Electronic Journal of Qualitative Theory of Differential Equations
2010, No. **25**, 1-11; **http://www.math.u-szeged.hu/ejqtde/**

OSCILLATION CRITERIA FOR SECOND ORDER NONLINEAR PERTURBED DIFFERENTIAL EQUATIONS

MOUSSADEK REMILI

Abstract. Sufficient conditions for the oscillation of the nonlinear second order differential equation $(a(t)x')' + Q(t, x') = P(t, x, x')$ are established where the coefficients are continuous and $a(t)$ is nonnegative.

1. INTRODUCTION

We are concerned here with the oscillatory behavior of solutions of the following second order nonlinear differential equation:

$$(a(t)x')' + Q(t, x) = P(t, x, x'), \tag{1.1}$$

where $a : [T_0, \infty) \to \mathbb{R}, Q : [T_0, \infty) \times \mathbb{R} \to \mathbb{R}$, and $P : [T_0, \infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous and $a(t) > 0$. Throughout the paper, we shall restrict our attention only to the solutions of the differential equation (1.1) which exist on some ray of the form $[T_0, \infty)$.

In this paper we give more general integral criteria to the oscillation of (1.1), which contain the results in [8] as particular cases.

A solution of (1.1) is said to be oscillatory if it has arbitrarily large zeros, and otherwise it is said to be nonoscillatory. If all solutions of (1.1) are oscillatory, (1.1) is called oscillatory. The oscillatory behavior of solutions of second order ordinary differential equation including the existence of oscillatory and nonoscillatory solutions has been the subject of intensive investigations. This problem has received the attention of many authors. Many criteria have been found which involve the average behavior of the integral of the alternating coefficient. Among numerous papers dealing with this subject we refer in particular to [1, 3, to 16 and 19, 20].

2. MAIN RESULTS

Assume that there exist continuous functions $p, q : [T_0, \infty) \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$, such that

$$xf(x) > 0 \quad \text{for} \quad x \neq 0, \tag{2.1}$$

$$f'(x) \geq k > 0 \quad \text{for} \quad x \neq 0, \tag{2.2}$$

1991 *Mathematics Subject Classification.* 34C10,34C15.

Key words and phrases. Oscillation, second order nonlinear differential equation.

$$(2.3) \qquad \frac{Q(t,x)}{f(x)} \geq q(t) \quad \text{and} \quad \frac{P(t,x,x')}{f(x)} \leq p(t) \quad \text{for } x \neq 0.$$

Theorem 1. *Suppose that conditions (2.1),(2.2), and (2.3) hold and let ρ be a positive continuously differentiable function on the interval $[T,\infty)$ such that $\rho' \geq 0$ on $[T_0,\infty)$. Equation (1.1) is oscillatory if*

$$(2.4) \qquad \lim_{t\to\infty} \int_{T_0}^{t} \frac{1}{\rho(s)a(s)} ds = \infty,$$

$$(2.5) \qquad \int_{T_0}^{\infty} R(s)ds = \infty,$$

where

$$R(t) = \rho(t)[q(t) - p(t)] \; - \frac{1}{4k}\frac{\rho'^2(t)}{\rho(t)} a(t).$$

Proof. Let x be a nonoscillatory solution on an interval $[T,\infty), T \geq T_0$ of the differential equation (1.1). Without loss of generality, this solution can be supposed such that $x(t) \neq 0$. We assume that $x(t)$ is positive on $[T,\infty)$ (the case $x(t) < 0$ can be treated similarly and will be omitted).

Then

$$(2.6) \qquad \left[\frac{a(t)x'(t)}{f[x(t)]}\right]' = \frac{P[t,x'(t),x(t)]}{f[x(t)]} - \frac{Q[t,x(t)]}{f[x(t)]} - \frac{a(t)f'(x(t)[x'(t)]^2}{f^2[x(t)]}.$$

Multiplying (2.6) by $\rho(t)$ and integrating from T to t , we obtain

(2.7)

$$\frac{\rho(t)a(t)x'(t)}{f[x(t)]} \leq C_T - \int_T^t \rho(s)[q(s)-p(s)]ds + \int_T^t \rho'(s)\frac{a(s)x'(s)}{f[x(s)]}ds - \int_T^t \rho(s)\frac{a(s)f'(x(s)[x'(s)^2]}{f^2[x(s)]}ds.$$

