



Explicit integral criteria for the existence of positive solutions of first order linear delay equations

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Abstract. It is well known that the linear differential equation $\dot{x}(t) + p(t)x(t - \tau(t)) = 0$ with continuous delay $\tau: [t_0 - r, \infty) \rightarrow (0, r]$, $r > 0$, $t_0 \in \mathbb{R}$, and $p: [t_0, \infty) \rightarrow (0, \infty)$ has a positive solution on $[t_0, \infty)$ if an explicit criterion of the integral type

$$\int_{t-\tau(t)}^t p(s) ds \leq \frac{1}{e}$$

holds for all $t \in [t_0, \infty)$. In this paper new integral explicit criteria, which essentially supplement related results in the literature are established. For example, if, for $t \in [t_0, \infty)$ and a fixed $\mu \in (0, 1)$, the integral inequality

$$\int_{t-\tau/2}^t p(s) ds \leq \frac{1}{2e} + \frac{\mu\tau^3}{96t^3e}$$

holds, then there exists a $t_0^* \geq t_0$ and a positive solution $x = x(t)$ on $[t_0^*, \infty)$. Examples illustrating the effectiveness of the results are given.

Keywords: time delay, linear differential equation, positive solution, integral criterion.

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1 Introduction

The purpose of the paper is to derive explicit integral criteria for the existence of eventually positive solutions to the equation

$$\dot{x}(t) + p(t)x(t - \tau(t)) = 0, \quad (1.1)$$

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with $t \geq t_0 \in \mathbb{R}$, in terms of inequalities of the integral of the coefficient p (without loss of generality, we will assume t_0 sufficiently large throughout the paper to ensure that the performed computations are well-defined) where $p: [t_0, \infty) \rightarrow (0, \infty)$ is a continuous function. We will also assume, that the delay $\tau(t)$ is continuous, positive and bounded on $[t_0 - r, \infty)$ by a constant r , i.e., $\tau(t) \leq r$. Next, define the set $\mathbb{R}_+ := [0, \infty)$.

A solution to (1.1) is defined as follows: a continuous function $x: [t^* - r, \infty) \rightarrow \mathbb{R}$ is called a solution of (1.1) corresponding to $t^* \in [t_0, \infty)$ if x is differentiable on $[t^*, \infty)$ (the derivative at t^* is regarded as the right-hand derivative) and satisfies (1.1) for all $t \geq t^*$. A solution of (1.1) corresponding to t^* is called oscillatory if it has arbitrarily large zeros. Otherwise, it is called non-oscillatory. A non-oscillatory solution x of (1.1) corresponding to t^* is called positive (negative) if $x(t) > 0$ ($x(t) < 0$) on $[t^* - r, \infty)$. A solution x of (1.1) corresponding to t^* is called eventually positive (eventually negative) if there exists $t^{**} > t^*$ such that $x(t) > 0$ ($x(t) < 0$) on $[t^{**}, \infty)$.

Repeated interest in studying the existence of positive solutions of delay differential equations and their systems can be observed recently (we refer, e.g., to the monograph [1] and the papers [3, 4, 7, 9, 18, 21, 25, 30, 33–35, 37, 40] and to the references therein). Classical results can be found, e.g., in monographs [2, 24, 26, 28] and further results, e.g., in papers [10, 11, 13–16, 19, 20, 23, 27, 29].

Equation (1.1) often serves, due to its simple form, as an equation prototype for testing and comparing new results. But equation (1.1) itself has interesting applications as well. It is, for example, well-known in number theory that what is called the Dickman–de Bruijn function (we refer to [36, 38] and to the references therein) is a positive solution of the initial problem

$$\dot{x}(t) + \frac{1}{t}x(t-1) = 0, \quad t \geq 1 \quad \text{if } x(t) = 1, \quad t \in [0, 1].$$

In [22, p. 226] equation (1.1), in the case of the coefficient and the delay in (1.1) being constant, $p(t) = p > 0$ and $\tau(t) = \tau > 0$, i.e.

$$\dot{x}(t) + px(t-\tau) = 0, \quad t \geq t_0 \tag{1.2}$$

models the amount of salt (expressed by a positive solution) in the brine in a tank diluted by fresh water. The same equation is used in an example of water temperature regulation by a showering person in [32, p. 74]. A well-known example of type (1.1) equation

$$\dot{x}(t) + 2te^{1-2t}x(t-1) = 0$$

with a solution $x(t) = e^{-t^2}$ illustrates the fact that linear equations with delay can have positive solutions decreasing for $t \rightarrow \infty$ to zero faster than an arbitrary exponential function $e^{-\alpha t}$ where $\alpha > 0$ (see, e.g. [31, p. 97]).

