|_{L^1(Q)}+\|b_n(x,u_{0n})\|_{L^1(\Omega)}) \le CK,$$

which implies by virtue of (3.6), (5.4) and (5.5) that,

$$\int_Q M(\nabla T_K(u_n))\,dx\,dt \le K(\|f\|_{L^1(Q)}+\|b(x,u_0)\|_{L^1(\Omega)}) \equiv CK. \tag{5.17}$$

We deduce from that above inequality (5.13) and (5.15) that

$$\int_\Omega B_K^n(x,u_n)(\tau)\,dx \le (\|f\|_{L^1(Q)}+\|b(x,u_0)\|_{L^1(\Omega)}) \equiv CK. \tag{5.18}$$

for almost any τ in $(0,T)$.

We prove (5.11). Indeed, thanks to lemma 5.7 of [21], there exist two positive constants δ,λ such that,

$$\int_Q M(v)\,dx\,dt \le \delta\int_Q M(\lambda|\nabla v|)\,dx\,dt \quad \text{for all} \quad v\in W_0^{1,x}L_M(Q). \tag{5.19}$$

Taking $v=\dfrac{T_K(u_n)}{\lambda}$ in (5.19) and using (5.17), one has

$$\int_Q M\Big(\frac{T_K(u_n)}{\lambda}\Big)\,dx\,dt \le CK, \tag{5.20}$$

where C is a constant independent of K and n. Which implies that,

$$meas\Big\{(x,t)\in Q:\ |u_n|>K\Big\} \le \frac{C'K}{M(\frac{K}{\lambda})}. \tag{5.21}$$

where C' is a constant independent of K and n. Finally,

$$\lim_{K\to\infty} meas\Big\{(x,t)\in Q:\ |u_n|>K\Big\}=0 \text{ uniformly with respect to } n.$$

□

We prove de following proposition:

Proposition 5.3. *Let u_n be a solution of the approximate problem (5.6)-(5.8), then*

$$u_n \to u \quad a.e. \ in \ \ Q, \tag{5.22}$$

$$b_n(x,u_n) \to b(x,u) \quad a.e. \ in \ \ Q, \tag{5.23}$$

$$b(x,u) \in L^\infty(0,T;L^1(\Omega)), \tag{5.24}$$

$$a_n\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big) \rightharpoonup \varphi_k \quad in \quad (L_{\overline{M}}(Q))^N \quad for \ \ \sigma(\Pi L_{\overline{M}},\Pi E_M) \tag{5.25}$$

for some $\varphi_k \in (L_{\overline{M}}(Q))^N$.

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{\{m\le|u_n|\le m+1\}} a_n(x,t,u_n,\nabla u_n)\nabla u_n \,dx\,dt=0. \tag{5.26}$$

Proof. Proceeding as in [5, 9, 7], we have for any $S\in W^{2,\infty}(\mathbb{R})$ such that S' has a compact support (supp $S'\subset[-K,K]$)

$$B_S^n(x,u_n) \text{ is bounded in } W_0^{1,x}L_M(Q), \tag{5.27}$$

and

$$\frac{\partial B_S^n(x,u_n)}{\partial t} \text{ is bounded in } L^1(Q)+W^{-1,x}L_{\overline{M}}(Q), \tag{5.28}$$

independently of n.

As a consequence of (4.6) and (5.17) we then obtain (5.27). To show that (5.28) holds true, we multiply the equation for u_n in (5.6) by $S'(u_n)$ to obtain

$$\frac{\partial B_S^n(x,u_n)}{\partial t} = \operatorname{div}\Big(S'(u_n)a_n(t,x,u_n,\nabla u_n)\Big) \tag{5.29}$$

$$-S''(u_n)a_n(x,t,u_n,\nabla u_n)\nabla u_n + \ div\Big(S'(u_n)\Phi_n(u_n)\Big)+f_nS'(u_n) \ \text{ in } D'(Q).$$