Where $C_T = \frac{\rho(T)a(T)x'(T)}{f[x(T)]}$. We use the following notation

$$\omega(t) = \frac{a(t)x'(t)}{f[x(t)]} \text{ and } W(t) = \omega(t) - \frac{\rho'(t)a(t)}{2k\rho(t)}.$$

Then we have by condition (2.2)

$$\frac{\rho(t)a(t)x'(t)}{f[x(t)]} \leq C_T - \int_T^t \rho(s)[q(s)-p(s)]ds + \int_T^t \left[\rho'(s)\omega(s) - k\frac{\rho(s)}{a(s)}\omega^2(s)\right] ds$$

$$\leq C_T - \int_T^t \rho(s)[q(s)-p(s)]ds - \int_T^t \frac{k\rho(s)}{a(s)}\left[W^2(s) - \left(\frac{\rho'(s)a(s)}{2k\rho(s)}\right)^2\right] ds$$

$$(2.8) \qquad \leq C_T - \int_T^t R(s)ds,$$

we see from (2.5) that

$$\lim_{t\to\infty}\frac{\rho(t)a(t)x'(t)}{f[x(t)]}=-\infty,$$

hence, there exist $T_1 \geq T$ such that

$$x'(t)<0 \text{ for } t\geq T_1.$$

Condition (2.5) also implies $\int_T^\infty \rho(s)[q(s)-p(s)]ds=\infty$ and there exists $T_2\geq T_1$ such that
$\int_{T_1}^{T_2}\rho(s)[q(s)-p(s)]ds=0$ and $\int_{T_2}^{t}\rho(s)[q(s)-p(s)]ds\geq 0$ for $t\geq T_2$. Now multiplying (1.1) by $\rho(t)$ and integrating by parts we obtain

$$\begin{aligned}
\rho(t)a(t)x'(t) &\leq C_{T_2}+\int_{T_2}^{t}\rho'(s)a(s)x'(s)ds-\int_{T_2}^{t}f[x(s)]\rho(s)[q(s)-p(s)]ds\\
&\leq C_{T_2}-f\left[x(t)\right]\int_{T_2}^{t}\rho(s)[q(s)-p(s)]ds\\
&\quad+\int_{T_2}^{t}x'(s)f'\left[x(s)\right]\int_{T_2}^{s}\rho(u)[q(u)-p(u)]duds\\
&\leq C_{T_2}\quad \text{for every } \ t\ \geq T_1,
\end{aligned}$$

where $C_{T_2}=\rho(T_2)a(T_2)x'(T_2)<0$. Thus

$$x(t)\leq C_{T_2}\int_{T_2}^{t}\frac{1}{\rho(s)a(s)}ds.$$

from (2.4) it follows that $x(t)\to-\infty$ as $t\to\infty$ which is a contradiction. ∎

Example 1. *Consider the equation*

$$[a(t)x']'+\left[\frac{1}{2}t^{-\frac{3}{2}}(2+\cos(t))+te^{x}\right]x=xt^{-\frac{1}{2}}\sin(t)+\frac{1}{t^3}\frac{x^3\cos(x')}{x^2+1}\quad \textit{for } t\geq\frac{\pi}{2}.$$

If we choose $f(x)=x$, $a(t)=Log(t)$ and $\rho(t)=t$, then

$$\frac{Q(t,x)}{f(x)}\geq\frac{1}{2}t^{-\frac{3}{2}}(2+\cos(t))=q(t);\ \frac{P(t,x,x')}{f(x)}\leq t^{-\frac{1}{2}}\sin(t)+\frac{1}{t^3}=p(t).$$

For every $t\geq T_0=\frac{\pi}{2}$ we obtain

$$\int_{T_0}^{t}R(s)ds=\int_{T_0}^{t}s(\frac{1}{2}s^{-\frac{3}{2}}(2+\cos(s))-s^{-\frac{1}{2}}\sin(s)-\frac{1}{s^3})-\frac{1}{4}\frac{Log(s)}{s})ds$$

$$=\int_{T_0}^{t}s(\frac{1}{2}s^{-\frac{3}{2}}(2+\cos(s))-s^{-\frac{1}{2}}\sin(s))ds-\int_{T_0}^{t}\frac{1}{s^2}ds-\int_{T_0}^{t}\frac{1}{4}\frac{Log(s)}{s}ds$$