It is well-known that either there exists an eventually positive solution of (1.1) or every solution of (1.1) is oscillatory. In the literature, by a critical case of the coefficient p in (1.1) is usually understood a boundary for p separating, in a sense, both the above mentioned asymptotically different qualitative cases of behavior of solutions to (1.1). We can give an explanation in the case of equation (1.2). In such a case, it is easy to show that there exists a positive solution if $p\tau \leq 1/e$ and that all solutions oscillate if $p\tau > 1/e$, the value $1/e$ is called the critical value.

In the paper, we develop some new explicit integral criteria related to the well-known classical sharp integral criterion

$$\int_{t-\tau(t)}^t p(s) ds \leq \frac{1}{e}, \quad \forall t \geq t_0 \tag{1.3}$$

for the existence of eventually positive solutions to (1.1) on $[t_0, \infty)$ (we refer, e.g., to [1, Corollary 2.15.], [2, Corollary 2.2.15.], [24, Corollary 2.2.1.], [28, Theorem 3.3.1.] and [29, Theorem 3]).

Now we give a short overview of known results stating the existence of a positive solution to (1.1).

1.1 Implicit criterion

The following well-known implicit criterion (with conditions adapted for (1.1)) on the existence of positive solutions is often cited in the literature.

Theorem 1.1. *Equation (1.1) has a positive solution with respect to t_0 if and only if there exists a continuous function $\lambda(t)$ on $[t_0 - r, \infty)$ such that $\lambda(t) > 0$ for $t \geq t_0$ and*

$$\lambda(t) \geq p(t)e^{\int_{t-\tau(t)}^t \lambda(s) ds}, \quad t \geq t_0. \quad (1.4)$$

This criterion can be found, e.g., in [27, Theorem 1, Assertion 7 and Corollary 2.1] and also in [1, 2] and [24, Theorem 2.1.4.]. Inequality (1.4) is of considerable importance since it often plays a crucial role in the process of deriving explicit criteria of positivity.

1.2 Explicit criteria

Some results cited below are formulated explicitly in terms of inequalities for the coefficient p or in terms of integrals containing p . These results deal with the critical case and are sharp (non-improvable) in various senses (often explained in the original papers). E.g., positive solutions might not exist if the cited inequalities are subject to certain small perturbations.

In some of the inequalities below appears what is called the iterated logarithm. We define iterated logarithms of k -th order as

$$\ln_k t := \underbrace{\ln \ln \dots \ln}_k t, \quad k \geq 1, \quad t > \exp_{k-2} 1, \quad \ln_0 t := t$$

and the iterated exponential

$$\exp_k t := \underbrace{(\exp(\exp(\dots \exp t)))}_k, \quad \exp_0 t := t, \quad \exp_{-1} t := 0$$

is used to determine the domain of the iterated logarithm.

1.2.1 Point-wise criteria

In [23] it is assumed that $p(t) = 1/e + a(t)$, $\tau(t) = 1$ and $t_0 = 1$. Then, the equation

$$\dot{x}(t) + \left(\frac{1}{e} + a(t)\right) x(t-1) = 0 \quad (1.5)$$

has a positive solution if

$$a(t) \leq 1/(8et^2) \quad (1.6)$$

for all sufficiently large t [23, Theorem 3]. This result is improved in [20] as follows. If

$$a(t) \leq \frac{1}{8et^2} \left(1 + \frac{1}{\ln^2 t}\right), \quad (1.7)$$

for all sufficiently large t , then (1.5) has a positive solution. A further generalization is given in [10], where it is proved that, for the existence of a positive solution to (1.1) if $\tau(t) = r$ the inequality

$$p(t) \leq \frac{1}{er} + \frac{r}{8et^2} + \frac{r}{8e(t \ln t)^2} + \frac{r}{8e(t \ln t \ln_2 t)^2} + \cdots + \frac{r}{8e(t \ln t \ln_2 t \cdots \ln_k t)^2} \quad (1.8)$$

for $t \rightarrow \infty$ and an integer $k \geq 0$ is sufficient. Obviously, criterion (1.3) is not applicable in the cases where the coefficient of the equation considered satisfies the inequalities described by (1.6)–(1.8).

