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Mezocontinuous operators and solutions of difference equations

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Abstract. We attempt to unify and extend the theory of asymptotic properties of solutions to difference equations of various types. Usually in difference equations some functions are used which generate transformations of sequences. We replace these functions by abstract operators and investigate some properties of such operators. We are interested in properties of operators which correspond to continuity or boundedness or local boundedness of functions. Next we investigate asymptotic properties of the set of all solutions to 'abstract' and 'functional' difference equations. Our approach is based on using the iterated remainder operator and the asymptotic difference pair. Moreover, we use the regional topology on the space of all real sequences and the 'regional' version of the Schauder fixed point theorem.

Keywords: difference equation, asymptotic difference pair, prescribed asymptotic behavior, paracontinuous operator, mezocontinuous operator.

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1 Introduction

Let $\mathbb{N}$, $\mathbb{R}$ denote the set of positive integers and the set of real numbers, respectively. Moreover, let $\mathrm{SQ} = \mathbb{R}^{\mathbb{N}}$ denote the space of all real sequences $x : \mathbb{N} \to \mathbb{R}$. In the paper we assume that

$$m \in \mathbb{N}, \qquad F : \mathrm{SQ} \to \mathrm{SQ}$$

and consider difference equations of the form

$$\Delta^m x_n = a_n F(x)(n) + b_n, \tag{E}$$

where $a_n, b_n \in \mathbb{R}$. We say that (E) is an abstract difference equation of order m.

Let $p \in \mathbb{N}$. We say that a sequence $x \in \mathrm{SQ}$ is a p-solution of equation (E) if equality (E) is satisfied for any $n \geq p$. We say that x is a solution if it is a p-solution for certain $p \in \mathbb{N}$. If x is a p-solution for any $p \in \mathbb{N}$, then we say that x is a full solution.

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As a special case of (E) we get equations of type:

$$\Delta^m x_n = a_n f(n, x_{\sigma_1(n)}, \dots, x_{\sigma_k(n)}) + b_n, \qquad f : \mathbb{N} \times \mathbb{R}^k \to \mathbb{R}, \tag{E1}$$

where $k \in \mathbb{N}$, $\sigma_1, \dots, \sigma_k : \mathbb{N} \to \mathbb{N}$, or

$$\Delta^m x_n = a_n f(n, x_n, \Delta x_n, \Delta^2 x_n, \dots, \Delta^k x_n) + b_n, \qquad f : \mathbb{N} \times \mathbb{R}^{k+1} \to \mathbb{R}, \tag{E2}$$

where k is an arbitrary natural number (the case $k > m$ is not excluded).

In the series of papers [15–23] a new method in the study of asymptotic properties of solutions to difference equations is presented. This method, based on using the iterated remainder operator, and the regional topology on the space of all real sequences, allows us to control the degree of approximation. In the paper [22], summarizing some earlier results, the notion of a difference asymptotic pair was introduced and the theory of such pairs was used to study the asymptotic properties of solutions to autonomous difference equations of the form

$$\Delta^m x_n = a_n f(x_{\sigma(n)}) + b_n.$$

In this paper we extend the results from [22] to more general classes of equations. Our approach to the study of asymptotic properties of solutions were inspired by the papers [1–14] and [24–33].

The paper is organized as follows. In Section 2, we introduce notation and terminology. In Section 3 we introduce the notion of mezocontinuous operator and we give some examples to show that the mezocontinuity of the operator F in equation (E) corresponds to the continuity of the function f in equations (E1) and (E2). Main results are obtained in Section 4. In Section 5 we apply our results to 'functional' equations (E1) and (E2).

2 Notation and terminology

If $p, k \in \mathbb{N}$, $p \le k$, then $\mathbb{N}(p)$, $\mathbb{N}(p,k)$ denote the sets defined by

$$\mathbb{N}(p) = \{p, p+1, \dots\}, \qquad \mathbb{N}(p,k) = \{p, p+1, \dots, k\}.$$

We use the symbols

$$\mathrm{Sol}(E), \qquad \mathrm{Sol}_p(E), \qquad \mathrm{Sol}_\infty(E)$$

to denote the set of all full solutions of (E), the set of all p-solutions of (E), and the set of all solutions of (E) respectively. If x, y in SQ, then xy and $|x|$ denote the sequences defined by $xy(n) = x_n y_n$ and $|x|(n) = |x_n|$ respectively. Let $a, b \in \mathrm{SQ}$, $p \in \mathbb{N}$. We will use the following notations

$$\mathrm{Fin}(p) = \{x \in \mathrm{SQ} : x_n = 0 \text{ for } n \ge p\}, \qquad \mathrm{Fin} = \bigcup_{p=1}^{\infty} \mathrm{Fin}(p),$$

$$\mathrm{o}(1) = \{x \in \mathrm{SQ} : \ x \text{ is convergent to zero}\}, \qquad \mathrm{O}(1) = \{x \in \mathrm{SQ} : \ x \text{ is bounded}\},$$

$$\mathrm{o}(a) = \{ax : \ x \in \mathrm{o}(1)\} + \mathrm{Fin}, \qquad \mathrm{O}(a) = \{ax : \ x \in \mathrm{O}(1)\} + \mathrm{Fin},$$

$$\Delta^{-m} b = \{y \in \mathrm{SQ} : \ \Delta^m y = b\}, \qquad \mathrm{Pol}(m-1) = \mathrm{Ker}\Delta^m.$$

For a subset Y of a metric space X and $c > 0$ let

$$\mathrm{B}^*(Y, c) = \bigcup_{y \in Y} \mathrm{B}^*(y, c),$$

where $\mathrm{B}^*(y,c)$ denotes the closed ball of radius c centered at y. For $y,\rho \in \mathrm{SQ}$ and $p \in \mathbb{N}$ we define

$$\mathrm{B}^*(y,\rho,p) = \{x \in \mathrm{SQ} : |x-y| \leq |\rho| \text{ and } x_n = y_n \text{ for } n < p\}.$$

Assume that Z is a linear subspace of a linear space X. We say that a subset W of X is Z-invariant if $W + Z \subset W$.