$$=\int_{T_0}^{t}d(s^{\frac{1}{2}}(2+\cos(s))+\frac{1}{t}-\frac{2}{\pi}-\frac{1}{8}Log^2(t)+\frac{1}{8}Log^2(\frac{\pi}{2})$$

$$=t^{\frac{1}{2}}(2+\cos t)-2(\frac{\pi}{2})^{\frac{1}{2}}+\frac{1}{t}-\frac{2}{\pi}-\frac{1}{8}Log^2(t)+\frac{1}{8}Log^2(\frac{\pi}{2})$$

$$\geq t^{\frac{1}{2}} - 2(\frac{\pi}{2})^{\frac{1}{2}} - \frac{2}{\pi} - \frac{1}{8}Log^2(t).$$

Thus we have

$$\int_{T_0}^{\infty} R(s) = \infty \quad \text{and} \quad \lim_{t\to\infty} \int_{T_0}^{t} \frac{1}{\rho(s)a(s)} ds = \int_{T_0}^{\infty} \frac{1}{sLog(s)} ds = \infty,$$

i.e. (2.1),(2.2),(2.3),(2.4) and (2.5) are satisfied. Hence the differential equation is oscillatory.

Theorem 2. *If the conditions (2.1),(2.2),(2.3) ,(2.4) hold, and let ρ be a positive continuously differentiable function on the interval $[T, \infty)$ such that $\rho' \geq 0$ on $[T_0, \infty)$ with*

$$\int_{T_0}^{\infty} \rho(s)[q(s) - p(s)]ds < \infty, \tag{2.9}$$

$$\liminf_{t\to\infty} \left[\int_{T}^{t} R(s)ds\right] \geq 0 \quad \textit{for all large } T, \tag{2.10}$$

$$\lim_{t\to\infty} \int_{T_0}^{t} \frac{1}{\rho(s)a(s)} \int_{s}^{\infty} R(u)duds = \infty, \tag{2.11}$$

and

$$\int_{\epsilon}^{\infty} \frac{dy}{f(y)} < \infty \quad \textit{and} \quad \int_{-\epsilon}^{-\infty} \frac{dy}{f(y)} < \infty \quad \textit{for every} \quad \epsilon > 0. \tag{2.12}$$

Then all solutions of (1.1) are oscillatory.

Remark 1. *Condition (2.9) implies that*

$$\int_{T}^{\infty} R(s)ds < \infty \quad \text{and} \quad \liminf_{t\to\infty} \left[\int_{T}^{t} R(s)ds\right] = \int_{T}^{\infty} R(s)ds,$$

hence (2.10) takes the form

$$\int_{T}^{\infty} R(s)ds \geq 0 \text{ for all large } T,$$

Proof. Let x be a nonoscillatory solution on an interval $[T,\infty)$ of the differential equation (1.1). We suppose, as in Theorem 1, that x is positive on $[T, \infty)$. We consider the following three cases for the behavior of $x'(t)$.

Case 1: $x'(t) > 0$ for $t \geq T_1$ for some $T_1 \geq T$, then from (2.8) we have

$$\int_{T_1}^{t} R(s)ds \leq \frac{\rho(T_1)a(T_1)x'(T_1)}{f[x(T_1)]} - \frac{\rho(t)a(t)x'(t)}{f[x(t)]}.$$

Hence, for all $t \geq T_1$

$$\int_{t}^{\infty} R(s)ds \leq \rho(t)\frac{a(t)x'(t)}{f[x(t)]}.$$

Using (2.12), we obtain

$$\begin{aligned}\int_{T_1}^{t} \frac{1}{\rho(s)a(s)} \int_{s}^{\infty} R(u)duds &\leq \int_{T_1}^{t} \frac{x'(s)}{f[x(s)]}ds \\ &\leq \int_{x(T_1)}^{\infty} \frac{dy}{f(y)} < \infty.\end{aligned}$$

This contradicts condition (2.11).