Assuming $t - \tau(t) \geq t_0 - \tau(t_0)$ if $t \geq t_0$ and

$$\int_{t-\tau(t)}^t \frac{1}{\tau(\xi)} d\xi \leq 1, \quad t \rightarrow \infty,$$

in [17], it is proved that, for the existence of an eventually positive solution of (1.1), it is sufficient if an integer $k \geq 0$ exists such that

$$\lim_{t \rightarrow \infty} \tau(t) \cdot \left(\frac{1}{t} \ln t \ln_2 t \cdots \ln_k t \right) = 0$$

and

$$p(t) \leq \frac{1}{e\tau(t)} + \frac{\tau(t)}{8et^2} + \frac{\tau(t)}{8e(t \ln t)^2} + \cdots + \frac{\tau(t)}{8e(t \ln t \ln_2 t \cdots \ln_k t)^2}.$$

Moreover, in [3], it is showed that, if (1.8) holds and $0 \leq \tau(t) \leq r$ for $t \rightarrow \infty$, then (1.1) has an eventually positive solution. We finish this short overview by including a general result published in [3]. Let $1/\tau(t)$ be a locally integrable function and

$$\lim_{t \rightarrow \infty} (t - \tau(t)) = \infty, \quad \int_{t_0}^{\infty} \frac{1}{\tau(\xi)} d\xi = \infty.$$

If there exists a $\delta \in (0, \infty)$ such that

$$\int_{t-\tau(t)}^t \frac{1}{\tau(\xi)} d\xi \leq \delta, \quad t \geq t_0$$

and, for a fixed integer $k \geq 0$,

$$p(t) \leq \frac{1}{e\delta\tau(t)} + \frac{\delta}{8e\tau(t)q^2(t)} + \frac{\delta}{8e\tau(t)(q(t) \ln q(t))^2} \\ + \cdots + \frac{\delta}{8e\tau(t)(q(t) \ln q(t) \ln_2 q(t) \cdots \ln_k q(t))^2}$$

where

$$q(t) = \int_{t_0}^t \frac{1}{\tau(\xi)} d\xi,$$

then there exists an eventually positive solution of (1.1).

1.2.2 Integral criteria

Note, that the choice $\lambda(t) = e p(t)$ in (1.4) turns this inequality into (1.3). The ideas how to utilize implicit criterion (1.4) to get new explicit integral criteria are brought from the papers [4] and [12]. A small modification of (1.4), transforming

$$\lambda(t) := e p(t) e^{-\omega(t)} \quad (1.9)$$

with $\omega := \lambda_i^*$, $i = 0, 1, \dots$, where λ_i^* are special functions defined as

$$\lambda_i^*(t) := \frac{\tau(t)}{2t} + \frac{\tau(t)}{2t \ln t} + \frac{\tau(t)}{2t \ln t \ln_2 t} + \dots + \frac{\tau(t)}{2t \ln t \ln_2 t \dots \ln_i t}$$

led to substantial progress in developing new positivity criteria. Theorem 4 in [12] states the following

Theorem 1.2. *Let us assume that, for a fixed $i \in \{0, 1, \dots\}$, the inequality*

$$\int_{t-\tau(t)}^t p(s) e^{-\lambda_i^*(s)} ds \leq \frac{1}{e} [1 - \lambda_i^*(t)]$$

holds for $t \in [t_0, \infty)$. Then there exists a positive solution $x = x(t)$ of (1.1) on $[t_0, \infty)$. Moreover,

$$x(t) < \exp \left(-e \int_{t_0}^t p(s) e^{-\lambda_i^*(s)} ds \right)$$

for $t \in [t_0, \infty)$.