Where $B_S^n(x,r)=\displaystyle\int_0^r S'(s)\frac{\partial b_n(x,s)}{\partial s}\,ds$. Since supp S' and supp S'' are both included in $[-K,K]$, u^ε may be replaced by $T_K(u_n)$ in each of these terms. As a consequence, each term in the right hand side of (5.29) is bounded either in $W^{-1,x}L_{\overline{M}}(Q)$ or in $L^1(Q)$. As a consequence of (3.2), (5.29) we then obtain (5.28). Arguing again as in [5, 7, 6, 9] estimates (5.27), (5.28) and (4.8), we can show (5.22) and (5.23).

We now establish that $b(x,u)$ belongs to $L^\infty(0,T;L^1(\Omega))$. To this end, recalling (5.23) makes it possible to pass to the limit-inf in (5.18) as n tends to $+\infty$ and to obtain

$$\frac{1}{K}\int_\Omega B_K(x,u)(\tau)\,dx \le (\|f\|_{L^1(Q)}+\|b(x,u_0)\|_{L^1(\Omega)})\equiv C,$$

for almost any τ in $(0,T)$. Due to the definition of $B_K(x,s)$, and because of the pointwise convergence of $\frac{1}{K}B_K(x,u)$ to $b(x,u)$ as K tends to $+\infty$, which shows that $b(x,u)$ belongs to $L^\infty(0,T;L^1(\Omega))$.

We prove (5.25). Let $\varphi\in(E_M(Q))^N$ with $\|\varphi\|_{M.Q}=1$. In view of the monotonicity of a one easily has,

(5.30)
$$\int_Q a_n\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)\varphi\,dx\,dt \le \int_Q a_n\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)\nabla T_k(u_n)\,dx\,dt + \int_Q a_n\Big(x,t,T_k(u_n),\varphi\Big)[\nabla T_k(u_n)-\varphi]\,dx\,dt.$$

and

(5.31)
$$-\int_Q a_n\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)\varphi\,dx\,dt \le \int_Q a_n\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)\nabla T_k(u_n)\,dx\,dt - \int_Q a_n\Big(x,t,T_k(u_n),-\varphi\Big)[\nabla T_k(u_n)+\varphi]\,dx\,dt,$$

since $T_k(u_n)$ is bounded in $W_0^{1,x}L_M(Q)$, one easily deduce that $a_n\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)$ is a bounded sequence in $(L_{\overline{M}}(Q))^N$, and we obtain (5.25).

Now we prove (5.26). We take of $T_1(u_n-T_m(u_n))$ as test function in (5.6), we obtain

$$(5.32)\quad \langle\frac{\partial b_n(x,u_n)}{\partial t},T_1(u_n-T_m(u_n))\rangle+\int_{\{m\le|u_n|\le m+1\}} a_n(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt + \int_Q \operatorname{div}\left[\int_0^{u_n}\Phi(r)T_1'(r-T_m(r))\right]dx\,dt = \int_Q f_nT_1(u_n-T_m(u_n))\,dx\,dt.$$

Using the fact that $\int_0^{u_n}\Phi(r)T_1'(r-T_m(r))\,dx\,dt\in W_0^{1,x}L_M(Q)$ and Stokes formula, we get

$$(5.33)\qquad \int_\Omega B_n^m(x,u_n(T))\,dx+\int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt \le \int_Q |f_nT_1(u_n-T_m(u_n))|\,dx\,dt+\int_\Omega B_n^m(x,u_{0n})\,dx,$$

where $B_n^m(x,r)=\int_0^r\frac{\partial b_n(x,s)}{\partial s}T_1(s-T_m(s))\,ds$.

In order to pass to the limit as n tends to $+\infty$ in (5.33), we use $B_n^m(x,u_n(T))\ge 0$ and (5.4)-(5.5), we obtain that

$$(5.34)\qquad \lim_{n\to+\infty}\int_{\{m\le|u_n|\le m+1\}} a_n(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt \le \int_{\{|u|>m\}}|f|\,dx\,dt+\int_{\{|u_0|>m\}}|b(x,u_0)|\,dx.$$

Finally by (3.8), (3.9) and (5.34) we obtain (5.26).