Case 2: $x'(t)$ changes signs, then there exists a sequence $(\alpha_n) \to \infty$ in $[T, \infty)$ such that $x'(\alpha_n) < 0$. Choose N large enough so that

$$\int_{\alpha_N}^{\infty} R(s)ds \geq 0$$

Then from (2.8) we have

$$\frac{\rho(t)a(t)x'(t)}{f[x(t)]} \leq C_{\alpha_N} - \int_{\alpha_N}^{t} R(s)ds.$$

So

$$\begin{aligned}\limsup_{t\to\infty} \frac{\rho(t)a(t)x'(t)}{f[x(t)]} &\leq C_{\alpha_N} + \limsup_{t\to\infty} \left[-\int_{\alpha_N}^{t} R(s)ds\right] \\ &= C_{\alpha_N} - \liminf_{t\to\infty} \left[\int_{\alpha_N}^{t} R(s)ds\right] \\ &< 0.\end{aligned}$$

Which contradicts the fact that $x'(t)$ oscillates.

Case 3: $x'(t) < 0.$ for $t \geq T_1$ for some $T_1 \geq T$, Wong[16] showed that (2.10) implies that for any $t_0 \geq T_0$ there exists $t_1 \geq t_0$ such that $\int_{t_1}^{\infty} \rho(s)[q(s)-p(s)]ds \geq 0$ for all $t \geq t_1$. Choosing $t_1 \geq T_1$ and then integrating (1.1) we have

$$\begin{aligned}\rho(t)a(t)x'(t) &\leq C_{t_1} + \int_{t_1}^{t} \rho'(s)a(s)x'(s)ds - \int_{t_1}^{t} f[x(s)]\rho(s)[q(s)-p(s)]ds \\ &\leq C_{t_1} - f\left[x(t)\right] \int_{t_1}^{t} \rho(s)[q(s)-p(s)]ds \\ &\quad + \int_{t_1}^{t} x'(s)f'\left[x(s)\right] \int_{t_1}^{t} \rho(u)[q(u)-p(u)]duds \\ &\leq C_{t_1} \quad \text{for every } t \geq t_1,\end{aligned}$$

where $C_{t_1} = \rho(t_1)a(t_1)x'(t_1) < 0.$

Thus

$$x(t) \leq C_{t_1} \int_{t_1}^{t} \frac{1}{\rho(s)a(s)}ds,$$

from (2.4) it follows that $x(t) \to -\infty$ as $t \to \infty$ which is a contradiction. ∎

Theorem 3. *Suppose (2.1),(2.2),(2.3) hold and assume that there exists a constant* $A > 0$ *such that*

$$\frac{a(t)}{\rho(t)} \le A, \tag{2.13}$$

$$\lim_{t\to\infty} \left[\int_T^t \frac{1}{\rho(s)} ds\right]^{-1} \int_T^t \frac{1}{\rho(s)} \int_T^s R(u)duds = \infty, \tag{2.14}$$

$$\lim_{t\to\infty} \int_T^t \frac{1}{s\rho(s)} ds = \infty. \tag{2.15}$$

Then (1) is oscillatory.

Proof. Let x be a nonoscillatory solution on an interval $[T, \infty)$, of the differential equation (1). Without loss of generality, this solution can be supposed such that $x(t) > 0$ for all $t \ge T$ (the case $x(t) < 0$ can be treated similarly and will be omitted).

defining for every $t \ge T$

$$g(t) = \left\{\int_T^t \frac{ds}{\rho(s)}\right\}^{-1}.$$

From (2.6) we have

$$\rho(t)\omega(t) + \int_T^t R(s)ds + \int_T^t \frac{k\rho(s)}{a(s)} W^2(s)ds \le C_T. \tag{2.16}$$

Therefore, for every $t \ge T$ we have

$$\begin{aligned} g(t)\int_T^t \omega(s)ds + \quad g(t)\int_T^t \frac{1}{\rho(s)} \int_T^s \frac{k\rho(s)}{a(s)} W^2(u)duds \\ \le C_T - g(t)\int_T^t \frac{1}{\rho(s)} \int_T^s R(u)duds. \end{aligned} \tag{2.17}$$

Now, by condition (2.14)

$$\lim_{t\to\infty} \left\{ g(t)\int_T^t \omega(s)ds + \quad g(t)\int_T^t \frac{1}{\rho(s)} \int_T^s \frac{k\rho(s)}{a(s)} W^2(u)duds \right\} = -\infty.$$