In particular, [12] demonstrates that Theorem 1.2 covers criteria (1.6)–(1.8).

2 New explicit integral criteria

Substituting (1.9) into (1.4), where $\omega: [t_0 - r, \infty) \rightarrow \mathbb{R}$ is a general function, results in the following statement (see [12, Theorem 3]).

Theorem 2.1. *Let $\omega: [t_0 - r, \infty) \rightarrow \mathbb{R}$ be a locally integrable function such that*

$$\int_{t-\tau(t)}^t p(s) e^{-\omega(s)} ds \leq \frac{1}{e} [1 - \omega(t)] \quad (2.1)$$

for $t \in [t_0, \infty)$. Then there exists a positive solution $x = x(t)$ of (1.1) on $[t_0, \infty)$ satisfying the inequality

$$x(t) < \exp \left(-e \int_{t_0}^t p(s) e^{-\omega(s)} ds \right) \quad (2.2)$$

for $t \in [t_0, \infty)$.

Theorem 2.1 is used in the proof of the following theorem.

Theorem 2.2. *Let $\omega: [t_0 - r, \infty) \rightarrow \mathbb{R}$ be a nonincreasing locally integrable function and let $\theta: [t_0, \infty) \rightarrow [0, 1]$ be a function. If*

$$e^{-\omega(t-\theta(t)\tau(t))} \int_{t-\tau(t)}^{t-\theta(t)\tau(t)} p(s) ds + e^{-\omega(t)} \int_{t-\theta(t)\tau(t)}^t p(s) ds \leq \frac{1}{e} [1 - \omega(t)] \quad (2.3)$$

for $t \in [t_0, \infty)$, then there exists a positive solution $x = x(t)$ of (1.1) on $[t_0, \infty)$.

Proof. For the left-hand side \mathcal{L} of inequality (2.1), we get

$$\begin{aligned}\mathcal{L} &= \int_{t-\tau(t)}^t p(s)e^{-\omega(s)} ds = \int_{t-\tau(t)}^{t-\theta(t)\tau(t)} p(s)e^{-\omega(s)} ds + \int_{t-\theta(t)\tau(t)}^t p(s)e^{-\omega(s)} ds \\ &\leq e^{-\omega(t-\theta(t)\tau(t))} \int_{t-\tau(t)}^{t-\theta(t)\tau(t)} p(s) ds + e^{-\omega(t)} \int_{t-\theta(t)\tau(t)}^t p(s) ds.\end{aligned}$$

Now, obviously, an estimate of the right-hand side \mathcal{R} of inequality (2.1), utilizing (2.3), is

$$\begin{aligned}\mathcal{R} &= \frac{1}{e} [1 - \omega(t)] \geq e^{-\omega(t-\theta(t)\tau(t))} \int_{t-\tau(t)}^{t-\theta(t)\tau(t)} p(s) ds + e^{-\omega(t)} \int_{t-\theta(t)\tau(t)}^t p(s) ds \\ &\geq \int_{t-\tau(t)}^t p(s)e^{-\omega(s)} ds = \mathcal{L}.\end{aligned}$$

Inequality (2.1) holds and from Theorem 2.1 the proof of Theorem 2.2 is complete. \square

Now we use Theorem 2.2 to get an easily verifiable explicit criterion.

Theorem 2.3. *Let*

$$\tau(t) \leq \tau(t - \tau(t)/2) \leq M\tau(t) \quad (2.4)$$

for all $t \geq t_0 > 0$ and a constant M . If there exists a function $\alpha: [t_0, \infty) \rightarrow \mathbb{R}_+$ such that

$$\int_{t-\tau(t)}^{t-\tau(t)/2} p(s) ds \leq \frac{1}{2e} + \alpha(t), \quad \int_{t-\tau(t)/2}^t p(s) ds \leq \frac{1}{2e} + \alpha(t) \quad (2.5)$$

for $t \in [t_0, \infty)$ and a constant $\mu \in (0, 1)$ such that

$$\alpha(t) \leq \frac{\mu(\tau^3(t - \tau(t)/2) + \tau^3(t))}{192t^3e}, \quad t \in [t_0, \infty), \quad (2.6)$$

then there exists a $t_0^* \in [t_0, \infty)$ and a positive solution $x = x(t)$ of (1.1) on $[t_0^*, \infty)$.