□

⋆ **Step 3**. This step is devoted to introduce for $K \geq 0$ fixed, a time regularization $w^i_{\mu,j}$ of the function $T_K(u)$ and to establish the following proposition:

Proposition 5.4. *Let u_n be a solution of the approximate problem (5.6)-(5.8). Then, for any $k \geq 0$:*

$$\nabla T_k(u_n) \to \nabla T_k(u) \quad a.e. \text{ in } \; Q, \tag{5.35}$$

$$a_n\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big) \rightharpoonup a\Big(x,t,T_k(u),\nabla T_k(u)\Big) \;\; \text{weakly in } \; (L_{\overline{M}}(Q))^N, \tag{5.36}$$

$$M(|\nabla T_k(u_n)|) \to M(|\nabla T_k(u)|) \;\; \text{strongly in } \; L^1(Q), \tag{5.37}$$

as n tends to $+\infty$.

Let use give the following lemma which will be needed later:

Lemma 5.5. *Under assumptions* (3.1) − (3.9)*, and let* (z_n) *be a sequence in* $W^{1,x}_0 L_M(Q)$ *such that,*

$$z_n \rightharpoonup z \text{ in } W^{1,x}_0 L_M(Q) \text{ for } \sigma(\Pi L_M(Q), \Pi E_{\overline{M}}(Q)), \tag{5.38}$$

$$(a_n(x,t,z_n,\nabla z_n))_n \text{ is bounded in } (L_{\overline{M}}(Q))^N, \tag{5.39}$$

$$\int_Q \Big[a_n(x,t,z_n,\nabla z_n) - a_n(x,t,z_n,\nabla z \chi_s)\Big]\Big[\nabla z_n - \nabla z \chi_s\Big]\,dx\,dt \longrightarrow 0, \tag{5.40}$$

as n and s tend to $+\infty$, and where χ_s is the characteristic function of

$$Q_s = \Big\{(x,t) \in Q \;;\; |\nabla z| \leq s\Big\}.$$

Then,

$$\nabla z_n \to \nabla z \quad a.e. \text{ in } Q, \tag{5.41}$$

$$\lim_{n\to\infty} \int_Q a_n(x,t,z_n,\nabla z_n)\nabla z_n\,dx\,dt = \int_Q a(x,t,z,\nabla z)\nabla z\,dx\,dt, \tag{5.42}$$

$$M(|\nabla z_n|) \to M(|\nabla z|) \quad \text{in } \; L^1(Q). \tag{5.43}$$

Proof. See [32]. □

Proof. (Proposition 5.4). The proof is almost identical of the one given in, e.g. [32]. where the result is established for $b(x,u) = u$ and where the growth of $a(x,t,u,Du)$ is controlled with respect to u. This proof is devoted to introduce for $k \geq 0$ fixed, a time regularization of the function $T_k(u)$, this notion, introduced by R. Landes (see Lemma 6 and Proposition 3, p. 230 and Proposition 4, p. 231 in [24]). More recently, it has been exploited in [10] and [15] to solve a few nonlinear evolution problems with L^1 or measure data.

Let $v_j \in D(Q)$ be a sequence such that $v_j \to u$ in $W^{1,x}_0 L_M(Q)$ for the modular convergence and let $\psi_i \in D(\Omega)$ be a sequence which converges strongly to u_0 in $L^1(\Omega)$.