Hence, there exist $T_1 \ge T$ such that

$$\int_T^t \omega(s)ds + \quad \int_T^t \frac{1}{\rho(s)} \int_T^s \frac{k\rho(s)}{a(s)} W^2(u)duds < 0 \text{ for } t \ge T_1, \tag{2.18}$$

Defining

$$\begin{aligned} H(t) &= \left|\int_T^t \frac{a(s)}{k\rho(s)} W(s)ds\right| \\ \Psi(t) &= \int_T^t \frac{H^2(s)}{s\rho(s)} ds \quad \text{for all } \ t \ge T, \end{aligned}$$

we may use the Schwart inequality to obtain

$$H^2(t) \leq \int_T^t \left[\frac{a(s)}{k\rho(s)}\right]^2 ds \int_T^t W^2(s)ds,$$

from (2.13) we have

$$H^{\ 2}(t) \leq Ct \int_T^t W^{\ 2}(s)ds,$$

where $C = \frac{A^2}{k^2}$.Thus, by condition (2.18) for $t \geq T_1$

$$\begin{aligned} -H(t)g(t) + g(t)\frac{1}{C}\int_T^t \frac{H^{\ 2}(s)}{s\rho(s)}ds &\leq g(t)\int_T^t \frac{a(s)}{k\rho(s)}W(s)ds + g(t)\int_T^t \frac{1}{\rho(s)}\int_T^s W^2(u)duds \\ &\leq 0, \end{aligned}$$

then

$$H^2(t) \geq \frac{1}{C^2}\left[\int_T^t \frac{H^2(s)}{s\rho(s)}ds\right]^2 \quad \text{for all} \quad t \geq T_1,$$

and

$$\frac{1}{C^2}\frac{1}{t\rho(t)} \leq \frac{\Psi'(t)}{\Psi^2(t)} \qquad \text{for all} \quad t \geq T_1.$$

So for any $t \geq T_1 \geq T$

$$\frac{1}{C^2}\int_{T_1}^t \frac{1}{s\rho(s)}ds \leq \int_{T_1}^t \frac{\Psi'(s)}{\Psi^2(s)}ds = \frac{1}{\Psi(T_1)} - \frac{1}{\Psi(t)} \leq \frac{1}{\Psi(T_1)} < \infty.$$

This contradicts condition (2.15). The proof of the theorem is now complete. ∎

Remark 2. *Theorem 3 generalizes Theorem 4 in [8].*

Theorem 4. *Suppose (2.1), (2.2), (2.3), hold and assume that there exist a constant $\lambda > 0$ such that*

$$\liminf_{t\to\infty} \int_T^t R(s)ds > -\infty \text{ for all large } T, \tag{2.19}$$

$$\limsup_{t\to\infty} \frac{1}{t}\int_T^t \frac{1}{\rho(s)}\int_T^s R(u)duds = \infty \quad \text{for all large } T, \tag{2.20}$$

$$\frac{a(t)}{\rho(t)} \leq \lambda t. \tag{2.21}$$

Then all solutions of (1) are oscillatory.

Proof. Let x be a nonoscillatory solution on an interval $[T,\infty)$, of the differential equation (1). Without loss of generality, this solution can be supposed such that $x(t) > 0$. for all $t \geq T$. We consider the following three cases for the behavior of x'.

Case 1: x' is oscillatory. Then there exists a sequence (t_n) in $[T,\infty)$ with $\lim_{n\to\infty} t_n = \infty$ and such that $x'(t_n) = 0.(n \geq 1)$. Thus (2.8) gives

$$\int_T^{t_n} \frac{k\rho(s)}{a(s)}W^2(s)ds \leq C_T - \int_T^{t_n} R(s)ds,$$

and hence, by taking into account condition (2.19), we conclude that

$$\int_T^\infty \frac{k\rho(s)}{a(s)} W^2(s)ds < \infty.$$

So, for some constant M we have

$$\int_T^t \frac{k\rho(s)}{a(s)} W^2(s)ds \le M \quad \text{for every} \quad t \ge T. \tag{2.22}$$

By the Schwarz's inequality, we have

$$\begin{aligned} \left| -\int_T^t W(s)ds \right|^2 &= \int_T^t \frac{k\rho(s)}{a(s)} W^2(s)ds \int_T^t \frac{a(s)}{k\rho(s)}ds \le M \int_T^t \frac{a(s)}{k\rho(s)}ds \\ &\le \frac{1}{2k} M\lambda t^2. \end{aligned}$$

and hence for every $t \ge T$

$$-\int_T^t W(s)ds = -\int_T^t \omega(s) - \frac{\rho'(s)a(s)}{2k\rho(s)}ds \le \sqrt{\frac{1}{2k}M\lambda} t.$$