Proof. In (2.3), put $\omega(t) = \tau(t)/(2t)$ (in accordance with recommendation (1.9) for $i = 0$) and $\theta(t) = 1/2$. Then, (2.3) equals

$$\begin{aligned}\mathcal{L}_1 &:= e^{-\tau(t-\tau(t)/2)/(2(t-\tau(t)/2))} \int_{t-\tau(t)}^{t-\tau(t)/2} p(s) ds \\ &\quad + e^{-\tau(t)/(2t)} \int_{t-\tau(t)/2}^t p(s) ds \leq \mathcal{R}_1 := \frac{1}{e} \left[1 - \frac{\tau(t)}{2t} \right].\end{aligned} \quad (2.7)$$

Obviously, due to the boundedness of $\tau(t)$,

$$\lim_{t \rightarrow \infty} \frac{\tau(t - \tau(t)/2)}{t - \tau(t)/2} = 0, \quad \lim_{t \rightarrow \infty} \frac{\tau(t)}{t} = 0, \quad (2.8)$$

and in view of (2.4),

$$\tau(t - \tau(t)/2) = O(\tau(t)). \quad (2.9)$$

Combining both properties (2.8), (2.9) we have

$$\frac{\tau^m(t - \tau(t)/2)}{t^s} = O\left(\frac{\tau^m(t)}{t^s}\right), \quad \frac{\tau^m(t - \tau(t)/2)}{(t - \tau(t)/2)^s} = O\left(\frac{\tau^m(t)}{t^s}\right) \quad (2.10)$$

for positive integers m and s . Due to (2.8)–(2.10), it is possible to asymptotically decompose both exponential functions in (2.7). For the first one, we get

$$\begin{aligned}
 & e^{-\tau(t-\tau(t)/2)/(2(t-\tau(t)/2))} \\
 &= 1 - \frac{\tau(t-\tau(t)/2)}{2(t-\tau(t)/2)} + \frac{1}{2} \left(\frac{\tau(t-\tau(t)/2)}{2(t-\tau(t)/2)} \right)^2 \\
 &\quad - \frac{1}{6} \left(\frac{\tau(t-\tau(t)/2)}{2(t-\tau(t)/2)} \right)^3 + O \left(\left(\frac{\tau(t-\tau(t)/2)}{t-\tau(t)/2} \right)^4 \right) \\
 &= 1 - \frac{1}{2t} \tau(t-\tau(t)/2) \left(1 + \frac{\tau(t)}{2t} + \frac{\tau^2(t)}{4t^2} + O \left(\frac{\tau^3(t)}{t^3} \right) \right) \\
 &\quad + \frac{1}{2} \frac{\tau^2(t-\tau(t)/2)}{4t^2} \left(1 + \frac{\tau(t)}{t} + O \left(\frac{\tau^2(t)}{t^2} \right) \right) \\
 &\quad - \frac{1}{6} \frac{\tau^3(t-\tau(t)/2)}{8t^3} \left(1 + O \left(\frac{\tau(t)}{t} \right) \right) + O \left(\frac{\tau^4(t)}{t^4} \right) \\
 &= 1 - \frac{\tau(t-\tau(t)/2)}{2t} - \frac{\tau(t-\tau(t)/2)\tau(t)}{4t^2} - \frac{\tau(t-\tau(t)/2)\tau^2(t)}{8t^3} \\
 &\quad + \frac{\tau^2(t-\tau(t)/2)}{8t^2} + \frac{\tau^2(t-\tau(t)/2)\tau(t)}{8t^3} - \frac{1}{6} \frac{\tau^3(t-\tau(t)/2)}{8t^3} + O \left(\frac{\tau^4(t)}{t^4} \right) \tag{2.11}
 \end{aligned}$$

and, for the second one, we derive

$$\begin{aligned}
 e^{-\tau(t)/(2t)} &= 1 - \frac{\tau(t)}{2t} + \frac{1}{2} \left(\frac{\tau(t)}{2t} \right)^2 - \frac{1}{6} \left(\frac{\tau(t)}{2t} \right)^3 + O \left(\frac{\tau^4(t)}{t^4} \right) \\
 &= 1 - \frac{\tau(t)}{2t} + \frac{\tau^2(t)}{8t^2} - \frac{\tau^3(t)}{48t^3} + O \left(\frac{\tau^4(t)}{t^4} \right). \tag{2.12}
 \end{aligned}$$