Let $w^\mu_{i,j} = T_k(v_j)_\mu + e^{-\mu t}T_k(\psi_i)$ where $T_k(v_j)_\mu$ is the mollification with respect to time of $T_k(v_j)$, note that $w^\mu_{i,j}$ is a smooth function having the following properties:

$$\frac{\partial w^\mu_{i,j}}{\partial t} = \mu(T_k(v_j) - w^\mu_{i,j}),\ \ w^\mu_{i,j}(0) = T_k(\psi_i),\ \ |w^\mu_{i,j}| \le k, \tag{5.44}$$

$$w^\mu_{i,j} \to T_k(u)_\mu + e^{-\mu t}T_k(\psi_i) \ \text{ in } \ W^{1,x}_0 L_M(Q), \tag{5.45}$$

for the modular convergence as $j \to \infty$.

$$T_k(u)_\mu + e^{-\mu t}T_k(\psi_i) \to T_k(u) \ \text{ in } \ W^{1,x}_0 L_M(Q), \tag{5.46}$$

for the modular convergence as $\mu \to \infty$.

Let now the function h_m defined on $\mathbb{R}$ with $m \ge k$ by: $h_m(r) = 1$ if $|r| \le m$, $h(r) = -|r| + m + 1$ if $m \le |r| \le m+1$ and $h(r) = 0$ if $|r| \ge m+1$.

Using the admissible test function $\varphi^{\mu,i}_{n,j,m} = (T_k(u_n) - w^\mu_{i,j})h_m(u_n)$ as test function in (5.6) leads to

$$\begin{aligned} &\langle \frac{\partial b_n(x,u_n)}{\partial t}, \varphi^{\mu,i}_{n,j,m}\rangle + \int_Q a_n(x,t,u_n,\nabla u_n)(\nabla T_k(u_n) - \nabla w^\mu_{i,j})h_m(u_n)\,dx\,dt \\ &+ \int_Q a_n(x,t,u_n,\nabla u_n)(T_k(u_n) - w^\mu_{i,j})\nabla u_n h'_m(u_n)\,dx\,dt \\ &+ \int_{\{m\le|u_n|\le m+1\}} \Phi_n(u_n)\nabla u_n h'_m(u_n)(T_k(u_n) - w^\mu_{i,j})\,dx\,dt \\ &+ \int_Q \Phi_n(u_n)h_m(u_n)(\nabla T_k(u_n) - \nabla w^\mu_{i,j})\,dx\,dt = \int_Q f_n\varphi^{\mu,i}_{n,j,m}\,dx\,dt. \end{aligned} \tag{5.47}$$

Denoting by $\epsilon(n,j,\mu,i)$ any quantity such that,

$$\lim_{i\to\infty}\lim_{\mu\to\infty}\lim_{j\to\infty}\lim_{n\to\infty}\epsilon(n,j,\mu,i) = 0.$$

The very definition of the sequence $w^\mu_{i,j}$ makes it possible to establish the following lemma.

Lemma 5.6. *Let $\varphi^{\mu,i}_{n,j,m} = (T_k(u_n) - w^\mu_{i,j})h_m(u_n)$, we have for any $k \ge 0$:*

$$\langle \frac{\partial b_n(x,u_n)}{\partial t}, \varphi^{\mu,i}_{n,j,m}\rangle \ge \epsilon(n,j,\mu,i), \tag{5.48}$$

where $\langle , \rangle$ denotes the duality pairing between $L^1(Q) + W^{-1,x}L_{\overline{M}}(Q)$ and $L^\infty(Q) \cap W^{1,x}_0 L_M(Q)$.

Proof. See [34, 32]. □

Now, we turn to complete the proof of proposition 5.4. First, it is easy to see that (see also [32]):

$$\int_Q f_n\varphi^{\mu,i}_{n,j,m}\,dx\,dt = \epsilon(n,j,\mu), \tag{5.49}$$

$$\int_Q \Phi_n(u_n)h_m(u_n)(\nabla T_k(u_n) - \nabla w^\mu_{i,j})\,dx\,dt = \epsilon(n,j,\mu), \tag{5.50}$$

and

$$\int_{\{m\le|u_n|\le m+1\}} \Phi_n(u_n)\nabla u_n(T_k(u_n)-w^\mu_{i,j})\,dx\,dt=\epsilon(n,j,\mu). \tag{5.51}$$