Furthermore, (2.16) gives

$$\frac{1}{\rho(t)} \int_T^t R(s)ds \le C_T - \omega(t),$$

and therefore for all $t \ge T$

$$\begin{aligned} \frac{1}{t}\int_T^t \frac{1}{\rho(s)} \int_T^s R(u)duds &\le \frac{C_T}{t}\int_T^t \frac{1}{\rho(s)}ds + \sqrt{\frac{1}{2k}M\lambda} \\ &\le \frac{C_T}{t}\frac{(t-T)}{\rho(T)} + \sqrt{\frac{1}{2k}M\lambda}, \end{aligned}$$

and

$$\limsup_{t\to\infty} \frac{1}{t}\int_T^t \frac{1}{\rho(s)} \int_T^s R(u)duds \le \frac{C_T}{\rho(T)} + \sqrt{\frac{1}{2k}M\lambda} < \infty.$$

This contradicts condition (2.20).

Case 2: $x' > 0$ on $[T_1, \infty), T_1 \ge T$. Using (2.8) we get

$$\int_T^t R(s)ds \le C_T,$$

and consequently

$$\limsup_{t\to\infty} \frac{1}{t}\int_T^t \frac{1}{\rho(s)} \int_T^s R(u)duds \le 0.$$

Which again contradicts (2.20).

Case 3: $x'(t) < 0$. From (2.7), and (2.19) it follows that

$$\frac{\rho(t)a(t)x'(t)}{f[x(t)]} \le C_T - \int_T^t \rho(s)[q(s)-p(s)]ds - \int_T^t \rho(s)\frac{a(s)[x'(s)]^2}{(f[x(s)])^2} f'(x(s))ds. \tag{2.23}$$

We distinguish two mutually exclusive cases where $-\int_T^\infty \rho(s)\frac{a(s)[x'(s)]^2}{(f[x(s)])^2} f'(x(s))ds$ is finite or infinite.

i) If $-\int_T^\infty \rho(s)\frac{a(s)[x'(s)]^2}{(f[x(s)])^2}f'(x(s))ds$ is finite. In this case, it follows that (2.22) holds for $t \geq T$. Once again, we can complete the proof by the procedure of the proof of Case 1.

ii) If $-\int_T^\infty R\rho(s)\frac{a(s)[x'(s)]^2}{(f[x(s)])^2}f'(x(s))ds$ is infinite. By Condition (2.19), and from (2.22) it follows that there exists a constant μ such that

$$-\frac{\rho(t)a(t)x'(t)}{f[x(t)]} \geq \mu + \int_T^t \left[\frac{x'(s)f'(x(s))}{f[x(s)]}\right]\frac{\rho(s)a(s)x'(s)}{f[x(s)]}ds \quad \text{for all } t \geq T.$$

Put

$$G(t) = \frac{x'(t)f'(x(t))}{f[x(t)]} \leq 0.$$

Furthermore, we choose a $T_1 \geq T$ so that

$$\mu + \int_T^{T_1} G(s)\frac{\rho(s)a(s)x'(s)}{f[x(s)]}ds = \mu_1 > 0,$$

and then for every $t \geq T_1$ we have

$$\frac{\rho(t)a(t)x'(t)}{f[x(t)]}G(t)\left[\mu + \int_T^t G(s)\frac{\rho(s)a(s)x'(s)}{f[x(s)]}ds\right]^{-1} \geq -G(t),$$

and integrating from $T_1$to t, we obtain

$$Log\frac{\left[\mu + \int_T^t G(s)\frac{\rho(s)a(s)x'(s)}{f[x(s)]}ds\right]}{\mu_1} \geq Log\frac{\rho(t)f(x(T))}{\rho(T)f(x(t))}.$$

Thus

$$\mu + \int_T^t G(s)\left(\frac{\rho(s)a(s)x'(s)}{f[x(s)]}\right)ds \geq \mu_1\frac{\rho(t)f(x(T))}{\rho(T)f(x(t))}.$$