Then, utilizing (2.5), (2.11) and (2.12), we can estimate the left-hand side \mathcal{L}_1 of (2.7),

$$\begin{aligned}
 \mathcal{L}_1 &= \left(1 - \frac{\tau(t-\tau(t)/2)}{2t} - \frac{\tau(t-\tau(t)/2)\tau(t)}{4t^2} - \frac{\tau(t-\tau(t)/2)\tau^2(t)}{8t^3} \right. \\
 &\quad + \frac{\tau^2(t-\tau(t)/2)}{8t^2} + \frac{\tau^2(t-\tau(t)/2)\tau(t)}{8t^3} - \frac{1}{6} \frac{\tau^3(t-\tau(t)/2)}{8t^3} \\
 &\quad \left. + O \left(\frac{\tau^4(t)}{t^4} \right) \right) \int_{t-\tau(t)}^{t-\tau(t)/2} p(s) \, ds \\
 &\quad + \left(1 - \frac{\tau(t)}{2t} + \frac{\tau^2(t)}{8t^2} - \frac{\tau^3(t)}{48t^3} + O \left(\frac{\tau^4(t)}{t^4} \right) \right) \int_{t-\tau(t)/2}^t p(s) \, ds \\
 &\leq \left(1 - \frac{\tau(t-\tau(t)/2)}{2t} - \frac{\tau(t-\tau(t)/2)\tau(t)}{4t^2} - \frac{\tau(t-\tau(t)/2)\tau^2(t)}{8t^3} \right. \\
 &\quad + \frac{\tau^2(t-\tau(t)/2)}{8t^2} + \frac{\tau^2(t-\tau(t)/2)\tau(t)}{8t^3} - \frac{1}{6} \frac{\tau^3(t-\tau(t)/2)}{8t^3} \\
 &\quad \left. + O \left(\frac{\tau^4(t)}{t^4} \right) \right) \left(\frac{1}{2e} + \alpha(t) \right)
 \end{aligned}$$

$$\begin{aligned}
& + \left(1 - \frac{\tau(t)}{2t} + \frac{\tau^2(t)}{8t^2} - \frac{\tau^3(t)}{48t^3} + O\left(\frac{\tau^4(t)}{t^4}\right) \right) \left(\frac{1}{2e} + \alpha(t) \right) \\
& = \frac{1}{e} - \frac{\tau(t - \tau(t)/2) + \tau(t)}{4te} + \frac{\tau^2(t - \tau(t)/2) + \tau^2(t) - 2\tau(t - \tau(t)/2)\tau(t)}{16t^2e} \\
& \quad + \frac{-6\tau(t - \tau(t)/2)\tau^2(t) + 6\tau^2(t - \tau(t)/2)\tau(t) - \tau^3(t - \tau(t)/2) - \tau^3(t)}{96t^3e} \\
& \quad + \alpha(t) \left(2 + O\left(\frac{\tau(t)}{t}\right) \right) + O\left(\frac{\tau^4(t)}{t^4}\right).
\end{aligned}$$

So, we have

$$\begin{aligned}
\mathcal{L}_1 & \leq \frac{1}{e} - \frac{\tau(t - \tau(t)/2) + \tau(t)}{4te} + \frac{\tau^2(t - \tau(t)/2) + \tau^2(t) - 2\tau(t - \tau(t)/2)\tau(t)}{16t^2e} \\
& \quad + \frac{-6\tau(t - \tau(t)/2)\tau^2(t) + 6\tau^2(t - \tau(t)/2)\tau(t) - \tau^3(t - \tau(t)/2) - \tau^3(t)}{96t^3e} \\
& \quad + 2\alpha(t) + O\left(\frac{\alpha(t)\tau(t)}{t}\right) + O\left(\frac{\tau^4(t)}{t^4}\right). \tag{2.13}
\end{aligned}$$