Concerning the third term of the right hand side of (5.47) we obtain that

$$\begin{aligned}\int_{\{m\le|u_n|\le m+1\}} & a_n(x,t,u_n,\nabla u_n)\nabla u_n h'_m(u_n)(T_k(u_n)-w^\mu_{i,j})\,dx\,dt \\ &\le 2k\int_{\{m\le|u_n|\le m+1\}} a_n(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt.\end{aligned} \tag{5.52}$$

Then by (5.26). we deduce that,

$$\int_{\{m\le|u_n|\le m+1\}} a_n(x,t,u_n,\nabla u_n)\nabla u_n h'_m(u_n)(T_k(u_n)-w^\mu_{i,j})\,dx\,dt\le\epsilon(n,\mu,m). \tag{5.53}$$

Finally, by means of (5.47)-(5.53), we obtain,

$$\int_Q a_n(x,t,u_n,\nabla u_n)(\nabla T_k(u_n)-\nabla w^\mu_{i,j})h_m(u_n)\,dx\,dt\le\epsilon(n,j,\mu,m). \tag{5.54}$$

Splitting the first integral on the left hand side of (5.54) where $|u_n|\le k$ and $|u_n|>k$, we can write,

$$\begin{aligned}&\int_Q a_n(x,t,u_n,\nabla u_n)(\nabla T_k(u_n)-\nabla w^\mu_{i,j})h_m(u_n)\,dx\,dt\\ &=\int_Q a_n(x,t,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla w^\mu_{i,j})h_m(u_n)\,dx\,dt\\ &\quad-\int_{\{|u_n|>k\}} a_n(x,t,u_n,\nabla u_n)\nabla w^\mu_{i,j}h_m(u_n)\,dx\,dt.\end{aligned}$$

Since $h_m(u_n)=0$ if $|u_n|\ge m+1$, one has

$$\begin{aligned}&\int_Q a_n(x,t,u_n,\nabla u_n)(\nabla T_k(u_n)-\nabla w^\mu_{i,j})h_m(u_n)\,dx\,dt\\ &=\int_Q a_n(x,t,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla w^\mu_{i,j})h_m(u_n)\,dx\,dt\\ &-\int_{\{|u_n|>k\}} a_n(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\nabla w^\mu_{i,j}h_m(u_n)\,dx\,dt=I_1+I_2\end{aligned} \tag{5.55}$$

In the following we pass to the limit in (5.55) as n tends to $+\infty$, then j then μ and then m tends to $+\infty$. We prove that

$$I_2=\int_Q \varphi_m\nabla T_k(u)_\mu h_m(u)_{\chi_{\{|u|>k\}}}\,dx\,dt+\epsilon(n,j,\mu).$$

Using now the term I_1 of (5.55), we conclude that, it is easy to show that,

$$\begin{aligned}&\int_Q a_n\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)(\nabla T_k(u_n)-\nabla w^\mu_{i,j})h_m(u_n)\,dx\,dt\\ &=\int_Q\Big[a_n(x,t,T_k(u_n),\nabla T_k(u_n))-a_n(x,t,T_k(u_n),\nabla T_k(v_j)\chi^s_j)\Big]\end{aligned} \tag{5.56}$$

$$\times\Big[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\Big] h_m(u_n)\, dx\, dt$$

$$+\int_Q a_n\Big(x,t,T_k(u_n),\nabla T_k(v_j)\chi_j^s\Big)\Big[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\Big] h_m(u_n)\, dx\, dt$$

$$+\int_Q a_n\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)\nabla T_k(v_j)\chi_j^s h_m(u_n)\, dx\, dt$$

$$-\int_Q a_n\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)\nabla w_{i,j}^{\mu} h_m(u_n)\, dx\, dt = J_1 + J_2 + J_3 + J_4,$$