2.1 Regional topology

Let X be a real vector space. We say that a function $\|\cdot\| : X \to [0,\infty]$ is a *regional norm* if the condition $\|x\| = 0$ is equivalent to $x = 0$ and for any $x,y \in X$ and $\alpha \in \mathbb{R}$ we have

$$\|\alpha x\| = |\alpha|\|x\|, \qquad \|x+y\| \leq \|x\| + \|y\|.$$

Hence, the notion of a regional norm generalizes the notion of a usual norm. Note that a regional norm may take the value ∞. If a regional norm on X is given, then we say that X is a *regional normed space*, if there exists a vector $x \in X$ such that $\|x\| = \infty$, then we say that X is *extraordinary*.

Assume X is a regional normed space. We say that a subset Z of X is *ordinary* if $\|x-y\| < \infty$ for any $x,y \in Z$. We regard any ordinary subset Z of X as a metric space with metric defined by

$$d(x,y) = \|x-y\|.$$

We say that a subset U of X is regionally open if $U \cap Z$ is open in Z for any ordinary subset Z of X. The family of all regionally open subsets is a topology on X which we call the regional topology. We regard any subset of X as a topological space with topology induced by the regional topology. Let

$$\mathrm{Reg}(0) = \{x \in X : \|x\| < \infty\}.$$

Obviously $\mathrm{Reg}(0)$ is a linear subspace of X. Moreover, the regional norm induces a usual norm on $\mathrm{Reg}(0)$. We say that X is a Banach regional space if $\mathrm{Reg}(0)$ is complete. Let $x \in X$. We say that the set

$$\mathrm{Reg}(x) = x + \mathrm{Reg}(0)$$

is a *region* of x. If $y \in X$ and $\|x-y\| < \infty$, then $\mathrm{Reg}(x) = \mathrm{Reg}(y)$. Any region is ordinary and open in X. Moreover, any region is connected and is metrically equivalent to the normed space $\mathrm{Reg}(0)$. From a topological point of view, the space X is a disjoint union of all regions. Note that if $x \in X$, then the region $\mathrm{Reg}(x)$ is the ordinary component of x and the connected component of x. Hence any ordinary subset of X is a subset of a certain region, and any connected subset of X is ordinary. Moreover, if $H : \mathrm{SQ} \to \mathrm{SQ}$ is continuous and $x \in \mathrm{SQ}$, then

$$H(\mathrm{Reg}(x)) \subset \mathrm{Reg}(H(x)).$$

We say that a subset Y of X is regional if $\mathrm{Reg}(y) \subset Y$ for any $y \in Y$. The basic properties of the regional topology are presented in [21]. We will use the following theorem (see [21, Theorem 3.1]).

Theorem 2.1 (Generalized Schauder theorem). *Assume Q is a closed and convex subset of a regional Banach space X, a map $H : Q \to Q$ is continuous and the set HQ is ordinary and totally bounded. Then there exists a point $x \in Q$ such that $Hx = x$.*

We will use the standard regional norm on SQ defined by

$$\|x\| = \sup\{|x_n| : n \in \mathbb{N}\}.$$

Moreover, we will use the following fixed point theorem.

Theorem 2.2. *Assume $y \in \mathrm{SQ}$, $\rho \in \mathrm{o}(1)$, $p \in \mathbb{N}$, and $B = \mathrm{B}^*(y, \rho, p)$. Then any continuous map $H : B \to \mathrm{B}$ has a fixed point.*

Proof. By [21, Theorem 3.3], B is ordinary, convex and compact. Hence the assertion is a consequence of Theorem 2.1. □

2.2 Remainder operator

For $t \in [1, \infty)$ and $m \in \mathbb{N}$ let

$$\mathrm{A}(t) := \left\{ a \in \mathrm{SQ} : \sum_{n=1}^{\infty} n^{t-1} |a_n| < \infty \right\},$$

$$r^m : \mathrm{A}(m) \to \mathrm{o}(1), \qquad r^m(a)(n) = \sum_{j=n}^{\infty} \binom{m-1+j-n}{m-1} a_j.$$

Then r^m is a linear operator which we call the iterated remainder operator of order m. The value $r^m(a)(n)$ we denote also by $r^m_n(a)$ or simply $r^m_n a$. The following lemma is a consequence of [20, Lemma 3.1], [20, Lemma 4.2], and [20, Lemma 4.8].

Lemma 2.3. *Assume $a \in \mathrm{A}(m)$, $u \in \mathrm{O}(1)$, $k \in \{0, 1, \dots, m\}$, and $p \in \mathbb{N}$. Then*

(a) $\mathrm{O}(a) \subset \mathrm{A}(m) \subset \mathrm{o}(n^{1-m})$, $|r^m(ua)| \le \|u\| r^m |a|$, $\Delta r^m |a| \le 0$,

(b) $|r^m_p a| \le r^m_p |a| \le \sum_{n=p}^{\infty} n^{m-1} |a_n|$, $r^k a \in \mathrm{A}(m-k)$,

(c) $\Delta^m r^m a = (-1)^m a$, $r^m \mathrm{Fin}(p) = \mathrm{Fin}(p) = \Delta^m \mathrm{Fin}(p)$.

For more information about the remainder operator see [20].

2.3 Asymptotic difference pairs

We say that a pair (A, Z) of linear subspaces of SQ is an *asymptotic difference pair* of order m or, simply, m-pair if

$$\mathrm{Fin} + Z \subset Z, \qquad \mathrm{O}(1)A \subset A, \qquad A \subset \Delta^m Z.$$

We say that an m-pair (A, Z) is *evanescent* if $Z \subset \mathrm{o}(1)$. The following lemma is a consequence of [22, Lemma 3.5 and Lemma 3.7].