The last inequality implies for $t \geq T_1$

$$x'(t) \leq -\frac{\eta}{a(t)},$$

where $\eta = \frac{\mu_1 + f(x(T))}{\rho(T)} > 0$. And consequently for $t \geq T_1$

$$x(t) \leq x(T_1) - \eta\int_{T_1}^t \frac{1}{a(s)}ds \leq -\frac{\eta}{b}(t - T_1).$$

Therefore, we conclude that $\lim_{t\to\infty} x(t) = -\infty$. This contradicts the assumption that $x(t) > 0$. This completes the proof of the theorem. ∎

Example 2. *Consider* $[a(t)x']' + \left[\frac{1}{2}t^{-\frac{5}{6}}(2 + \cos(t) + tx^2\right]x = xt^{-\frac{1}{6}}\sin(t) + \frac{1}{t^3}\frac{x^3\cos^2(x')}{x^2+1}$ *for* $t \geq \frac{\pi}{2}$, *with* $f(x) = x$ $a(t) = t^{2/3}$, $\rho(t) = t^{1/3}$ *then*

$$\frac{Q(t,x)}{f(x)} \geq \frac{1}{2}t^{-\frac{5}{6}}(2+\cos(t) = q(t); \quad \frac{P(t,x,x')}{f(x)} \leq t^{-\frac{1}{6}}\sin(t) + \frac{1}{t^3} = p(t).$$

For every $t \geq T_0 = \frac{\pi}{2}$, we obtain

$$\begin{aligned}
\int_{T_0}^{t} R(s)ds &= \int_{T_0}^{t} s(\frac{1}{2}s^{-\frac{5}{6}}(2+\cos(s) - s^{-\frac{1}{6}}\sin(s) - \frac{1}{s^3}) - \frac{1}{36}\frac{1}{s})ds \\
&= \int_{T_0}^{t} s(\frac{1}{2}s^{-\frac{3}{2}}(2+\cos(s) - s^{-\frac{1}{2}}\sin(s))ds - \int_{T_0}^{t}\frac{1}{s^2}ds - \int_{T_0}^{t}\frac{1}{36}\frac{1}{s}ds \\
&= \int_{T_0}^{t} d(s^{\frac{1}{2}}(2+\cos(s)) + \frac{1}{t} - \frac{2}{\pi} - \frac{1}{36}Log(t) + \frac{1}{36}Log(\frac{\pi}{2}) \\
&\geq t^{\frac{1}{2}} - 2(\frac{\pi}{2})^{\frac{1}{2}} - \frac{2}{\pi} - \frac{1}{36}Log(t).
\end{aligned}$$

Thus we have

$$\begin{aligned}
\frac{1}{t}\int_{T}^{t}\frac{1}{\rho(s)}\int_{T}^{s} R(u)duds &\geq \frac{1}{t}\int_{T}^{t} s^{\frac{-1}{3}}\left[s^{\frac{1}{2}} - 2(\frac{\pi}{2})^{\frac{1}{2}} - \frac{2}{\pi} - \frac{1}{36}Log(s)\right]ds \\
&\geq \frac{1}{t}\int_{T}^{t} s^{\frac{-1}{3}}\left[s^{\frac{1}{2}} - 2(\frac{\pi}{2})^{\frac{1}{2}} - \frac{2}{\pi} - \frac{1}{36}s^{\frac{1}{3}}\right]ds \\
&\geq \frac{6}{7}t^{\frac{1}{6}} - \left[2(\frac{\pi}{2})^{\frac{1}{2}} + \frac{2}{\pi}\right]t^{\frac{-1}{3}} - \frac{1}{36} - \frac{6}{7}(\frac{\pi}{2})^{\frac{7}{6}},
\end{aligned}$$

and consequently,
$\lim_{t\to\infty}\inf \int_T^t R(s)ds > -\infty$; $\limsup_{t\to\infty} \frac{1}{t}\int_T^t \frac{1}{\rho(s)}\int_T^s R(u)duds = \infty$; and $\frac{a(t)}{\rho(t)} \leq t^{1/3} \leq t$.

This means that (2.19), (2.20) hold. Thus, from Theorem 4 it follows that, when (2.21) is satisfied, our differential equation is oscillatory.

Acknowledgement

The author wishes to thank the referee for all his/her suggestions and remarks.