where χ_j^s denotes the characteristic function of the subset

$$\Omega_s^j = \Big\{(x,t) \in Q \ :\ |\nabla T_k(v_j)| \le s\Big\}$$

In the following we pass to the limit in (5.56) as n tends to $+\infty$, then j then μ then m tends and then s tends to $+\infty$ in the last three integrals of the last side. We prove that

$$J_2 = \epsilon(n,j), \tag{5.57}$$

$$J_3 = \int_Q \varphi_k \nabla T_k(u)\chi_s\, dx\, dt + \epsilon(n,j), \tag{5.58}$$

and

$$J_4 = -\int_Q \varphi_k \nabla T_k(u)\, dx\, dt + \epsilon(n,j,\mu,s). \tag{5.59}$$

We conclude then that,

(5.60)

$$\int_Q \Big[a_n\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big) - a_n\Big(x,t,T_k(u_n),\nabla T_k(u)\chi_s\Big)\Big]\Big[\nabla T_k(u_n) - \nabla T_k(u)\chi_s\Big]\, dx\, dt$$

$$= \int_Q \Big[a_n\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big) - a_n\Big(x,t,T_k(u_n),\nabla T_k(u)\chi_s\Big)\Big]$$

$$\times\Big[\nabla T_k(u_n) - \nabla T_k(u)\chi_s\Big] h_m(u_n)\, dx\, dt$$

$$+\int_Q a_n\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)\Big[\nabla T_k(u_n) - \nabla T_k(u)\chi^s\Big](1 - h_m(u_n))\, dx\, dt$$

$$-\int_Q a_n\Big(x,t,T_k(u_n),\nabla T_k(u)\chi_s\Big)\Big[\nabla T_k(u_n) - \nabla T_k(u)\chi_s\Big](1 - h_m(u_n))\, dx\, dt.$$

Combining (5.48), (5.56), (5.57), (5.58), (5.59) and (5.60) we deduce,

(5.61)

$$\int_Q \Big[a_n\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big) - a_n\Big(x,t,T_k(u_n),\nabla T_k(u)\chi_s\Big)\Big]\Big[\nabla T_k(u_n) - \nabla T_k(u)\chi_s\Big]\, dx\, dt$$

$$\le \epsilon(n,j,\mu,m,s).$$

To pass to the limit in (5.61) as $n,\ j,\ m,\ s$ tends to infinity, we obtain

$$\lim_{s\to\infty}\lim_{n\to\infty}\int_Q \Big[a_n\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big) - a_n\Big(x,t,T_k(u_n),\nabla T_k(u)\chi_s\Big)\Big] \tag{5.62}$$

$$\times\Big[\nabla T_k(u_n) - \nabla T_k(u)\chi_s\Big]\, dx\, dt = 0.$$

This implies by the lemma 5.5, the desired statement and hence the proof of Proposition 5.4 is achieved. ☐

⋆ **Step 4**. In this step we prove that u satisfies (4.2).

Lemma 5.7. *The limit u of the approximate solution u_n of (5.6)-(5.8) satisfies*

$$\lim_{m\to+\infty}\int_{\{m\le|u|\le m+1\}} a(x,t,u,\nabla u)\nabla u\,dx\,dt = 0. \tag{5.63}$$

Proof. Remark that for any fixed $m \ge 0$ one has

$$\begin{aligned}&\int_{\{m\le|u_n|\le m+1\}} a_n(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt\\ &= \int_Q a_n(x,t,u_n,\nabla u_n)\Big[\nabla T_{m+1}(u_n) - \nabla T_m(u_n)\Big]\,dx\,dt\\ &= \int_Q a_n\Big(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n)\Big)\nabla T_{m+1}(u_n)\,dx\,dt\\ &\quad - \int_Q a_n\Big(x,t,T_m(u_n),\nabla T_m(u_n)\Big)\nabla T_m(u_n)\,dx\,dt\end{aligned}$$