Lemma 2.4. *Assume (A, Z) is an m-pair, $a, b, x \in \mathrm{SQ}$. Then*

(a) *if $a - b \in A$, then $\Delta^{-m} a + Z = \Delta^{-m} b + Z$,*

(b) *if $a \in A$ and $\Delta^m x \in \mathrm{O}(a) + b$, then $x \in \Delta^{-m} b + Z$,*

(c) *if $Z \subset \mathrm{o}(1)$, then $A \subset \mathrm{A}(m)$ and $r^m A \subset Z$.*

Example 2.5. Assume $s \in \mathbb{R}$, $t \in (-\infty, m-1]$, $\lambda \in (1, \infty)$, and

$$(s+1)(s+2)\cdots(s+m) \neq 0.$$

Then

$$(\mathrm{o}(n^s), \mathrm{o}(n^{s+m})), \qquad (\mathrm{O}(n^s), \mathrm{O}(n^{s+m})), \qquad (\mathrm{A}(m-t), \mathrm{o}(n^t)),$$
$$(\mathrm{o}(\lambda^n), \mathrm{o}(\lambda^n)), \qquad (\mathrm{O}(\lambda^n), \mathrm{O}(\lambda^n))$$

are m-pairs.

Example 2.6. Assume $s \in (-\infty, -m)$, $t \in (-\infty, 0]$, $u \in [1, \infty)$, and $\lambda \in (0,1)$. Then

$$(\mathrm{o}(n^s), \mathrm{o}(n^{s+m})), \qquad (\mathrm{O}(n^s), \mathrm{O}(n^{s+m})), \qquad (\mathrm{A}(m-t), \mathrm{o}(n^t)),$$
$$(\mathrm{A}(m+u), \mathrm{A}(u)), \qquad (\mathrm{o}(\lambda^n), \mathrm{o}(\lambda^n)), \quad (\mathrm{O}(\lambda^n), \mathrm{O}(\lambda^n))$$

are evanescent m-pairs.

For more information about difference pairs see [22].

3 Mezocontinuous operators

Assume $W \subset X \subset \mathrm{SQ}$ and $H : X \to \mathrm{SQ}$. We define $\|H\| \in [0, \infty]$ by

$$\|H\| = \sup\{|H(x)(n)| : x \in X, n \in \mathbb{N}\}.$$

We say, that H is bounded if $\|H\| < \infty$. Let P be a property of operators. We say that H has the property P on W if the restriction $H|W$ has the property P. Recall that we regard any subset of SQ as a topological space with topology induced by the regional topology. Hence, the continuity of H is defined by a standard way. We say, that H is:

paracontinuous if for any $\varepsilon > 0$ and any $n \in \mathbb{N}$ there exists a $\delta = \delta(n, \varepsilon) > 0$ such that if $x, z \in X$, and $\|x - z\| < \delta$, then $|H(x)(n) - H(z)(n)| < \varepsilon$,

mezocontinuous if it is paracontinuous on any bounded subset of X,

regionally bounded if it is bounded on $X \cap \mathrm{Reg}(x)$ for any $x \in X$.

We say that a map $G : X \to Y$ from a subset X of SQ to a metric space Y is *uniformly continuous* if it is uniformly continuous on any ordinary subset Z of X.

Remark 3.1. For $n \in \mathbb{N}$ let ev_n denote the evaluation (projection operator) defined by

$$\mathrm{ev}_n : \mathrm{SQ} \to \mathbb{R}, \qquad \mathrm{ev}_n(x) = x_n.$$

If $X \subset \mathrm{SQ}$, then an operator $H : X \to \mathrm{SQ}$ is paracontinuous if and only if for any n the function $\mathrm{ev}_n \circ H : \mathrm{SQ} \to \mathbb{R}$ is uniformly continuous.

Remark 3.2. If $X \subset \mathrm{SQ}$, $H_1, H_2, \ldots, H_k : X \to \mathrm{SQ}$ are paracontinuous, and a function $\varphi : \mathbb{R}^k \to \mathbb{R}$ is uniformly continuous, then the operator $H : X \to \mathrm{SQ}$ defined by

$$H(x)(n) = \varphi(H_1(x)(n), \ldots, H_k(x)(n))$$

is paracontinuous.

Example 3.3. Assume $\varphi_n : \mathbb{R} \to \mathbb{R}$ is a sequence of uniformly continuous functions. Then the operator $H : \mathrm{SQ} \to \mathrm{SQ}$ defined by $H(x)(n) = \varphi_n(x_n)$ is paracontinuous.

Example 3.4. Assume $k \in \mathbb{N}$, $f : \mathbb{N} \times \mathbb{R}^k \to \mathbb{R}$ is continuous and $\sigma_1, \sigma_2, \dots, \sigma_k : \mathbb{N} \to \mathbb{N}$. Then the operator

$$H : \mathrm{SQ} \to \mathrm{SQ}, \quad H(x)(n) = f(n, x_{\sigma_1(n)}, \dots, x_{\sigma_k(n)})$$

is mezocontinuous.

Justification. Assume S is a bounded subset of SQ. Choose a positive ε and an index n. Let

$$S_n = \{(x_{\sigma_1(n)}, \dots, x_{\sigma_k(n)}) : x \in S\}.$$

Since S is bounded, the set S_n is a bounded subset of $\mathbb{R}^k$. Choose a compact interval I such that $S_n \subset I^k$. The function

$$g : \mathbb{R}^k \to \mathbb{R}, \qquad g(t_1, \dots, t_k) = f(n, t_1, \dots, t_k)$$

is uniformly continuous on I^k. Choose $\delta > 0$ such that if $\alpha, \beta \in I^k$, and $\|\alpha - \beta\| < \delta$, then $|g(\alpha) - g(\beta)| < \varepsilon$. Now, assume $x, z \in S$ and $\|x - z\| < \delta$. Then

$$x^* := (x_{\sigma_1(n)}, \dots, x_{\sigma_k(n)}) \in S_n, \qquad y^* := (y_{\sigma_1(n)}, \dots, y_{\sigma_k(n)}) \in S_n$$

and $\|x^* - y^*\| \leq \|x - y\| < \delta$. Hence $|g(x^*) - g(y^*)| < \varepsilon$. This means that

$$|H(x)(n) - H(y)(n)| < \varepsilon.$$

Therefore H is paracontinuous on S. □

Example 3.5. If $k \in \mathbb{N}$ and $f : \mathbb{N} \times \mathbb{R}^{k+1} \to \mathbb{R}$ is continuous, then the operator

$$H : \mathrm{SQ} \to \mathrm{SQ}, \qquad H(x)(n) = f(n, x_n, \Delta x_n, \Delta^2 x_n, \dots, \Delta^k x_n)$$

is mezocontinuous.