References

[1] B. Ayanlar and A. Tiryaki, *Oscillation theorems for nonlinear second-order differential equations.* Comp and Maths with Applications 44(2002) 529-538.

[2] G. J. Butler, *Integral averages and the oscillation of second order ordinary differential equations.* SIAM J. Math. Anal 11(1980)190-200.

[3] W. J. Coles, *An oscillation criterion for the second-order equations.* Proc. Amer. Math. Soc. 19(1968),755-759

[4] W. J. Coles, *Oscillation criteria for nonlinear second-order equations.* Ann. Mat. Pura. Appl 82(1969), 132-134

[5] E. M. Elabbasy, T. S. Hassan, S. H. Saker, *Oscillation of second-order nonlinear differential equations with a damping term.* Electronic Journal of Differential Equations, Vol.2005(2005), N° 76, pp. 1-13.

[6] H. L. Hong, *On the oscillatory behavior of solutions of second order non-linear differential equations.* Publ. Math. Debrecen 52, 55-68, (1998).
[7] J. R. Graef and P. W. Spikes, *On the oscillatory behavior of solutions of second order non-linear differential equations.* Czech, Math. J, 36(1986), 275-284
[8] J. R. Graef, S. M. Rankin and P. W. Spikes, *Oscillation Theorems for Perturbed nonlinear Differential Equation.* J. Math. Anal. Appl. 65, 375-390(1978)
[9] M. K. Kwang and J. S. W. Wong, *An application of integral inequality to second order non-linear oscillation.* J. Differential Equations 46, 63-67, (1992)
[10] I. V. Kamenev, *An integral criterion for oscillation of linear differential equation of second order.* Mat. Zametki 23, 249-251,(1978)
[11] CH. G. Philos, *Oscillation criteria for second order superlinear differential equations.* Can. J. Math. Vol. XLI, N°2, 1989, pp. 321-340.
[12] CH. G. Philos, *An oscillation criterion for superlinear differential Equations of second order.* J. Math Anal. Appl vol 148, N°2, May 15, 1990 306-316
[13] CH. G. Philos, *Integral averages and oscillation of second order sublinear differential equations.* Differential and integral equations, Vol 4, N°1, January 1991, pp. 205-213
[14] CH. G. Philos, O*scillation theorems for linear differential equation of second order.* Arch. Math. 53(1989), 483-492
[15] Y. G. Sun, *New Kamenev-type oscillation criteria for second-order nonlinear differential equations with damping.* J. Math. Anal. Appl. 291(2004), 341-351.
[16] J. S. W. Wong, *Oscillation theorems for second order nonlinear differential equations.* Bull. Inst. Math. Acad. Sinica (1975), 263-309.
[17] P. J. Y. Wong and R. P. Agarwal, *The oscillation and asymptotically monotone solutions of second order quasi linear differential equations.* Appl. Math. Compt. 79, 207-237,(1996).
[18] P. J. Y. Wong and R. P. Agarwal, *Oscillatory behavior of solutions of certain second order nonlinear differential equations.* J. Math. Anal. Appl. 198, 397-354,(1996).
[19] P. J. Y. Wong and R. P. Agarwal, *Oscillation criteria for half-linear differential equations.* Adv. Math. Sci. Appl. 9 (2), 649-663, (1999)
[20] C. C. Yeh, *Oscillation theorems for nonlinear second order differential equations with damped term.* Proc. Amer. Math. Soc. 84(1982), 397-402.
[21] C. C. Yeh, *An oscillation criterion for second order nonlinear differential Equations with functional arguments.* J. Math. Anal. Appl. 76, 72-76, (1980).
[22] Zhiting Xu, Yong Xia, *Kamenev-Type oscillation criteria for second-order quasilinear differential equations.* Electronic Journal of Differential Equations, Vol. 2005(2005), N°27, pp. 1-9.

Editorial Note (February 4, 2011): One of our readers has brought to our attention that the author needs to reference one of his earlier papers:
M. Remili, *Oscillation theorem for perturbed nonlinear differential equations*, International Mathematical Forum, 3, 2008, no. 11, 513-524.
We agree that it is critical for the interested reader to consult the earlier paper.

(Received June 23, 2009)

Department of Mathematics Faculty of Sciences University of Oran BP 1524 Algeria
E-mail address: m.remili@yahoo.fr