According to (5.42) (with $z_n = T_m(u_n)$ or $z_n = T_{m+1}(u_n)$), one is at liberty to pass to the limit as n tends to $+\infty$ for fixed $m \ge 0$ and to obtain

$$\begin{aligned}&\lim_{n\to+\infty}\int_{\{m\le|u_n|\le m+1\}} a_n(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt\\ &= \int_Q a\Big(x,t,T_{m+1}(u),\nabla T_{m+1}(u)\Big)\nabla T_{m+1}(u)\,dx\,dt\\ &\quad - \int_Q a\Big(x,t,T_m(u),\nabla T_m(u)\Big)\nabla T_m(u)\,dx\,dt\\ &= \int_{\{m\le|u|\le m+1\}} a(x,t,u,\nabla u)\nabla u\,dx\,dt\end{aligned} \tag{5.64}$$

Taking the limit as m tends to $+\infty$ in (5.64) and using the estimate (5.26) it possible to conclude that (5.63) holds true and the proof of Lemma 5.7 is complete. ☐

⋆ **Step 5**. In this step, u is shown to satisfies (4.3) and (4.4). Let S be a function in $W^{2,\infty}(\mathbb{R})$ such that S' has a compact support. Let K be a positive real number such that $supp(S') \subset [-K,K]$. Pointwise multiplication of the approximate equation (5.6) by $S'(u_n)$ leads to

$$\begin{aligned}&\frac{\partial B_S^n(x,u_n)}{\partial t} - div\Big(S'(u_n)a_n(x,t,u_n,\nabla u_n)\Big) + S''(u_n)a_n(x,t,u_n,\nabla u_n)\nabla u_n\\ &\quad - div\Big(S'(u_n)\Phi(u_n)\Big) + S''(u_n)\Phi(u_n)\nabla u_n = fS'(u_n) \quad \text{in } D'(Q),\end{aligned} \tag{5.65}$$

where $B_S^n(x,z) = \displaystyle\int_0^z S'(r)\frac{\partial b_n(x,r)}{\partial r}\,dr$.

It what follows we pass to the limit as n tends to $+\infty$ in each term of (5.65).

$\star$ Since S' is bounded, and $B_S^n(x,u_n)$ converges to $B_S(x,u)$ a.e. in Q and in $L^\infty(Q)$ weak $\star$. Then $\frac{\partial B_S^n(x,u_n)}{\partial t}$ converges to $\frac{\partial B_S(x,u)}{\partial t}$ in $D'(Q)$ as n tends to $+\infty$.

$\star$ Since $suppS \subset [-K,K]$, we have

$$S'(u_n)a_n(x,t,u_n,\nabla u_n) = S'(u_n)a_n\Big(x,t,T_K(u_n),\nabla T_K(u_n)\Big) \text{ a.e. in } Q.$$

The pointwise convergence of u_n to u as n tends to $+\infty$, the bounded character of S', (5.22) and (5.36) of Lemma 5.4 imply that

$$S'(u_n)a_n\Big(x,t,T_K(u_n),\nabla T_K(u_n)\Big) \rightharpoonup S'(u)a\Big(x,t,T_K(u),\nabla T_K(u)\Big) \text{ weakly in } (L_{\overline{M}}(Q))^N,$$

for $\sigma(\Pi L_{\overline{M}}, \Pi E_M)$ as n tends to $+\infty$, because $S(u)=0$ for $|u|\geq K$ a.e. in Q. And the term $S'(u)a\Big(x,t,T_K(u),\nabla T_K(u)\Big) = S'(u)a(x,t,u,\nabla u)$ a.e. in Q.