Justification. Assume S is a bounded subset of SQ. Choose a positive ε and an index n. Let

$$S_n = \{(x_n, \Delta x_n, \Delta^2 x_n, \dots, \Delta^k x_n) : x \in S\}.$$

Since S is bounded, the set S_n is a bounded subset of $\mathbb{R}^{k+1}$. Choose a compact interval I such that $S_n \subset I^{k+1}$. The function

$$g : \mathbb{R}^{k+1} \to \mathbb{R}, \qquad g(t_0, t_1, \dots, t_k) = f(n, t_0, t_1, \dots, t_k)$$

is uniformly continuous on I^{k+1}. Note that if $x, y \in \mathrm{SQ}$, then

$$\|\Delta x - \Delta y\| \leq 2\|x - y\|, \dots, \|\Delta^k x - \Delta^k y\| \leq 2^k \|x - y\|.$$

Choose $\delta > 0$ such that if $\alpha, \beta \in I^{k+1}$, and $\|\alpha - \beta\| < 2^k \delta$, then $|g(\alpha) - g(\beta)| < \varepsilon$. Now, assume $x, z \in S$ and $\|x - z\| < \delta$. Then

$$x^* := (x_n, \Delta x_n, \Delta^2 x_n, \dots, \Delta^k x_n) \in S_n, \qquad y^* := (y_n, \Delta y_n, \Delta^2 y_n, \dots, \Delta^k y_n) \in S_n$$

and $\|x^* - y^*\| < 2^k \delta$. Hence $|g(x^*) - g(y^*)| < \varepsilon$. This means that

$$|H(x)(n) - H(y)(n)| < \varepsilon.$$

Therefore H is paracontinuous on S. □

Example 3.6. Assume B is a bounded subset of $\mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$, the restriction $f|B$ is continuous but not uniformly continuous, $p \in \mathbb{N}$, $W = \{x \in \mathrm{SQ} : x(\mathbb{N}) \subset B\}$, and

$$H : W \to \mathrm{SQ}, \qquad H(x)(n) := \begin{cases} f(x_p) & \text{for } n = p \\ x_n & \text{for } n \neq p. \end{cases}$$

Then H is continuous but not mezocontinuous.

Justification. Let $x \in W$ and $\varepsilon > 0$. Choose $\delta \in (0, \varepsilon)$ such that if $t \in X$ and $|t - x_p| < \delta$, then $|f(t) - f(x_p)| < \varepsilon$. Now, if $z \in B$ and $\|z - x\| < \delta$, then $\|Hz - Hx\| < \varepsilon$. Hence H is continuous. Choose positive ε and δ. Since $f|B$ is not uniformly continuous, there exist $s, t \in B$ such that $|s - t| < \delta$ and $|f(s) - f(t)| \geq \varepsilon$. Define sequences x, z by: $x_n = z_n = s$ for $n \neq p$, $x_p = s$, $z_p = t$. Then $x, z \in W$, $\|z - x\| = |z_p - x_p| = |t - s| < \delta$, and

$$|H(z)(p) - H(x)(p)| = |f(t) - f(s)| \geq \varepsilon.$$

Hence H is not paracontinuous. Since W is bounded, H is not mezocontinuous. □

Example 3.7. Let $f : \mathbb{R} \to \mathbb{R}$ and $H : \mathrm{SQ} \to \mathrm{SQ}$ is given by $H(x)(n) = f(x_n)$. Then:

(a) if f is uniformly continuous then H is uniformly continuous,

(b) if f is continuous then H is mezocontinuous,

(c) if f is not uniformly continuous then H is discontinuous.

Justification. The assertion (a) is obvious, and (b) is a consequence of Example 3.4. Assume f is not uniformly continuous. Then there exists a positive ε such that for any $n \in \mathbb{N}$ there exist $x_n, z_n \in \mathbb{R}$ satisfying $|x_n - z_n| < 1/n$ and $|f(x_n) - f(z_n)| \geq \varepsilon$. Let $\delta > 0$. Choose $k \in \mathbb{N}$ such that $\delta < 1/k$ and define $y \in \mathrm{SQ}$ by

$$y_n = \begin{cases} x_n & \text{for } n \leq k, \\ z_n & \text{for } n > k. \end{cases}$$

Then $\|x - y\| < \delta$ and $\|Hx - Hy\| \geq \varepsilon$. Hence H is discontinuous at x. □

4 Solutions of abstract equations

Let $W \subset \mathrm{SQ}$, $a \in \mathrm{A}(m)$, and $p \in \mathbb{N}$. We say that W is

(F, a, p)-*regular* if for any $y \in W$ there exists a positive constant M such that F is paracontinuous on $B = \mathrm{B}^*(y, Mr^m|a|, p)$ and $\|F|B\| \leq M$,

F-regular if for any $y \in W$ there exist a positive constant c and an index q such that $F|\mathrm{B}^*(y, c, q)$ is paracontinuous and bounded,

F-optimal if W is $\mathrm{o}(1)$-invariant and $F|W$ is mezocontinuous and regionally bounded,

F-ordinary if $F(W) \subset \mathrm{O}(1)$.

Theorem 4.1. *Assume (A, Z) is an evanescent m-pair, $a \in \mathrm{A}$, $p \in \mathbb{N}$, $M > 0$,*

$$y \in \Delta^{-m} b, \qquad \rho = Mr^m|a|, \qquad B = \mathrm{B}^*(y, \rho, p), \qquad \|F|B\| \leq M,$$

and F is paracontinuous or continuous on B. Then $y \in \mathrm{Sol}_p(\mathrm{E}) + Z$.

Proof. If $x \in B$, then the sequence Fx is bounded. Hence $aFx \in \mathrm{O}(a) \subset \mathrm{A}(m)$. Define

$$H : S \to \mathrm{SQ}, \qquad H(x)(n) = \begin{cases} y_n & \text{for } n < p, \\ y_n + (-1)^m r_n^m(aFx) & \text{for } n \geq p. \end{cases}$$

If $n \geq p$, then using Lemma 2.3 we have

$$|H(x)(n) - y_n| = |r_n^m(aFx)| \leq r_n^m|aFx| \leq Mr_n^m|a| = \rho_n.$$

Hence $HB \subset B$. Let $\varepsilon > 0$. Assume that F is paracontinuous on B. There exist $q \geq p$ and $\alpha > 0$ such that

$$M \sum_{n=q}^{\infty} n^{k-1}|a_n| < \varepsilon \quad \text{and} \quad \alpha \sum_{n=p}^{q} n^{k-1}|a_n| < \varepsilon.$$

For any $n \in \{p, \dots, q\}$ there exists $\delta_n > 0$ such that if $x, z \in B$ and $\|x - z\| < \delta_n$, then

$$|F(x)(n) - F(z)(n)| < \alpha.$$

Let $\delta = \min(\delta_p, \delta_{p+1}, \dots, \delta_q)$. If $x, z \in B$ and $\|x - z\| < \delta$, then using Lemma 2.3, we obtain

$$\begin{aligned} \|Hx - Hz\| &= \sup_{n\geq 1} |H(x)(n) - H(z)(n)| = \sup_{n \geq p} |r_n^m(aFx) - r_n^m(aFz)| \\ &= \sup_{n\geq p} |r_n^m(aFx - aFz)| \leq \sup_{n\geq p} r_n^m|aFx - aFz| \\ &= r_p^m|aFx - aFz| \leq \sum_{n=p}^{\infty} n^{m-1}|a_nF(x)(n) - a_nF(z)(n)| \\ &\leq \sum_{n=p}^{q} n^{m-1}|a_nF(x)(n) - a_nF(z)(n)| + \sum_{n=q}^{\infty} n^{m-1}|a_nF(x)(n) - a_nF(z)(n)| \\ &\leq \alpha \sum_{n=p}^{q} n^{m-1}|a_n| + \sum_{n=q}^{\infty} n^{m-1}|a_nF(x)(n)| + \sum_{n=q}^{\infty} n^{m-1}|a_nF(z)(n)| < 3\varepsilon. \end{aligned}$$

Hence H is continuous. Now assume that F is continuous on B and $x \in B$. There exists a $\delta(x, \varepsilon) > 0$ such that the condition $\|z - x\| < \delta(x, \varepsilon)$ implies $|Fx - Fz| < \varepsilon$. If $z \in B$, $\|z - x\| < \delta(x, \varepsilon)$, and $n \geq p$, then, we obtain

$$\begin{aligned} |H(x)(n) - H(z)(n)| &= |r_n^m(aF(x)) - r_n^m(aF(z))| \\ &\leq r_n^m(|a||Fx - Fz|) \leq r_n^m(\varepsilon|a|) \leq \varepsilon r_1^m|a| = M^{-1}\rho_1\varepsilon. \end{aligned}$$

Hence $\|Hx - Hz\| \leq M^{-1}\rho_1\varepsilon$. Therefore H is continuous. By Theorem 2.2, there exists an $x \in B$ such that $Hx = x$. Then $x_n = y_n + (-1)^m r_n^m aF(x)$ for $n \geq p$. Hence

$$x - y - (-1)^m r^m aFx \in \mathrm{Fin}(p). \tag{4.1}$$

Using Lemma 2.3 we have

$$\Delta^m((-1)^m r^m aF(x)) = aF(x), \quad \text{and} \quad \Delta^m \mathrm{Fin}(p) = \mathrm{Fin}(p). \tag{4.2}$$

Using (4.1), (4.2), and the equality $\Delta^m y = b$, we get

$$\Delta^m x - aF(x) - b \in \mathrm{Fin}(p).$$

Hence $x \in \mathrm{Sol}_p(\mathrm{E})$. Moreover, we have $(-1)^m aFx \in \mathrm{O}(a)$. Hence, by (4.1),

$$y \in x + r^m \mathrm{O}(a) + \mathrm{Fin}(p).$$

By Lemma 2.3 we get

$$\begin{aligned} r^m \mathrm{O}(a) + \mathrm{Fin}(p) &= r^m \mathrm{O}(a) + r^m \mathrm{Fin}(p) = r^m (\mathrm{O}(a) + \mathrm{Fin}(p)) \\ &= r^m \mathrm{O}(a) \subset r^m A \subset Z. \end{aligned}$$

□

Corollary 4.2. *Assume (A, Z) is an evanescent m-pair, $a \in \mathrm{A}$, $p \in \mathbb{N}$, and W is an (F, a, p)-regular subset of* SQ*. Then*

$$W \cap \Delta^{-m} b \subset \mathrm{Sol}_p(\mathrm{E}) + Z.$$

Proof. The assertion is an immediate consequence of Theorem 4.1. □

Corollary 4.3. *Assume (A, Z) is an evanescent m-pair, $a \in \mathrm{A}$, and W is an F-regular subset of* SQ*. Then*

$$W \cap \Delta^{-m} b \subset \mathrm{Sol}_\infty(\mathrm{E}) + Z. \tag{4.3}$$

Proof. Let $y \in W \cap \Delta^{-m} b$. Choose a positive constant c and an index q such that F is paracontinuous and bounded on $\mathrm{B}^*(y, c, q)$. Let

$$M = \| F | \mathrm{B}^*(y, c, q) \|.$$

Since $r^m |a| \in \mathrm{o}(1)$, there exists an index $k \geq q$ such that $M r_k^m |a| \leq c$. Since the sequence $r^m |a|$ is nonincreasing, we have

$$\mathrm{B}^*(y, M r^m |a|, k) \subset \mathrm{B}^*(y, c, q).$$

By Theorem 4.1, $y \in \mathrm{Sol}_k(\mathrm{E}) + Z \subset \mathrm{Sol}_\infty(\mathrm{E}) + Z$ and we obtain (4.3). □

Example 4.4. Assume $g : \mathbb{R} \to \mathbb{R}$ is positive, continuous, and nondecreasing,

$$f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}, \qquad f(n, t) = g(t), \qquad F : \mathrm{SQ} \to \mathrm{SQ}, \qquad F(x)(n) = f(n, x_n).$$

Moreover, assume $a \in \mathrm{A}(m)$, $p \in \mathbb{N}$, $M, \alpha, \lambda \in \mathbb{R}$, $g(\lambda) = M$, $\lambda > \alpha$, and

$$\sum_{n=p}^{\infty} n^{m-1} |a_n| \leq \frac{\lambda - \alpha}{M}$$

Then the set $W = \{ y \in \mathrm{SQ} : \ y(\mathbb{N}) \subset (-\infty, \alpha) \}$ is (F, a, p)-regular.