$\star$ Since $suppS' \subset [-K,K]$, we have

$$S''(u_n)a_n(x,t,u_n,\nabla u_n)\nabla u_n = S''(u_n)a_n\Big(x,t,T_K(u_n),\nabla T_K(u_n)\Big)\nabla T_K(u_n) \text{ a.e. in } Q.$$

The pointwise convergence of $S''(u_n)$ to $S''(u)$ as n tends to $+\infty$, the bounded character of S'' and (5.22)-(5.36) of Lemma 5.4 allow to conclude that

$$S'(u_n)a_n(x,t,u_n,\nabla u_n)\nabla u_n \rightharpoonup S'(u)a\Big(x,t,T_K(u),\nabla T_K(u)\Big)\nabla T_K(u) \text{ weakly in } L^1(Q),$$

as n tends to $+\infty$. And

$$S''(u)a\Big(x,t,T_K(u),\nabla T_K(u)\Big)\nabla T_K(u) = S''(u)a(x,t,u,\nabla u)\nabla u \text{ a.e. in } Q.$$

$\star$ Since $suppS' \subset [-K,K]$, we have $S'(u_n)\Phi_n(u_n) = S'(u_n)\Phi_n(T_K(u_n))$ a.e. in Q. As a consequence of (3.7), (5.3) and (5.22), it follows that:

$$S'(u_n)\Phi_n(u_n) \to S'(u)\Phi(T_K(u)) \text{ strongly in } (E_M(Q))^N,$$

as n tends to $+\infty$. The term $S'(u)\Phi(T_K(u))$ is denoted by $S'(u)\Phi(u)$.

$\star$ Since $S \in W^{1,\infty}(\mathbb{R})$ with $suppS' \subset [-K,K]$, we have $S''(u_n)\Phi_n(u_n)\nabla u_n = \Phi_n(T_K(u_n))\nabla S''(u_n)$ a.e. in Q, we have, $\nabla S''(u_n)$ converges to $\nabla S''(u)$ weakly in $L_M(Q)^N$ as n tends to $+\infty$, while $\Phi_n(T_K(u_n))$ is uniformly bounded with respect to n and converges a.e. in Q to $\Phi(T_K(u))$ as n tends to $+\infty$. Therefore

$$S''(u_n)\Phi_n(u_n)\nabla u_n \rightharpoonup \Phi(T_K(u))\nabla S''(u) \text{ weakly in } L_M(Q).$$

$\star$ Due to (5.4) and (5.22), we have $f_nS(u_n)$ converges to $fS(u)$ strongly in $L^1(Q)$, as n tends to $+\infty$.

As a consequence of the above convergence result, we are in a position to pass to the limit as n tends to $+\infty$ in equation (5.65) and to conclude that u satisfies (4.3).

It remains to show that $B_S(x,u)$ satisfies the initial condition (4.4). To this end, firstly remark that, S' has a compact support, we have $B_S^n(x,u_n)$ is bounded in $L^\infty(Q)$. Secondly, (5.65) and the above considerations on the behavior of the terms of this equation show that $\dfrac{\partial B_S^n(x,u_n)}{\partial t}$ is bounded in $L^1(Q) + W^{-1,x}L_{\overline{M}}(Q)$. As a consequence, an Aubin's type Lemma (see e.g., [36], Corollary 4) (see also [16]) implies that $B_S^n(x,u^n)$ lies in a compact set of $C^0([0,T];L^1(\Omega))$. It follows that, $B_S^n(x,u_n)(t=0)$ converges to $B_S(x,u)(t=0)$ strongly in $L^1(\Omega)$. Due to (4.8) and (5.5), we conclude that $B_S^n(x,u_n)(t=0) = B_S^n(x,u_{0n})$ converges to $B_S(x,u)(t=0)$ strongly in $L^1(\Omega)$. Then we conclude that

$$B_S(x,u)(t=0) = B_S(x,u_0) \text{ in } \Omega.$$

As a conclusion of step 1 to step 5, the proof of theorem 5.1 is complete.

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Faculté des Sciences Juridiques, Économiques et Sociales. Université Hassan 1, B.P. 764. Settat. Morocco

E-mail address: redwane_hicham@yahoo.fr