Justification. By Lemma 2.3

$$r_n^m |a| \leq r_p^m |a| \leq \sum_{n=p}^{\infty} n^{m-1} |a_n| \leq \frac{\lambda - \alpha}{M}$$

for any $n \geq p$. Let $y \in W$ and let

$$B = \mathrm{B}^*(y, M r^m |a|, p).$$

If $x \in B$, then $|x_n - y_n| \leq M r_p^m |a|$ for any n. Hence $x_n \leq y_n + M r_p^m |a|$ and

$$|F(x)(n)| = |f(n, x_n)| = g(x_n) \leq g(y_n + M r_p^m |a|) \leq g(\alpha + \lambda - \alpha) = g(\lambda) = M$$

for any $x \in B$ and any $n \in \mathbb{N}$. Hence $\|F|B\| \leq M$. By Example 3.4, F is paracontinuous on B. Therefore W is (F, a, p)-regular. □

The following theorem is a consequence of Lemma 2.4. Note that in this theorem we do not assume that a pair (A, Z) is evanescent.

Theorem 4.5. *Assume (A, Z) is an m-pair, $a \in \mathrm{A}$, and W is an F-ordinary subset of* SQ. *Then*

$$W \cap \mathrm{Sol}_\infty(\mathrm{E}) \subset \Delta^{-m} b + Z. \tag{4.4}$$

Proof. If $x \in \mathrm{Sol}_\infty(\mathrm{E})$ and $Fx \in \mathrm{O}(1)$, then

$$\Delta^m x \in aF(x) + b + \mathrm{Fin} \subset \mathrm{O}(a) + b + \mathrm{Fin} = \mathrm{O}(a) + b.$$

Hence, using Lemma 2.4, we obtain (4.4). □

Lemma 4.6. *Assume X is a linear space, $W, S, Y \subset X$, Z is a linear subspace of X, W is Z-invariant and $W \cap S \subset Y + Z$. Then $W \cap S \subset W \cap Y + Z$.*

Proof. Let $w \in W \cap S$. By assumption, $w = y + z$ for some $y \in Y$ and $z \in Z$. Since W is Z-invariant, we have $y = w - z \in W$. Hence $y \in W \cap Y$ and we obtain

$$w = y + z \in W \cap Y + Z.$$ □

Theorem 4.7. *Assume (A, Z) is an evanescent m-pair, $a \in \mathrm{A}$, and W is a Z-invariant subset of* SQ. *Then*

(a) *if W is F-regular, then*
$$W \cap \mathrm{Sol}_\infty(\mathrm{E}) + Z = W \cap \Delta^{-m} b + Z,$$

(b) *if W is F-optimal, then*

$$W \cap \mathrm{Sol}(\mathrm{E}) + Z = W \cap \mathrm{Sol}_\infty(\mathrm{E}) + Z = W \cap \Delta^{-m} b + Z.$$

Proof. Assume W is F-regular. Then W is F-ordinary and, by Theorem 4.5,

$$W \cap \mathrm{Sol}_\infty(\mathrm{E}) \subset \Delta^{-m} b + Z. \tag{4.5}$$

By Corollary 4.3, we have
$$W \cap \Delta^{-m} b \subset \mathrm{Sol}_\infty(\mathrm{E}) + Z. \tag{4.6}$$

Using (4.6), (4.5), and Lemma 4.6 we obtain (a).

Assume W is F-optimal. Then, W is F-ordinary and, using Theorem 4.5,

$$W \cap \mathrm{Sol}(\mathrm{E}) \subset W \cap \mathrm{Sol}_\infty(\mathrm{E}) \subset \Delta^{-m} b + Z. \tag{4.7}$$

Assume $y \in W \cap \Delta^{-m} b$. Let $M = \|F|\mathrm{Reg}(y)\|$. Since $a \in A \subset \mathrm{A}(m)$, we have

$$\sum_{n=1}^{\infty} n^{m-1} |a_n| < \infty.$$

Choose a positive c such that

$$M \sum_{n=1}^{\infty} n^{m-1} |a_n| < c.$$

Then
$$\mathrm{B}^*(y, Mr^m|a|, 1) \subset \mathrm{B}^*(y, c, 1) \subset y + \mathrm{O}(1) = \mathrm{Reg}(y).$$

By Theorem 4.1,

$$y \in \mathrm{Sol}_1(\mathrm{E}) + Z = \mathrm{Sol}(\mathrm{E}) + Z.$$

Hence

$$W \cap \Delta^{-m} b \subset \mathrm{Sol}(\mathrm{E}) + Z. \tag{4.8}$$

Using (4.8), (4.7), and Lemma 4.6 we obtain (b). □

Let $X \subset \mathrm{SQ}$. We say that an operator $H : X \to \mathrm{SQ}$ is *unbounded* at a point $p \in [-\infty, \infty]$ if there exists a sequence $x \in X$ and an increasing sequence $\alpha : \mathbb{N} \to \mathbb{N}$ such that

$$\lim_{n\to\infty} x(\alpha(n)) = p \quad \text{and} \quad \lim_{n\to\infty} |H(x)(\alpha(n))| = \infty.$$

Let

$$\mathrm{U}(H) = \{p \in [-\infty, \infty] : H \text{ is unbounded at } p\}.$$

For $x \in \mathrm{SQ}$ let

$$\mathrm{L}(x) = \{p \in [-\infty, \infty] : p \text{ is a limit point of } x\}.$$

The following theorem extends [22, Theorem 4.2].

Theorem 4.8. *Assume* (A, Z) *is an m-pair,* $a \in A$*, and* $x \in \mathrm{Sol}_\infty(\mathrm{E})$*. Then*

$$x \in \Delta^{-m} b + Z \quad \textit{or} \quad \mathrm{L}(x) \cap \mathrm{U}(F) \neq \emptyset.$$

Proof. Assume $\mathrm{L}(x) \cap \mathrm{U}(F) = \emptyset$. If the sequence $F(x)$ is unbounded from above, then there exists an increasing sequence $\beta : \mathbb{N} \to \mathbb{N}$ such that

$$\lim_{n\to\infty} F(x)(\beta(n)) = \infty. \tag{4.9}$$

Let $y = x \circ \beta$ and let $p \in \mathrm{L}(y)$. There exists an increasing sequence $\gamma : \mathbb{N} \to \mathbb{N}$ such that

$$\lim_{n\to\infty} y(\gamma(n)) = p. \tag{4.10}$$

Let $\alpha = \beta \circ \gamma$. Then

$$x(\alpha(n)) = (x \circ \beta \circ \gamma)(n) = (x \circ \beta)(\gamma(n)) = y(\gamma(n)).$$

Hence, by (4.10), $\lim_{n\to\infty} x(\alpha(n)) = p$. Moreover, using (4.9), we get

$$\lim_{n\to\infty} F(x)(\alpha(n)) = \lim_{n\to\infty} F(x)(\beta(\gamma(n))) = \infty.$$

Therefore $p \in \mathrm{U}(F)$. Since y is a subsequence of x, we have $p \in \mathrm{L}(x)$. Thus

$$p \in \mathrm{L}(x) \cap \mathrm{U}(F).$$

Analogously, if the sequence $F(x)$ is unbounded from below, then $\mathrm{L}(x) \cap \mathrm{U}(F) \neq \emptyset$. Therefore $F(x)$ is bounded. Since $x \in \mathrm{Sol}_\infty(\mathrm{E})$, we have

$$\Delta^m x \in aF(x) + b + \mathrm{Fin} \subset a\mathrm{O}(1) + \mathrm{Fin} + b = \mathrm{O}(a) + b.$$

By Lemma 2.4 (b) we get $x \in \Delta^{-m} b + Z$. □

5 Solutions of functional equations

For a subset V of $\mathbb{R}$ we denote by $\overline{V}$ the closure of V in the extended line $[-\infty,\infty]$.

Theorem 5.1. *Assume (A,Z) is an evanescent m-pair, $a \in A$, $k,p \in \mathbb{N}$, $c > 0$,*

$$f : \mathbb{N} \times \mathbb{R}^k \to \mathbb{R}, \qquad \sigma_1,\dots,\sigma_k : \mathbb{N} \to \mathbb{N}, \qquad \sigma_i(n) \to \infty \quad \text{for } i = 1,\dots,k,$$

$$V \subset \mathbb{R}, \qquad W = \{x \in \mathrm{SQ} : \mathrm{L}(x) \subset \overline{V}\}, \qquad U = \mathbb{N}(p) \times \mathrm{B}^*(V,c)^k,$$

and f is continuous. Then W is $\mathrm{o}(1)$*-invariant and*

(*a*) *if f is bounded on U, then for any Z-invariant subset Q of W we have*

$$Q \cap \Delta^{-m} b + Z = Q \cap \mathrm{Sol}_\infty(\mathrm{E1}) + Z; \tag{5.1}$$

(*b*) *if f is bounded, then for any Z-invariant subset Q of* SQ *we have*

$$Q \cap \mathrm{Sol}(\mathrm{E1}) + Z = Q \cap \mathrm{Sol}_\infty(\mathrm{E1}) + Z = Q \cap \Delta^{-m} b + Z. \tag{5.2}$$

Proof. If $y \in \mathrm{SQ}$ and $z \in \mathrm{o}(1)$, then $\mathrm{L}(y+z) = \mathrm{L}(y)$. Hence the set W is $\mathrm{o}(1)$-invariant. Let

$$F : \mathrm{SQ} \to \mathrm{SQ}, \qquad F(x)(n) = f(n, x_{\sigma_1(n)},\dots,x_{\sigma_k(n)}).$$

Assume f is bounded on U. Choose $y \in W$ and $\varepsilon \in (0, c/2)$. It is easy to see that the set

$$y(\mathbb{N}) \setminus \mathrm{B}^*(V,\varepsilon)$$

is finite. Hence there exists an index $p_1 \ge p$ such that $y_n \in \mathrm{B}^*(V,\varepsilon)$ for any $n \ge p_1$. Choose $q \in \mathbb{N}$ such that $\sigma_i(n) \ge p_1$ for any $n \ge q$ and any $i \in \{1,\dots,k\}$. Let

$$q_1 = \max \bigcup_{i=1}^{k} \sigma_i(\mathbb{N}(1,q)), \qquad Y = y(\mathbb{N}(1,q_1)), \qquad C = \mathbb{N}(1,q) \times \mathrm{B}^*(Y,\varepsilon)^k.$$

Then C is compact and f is bounded on C. Define

$$M_1 = \|f|U\| \quad \text{and} \quad M_2 = \|f|C\|.$$

Let $x \in \mathrm{B}^*(y,\varepsilon,q)$. Then $|x_n - y_n| \le \varepsilon$ for any n. If $n \ge q$ and $i \in \mathbb{N}(1,k)$, then $\sigma_i(n) \ge p_1$ and

$$y_{\sigma_i(n)} \in \mathrm{B}^*(V,\varepsilon).$$

Hence there exists $u \in V$ such that $|u - y_{\sigma_i(n)}| \le \varepsilon$. Then

$$|x_{\sigma_i(n)} - u| \le |x_{\sigma_i(n)} - y_{\sigma_i(n)}| + |y_{\sigma_i(n)} - u| \le 2\varepsilon < c.$$

Hence $(x_{\sigma_1(n)},\dots,x_{\sigma_k(n)}) \in \mathrm{B}^*(V,c)^k$ and for $n \ge q$ we get

$$|F(x)(n)| = |f(n, x_{\sigma_1(n)},\dots,x_{\sigma_k(n)})| \le M_1.$$

If $n \le q$ and $i \in \mathbb{N}(1,k)$, then

$$x_{\sigma_i(n)} \in \mathrm{B}^*(y_{\sigma_i(n)},\varepsilon) \subset \mathrm{B}^*(Y,\varepsilon)$$

Hence

$$(n, x_{\sigma_1(n)}, \dots, x_{\sigma_k(n)}) \in C \quad \text{and} \quad |F(x)(n)| \le M_2.$$

Therefore F is bounded on $\mathrm{B}^*(y, \varepsilon, q)$. By Example 3.4, F is paracontinuous on $\mathrm{B}^*(y, \varepsilon, q)$. Thus W is F-regular. Hence any subset Q of W is F-regular. If, moreover Q is Z-invariant, then, by Theorem 4.7 (a), we obtain (5.1). Now, assume f is bounded on $\mathbb{N} \times \mathbb{R}^k$ and Q is a Z-invariant subset of SQ. Then we can take $V = \mathbb{R}$, $W = \mathrm{SQ}$, and by (5.1), we obtain

$$Q \cap \mathrm{Sol}(\mathrm{E1}) + Z \subset Q \cap \mathrm{Sol}_\infty(\mathrm{E1}) + Z = Q \cap \Delta^{-m} b + Z. \tag{5.3}$$

Assume $y \in Q \cap \Delta^{-m} b$. Let $M = \|f\|$. Then $\|F\| \le M$ and

$$\|F|\mathrm{B}^*(y, Mr^m|a|, 1)\| \le M.$$

By Example 3.4, F is paracontinuous on $\mathrm{B}^*(y, Mr^m|a|, 1)$. Hence, by Theorem 4.1,

$$y \in \mathrm{Sol}_1(\mathrm{E1}) + Z = \mathrm{Sol}(\mathrm{E1}) + Z.$$

Therefore

$$Q \cap \Delta^{-m} b \subset \mathrm{Sol}(\mathrm{E1}) + Z. \tag{5.4}$$

Using (5.3), (5.4) and Lemma 4.6 we obtain (5.2). □

Theorem 5.2. *Assume (A, Z) is an evanescent m-pair, $a \in A$, $k, p \in \mathbb{N}$, $c > 0$,*

$$V_0, V_1, \dots, V_k \subset \mathbb{R}, \qquad U = \mathbb{N}(p) \times \mathrm{B}^*(V_0, c) \times \cdots \times \mathrm{B}^*(V_k, c),$$

$$W = \{x \in \mathrm{SQ} : \mathrm{L}(\Delta^i x) \subset \overline{V_i} \quad \text{for } i = 0, 1, \dots, k\},$$

and $f : \mathbb{N} \times \mathbb{R}^{k+1} \to \mathbb{R}$ is continuous. Then W is $\mathrm{o}(1)$-invariant and

(a) *if f is bounded on U, then for any Z-invariant subset Q of W we have*

$$Q \cap \Delta^{-m} b + Z = Q \cap \mathrm{Sol}_\infty(\mathrm{E2}) + Z; \tag{5.5}$$

(b) *if f is bounded, then for any Z-invariant subset Q of SQ we have*

$$Q \cap \mathrm{Sol}(\mathrm{E2}) + Z = Q \cap \mathrm{Sol}_\infty(\mathrm{E2}) + Z = Q \cap \Delta^{-m} b + Z. \tag{5.6}$$

Proof. Obviously W is $\mathrm{o}(1)$-invariant. Let

$$F : \mathrm{SQ} \to \mathrm{SQ}, \qquad F(x)(n) = f(n, x_n, \Delta x_n, \Delta^2 x_n, \dots, \Delta^k x_n).$$

Assume f is bounded on U. Choose $y \in W$ and $\varepsilon \in (0, c/2^{k+1})$. It is easy to see that, for any $i \in \mathbb{N}(0, k)$, the set

$$\Delta^i y(\mathbb{N}) \setminus \mathrm{B}^*(V_i, \varepsilon)$$

is finite. Hence there exists an index p such that $\Delta^i y_n \in \mathrm{B}^*(V_i, \varepsilon)$ for any $i \in \mathbb{N}(0, k)$ and any $n \ge p$. Let $M = \|F|U\|$, $x \in \mathrm{B}^*(y, \varepsilon, p)$ and $n \ge p$. If $i \in \mathbb{N}(0, k)$, then

$$|\Delta^i x_n - \Delta^i y_n| \le 2^i \varepsilon$$

and there exists a point $u_i \in V_i$ such that $|\Delta^i y_n - u_i| < \varepsilon$. Hence

$$\Delta^i x_n \in \mathrm{B}^*(V_i, 2^{i+1}\varepsilon) \subset \mathrm{B}^*(V_i, c).$$

Therefore $|F(x)(n)| \leq M$ for any $x \in \mathrm{B}^*(y, \varepsilon, p)$ and any $n \geq p$. Thus F is bounded on $\mathrm{B}^*(y, \varepsilon, p)$. By Example 3.5, F is paracontinuous on $\mathrm{B}^*(y, \varepsilon, p)$. Hence W is F-regular. Assume Q is a Z-invariant subset of W. Then Q is F-regular and, by Theorem 4.7 (a), we obtain (5.5). Now, assume f is bounded and Q is a Z-invariant subset of SQ. Then we can take

$$V_0 = V_1 = \cdots = V_k = \mathbb{R}, \qquad W = \mathrm{SQ},$$

and, by (5.5) we obtain

$$Q \cap \mathrm{Sol}(\mathrm{E2}) + Z \subset Q \cap \mathrm{Sol}_\infty(\mathrm{E2}) + Z = Q \cap \Delta^{-m} b + Z. \tag{5.7}$$

Assume $y \in Q \cap \Delta^{-m} b$. Let $M = \|f\|$. Then $\|F\| \leq M$ and

$$\|F|\mathrm{B}^*(y, Mr^m|a|, 1)\| \leq M.$$

By Example 3.4, F is paracontinuous on $\mathrm{B}^*(y, Mr^m|a|, 1)$. Hence, by Theorem 4.1,

$$y \in \mathrm{Sol}_1(\mathrm{E2}) + Z = \mathrm{Sol}(\mathrm{E2}) + Z.$$

Therefore

$$Q \cap \Delta^{-m} b \subset \mathrm{Sol}(\mathrm{E2}) + Z. \tag{5.8}$$

Using (5.7), (5.8) and Lemma 4.6 we obtain (5.6). □

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