Then, for $\mathcal{L}_1 \leq \mathcal{R}_1$,

$$\begin{aligned}
& \frac{1}{e} - \frac{\tau(t - \tau(t)/2) + \tau(t)}{4te} + \frac{\tau^2(t - \tau(t)/2) + \tau^2(t) - 2\tau(t - \tau(t)/2)\tau(t)}{16t^2e} \\
& \quad + \frac{-6\tau(t - \tau(t)/2)\tau^2(t) + 6\tau^2(t - \tau(t)/2)\tau(t) - \tau^3(t - \tau(t)/2) - \tau^3(t)}{96t^3e} \\
& \quad + 2\alpha(t) + O\left(\frac{\alpha(t)\tau(t)}{t}\right) + O\left(\frac{\tau^4(t)}{t^4}\right) < \frac{1}{e} \left[1 - \frac{\tau(t)}{2t} \right]
\end{aligned}$$

or

$$\begin{aligned}
& \frac{\tau(t) - \tau(t - \tau(t)/2)}{4te} + \frac{(\tau(t - \tau(t)/2) - \tau(t))^2}{16t^2e} + \frac{-6\tau(t - \tau(t)/2)\tau(t)(\tau(t) - \tau(t - \tau(t)/2))}{96t^3e} \\
& \quad - \frac{\tau^3(t - \tau(t)/2) + \tau^3(t)}{96t^3e} + 2\alpha(t) + O\left(\frac{\alpha(t)\tau(t)}{t}\right) + O\left(\frac{\tau^4(t)}{t^4}\right) \\
& = \frac{\tau(t) - \tau(t - \tau(t)/2)}{4te} \left(1 + \frac{\tau(t) - \tau(t - \tau(t)/2)}{4t} - \frac{6\tau(t - \tau(t)/2)\tau(t)}{24t^2} \right) \\
& \quad - \frac{\tau^3(t - \tau(t)/2) + \tau^3(t)}{96t^3e} + 2\alpha(t) + O\left(\frac{\alpha(t)\tau(t)}{t}\right) + O\left(\frac{\tau^4(t)}{t^4}\right) < 0 \tag{2.14}
\end{aligned}$$

is sufficient. From (2.4), we have $-\tau(t - \tau(t)/2) + \tau(t) \leq 0$. As the delay τ is bounded, (assuming t_0 sufficiently large),

$$\frac{\tau(t) - \tau(t - \tau(t)/2)}{4te} \left(1 + \frac{\tau(t) - \tau(t - \tau(t)/2)}{4t} - \frac{6\tau(t - \tau(t)/2)\tau(t)}{24t^2} \right) \leq 0. \tag{2.15}$$

Finally, utilizing (2.4), (2.6) and (2.15), we conclude that (2.14) will be satisfied if

$$\begin{aligned}
 & -\frac{\tau^3(t - \tau(t)/2) + \tau^3(t)}{96t^3} + 2\alpha(t) + O\left(\frac{\alpha(t)\tau(t)}{t}\right) + O\left(\frac{\tau^4(t)}{t^4}\right) \\
 & \leq -\frac{(1 - \mu)(\tau^3(t - \tau(t)/2) + \tau^3(t))}{96t^3e} + O\left(\frac{\alpha(t)\tau(t)}{t}\right) + O\left(\frac{\tau^4(t)}{t^4}\right) \\
 & = -\frac{(1 - \mu)(\tau^3(t - \tau(t)/2) + \tau^3(t))}{96t^3e} + O\left(\frac{(\tau^3(t - \tau(t)/2) + \tau^3(t))\tau(t)}{t^4}\right) + O\left(\frac{\tau^4(t)}{t^4}\right) \\
 & < 0.
 \end{aligned}$$

Since $1 - \mu > 0$, the last inequality is valid and $\mathcal{L}_1 < \mathcal{R}_1$. Without loss of generality, assume that above inequalities are valid on $[t_0^*, \infty)$ where $t_0^* \geq t_0$ is sufficiently large. Inequality (2.3) holds on $[t_0^*, \infty)$, Theorem 2.2 is applicable, and a positive solution $x = x(t)$ of (1.1) on $[t_0^*, \infty)$ exists. \square

A minor modification in the proof of Theorem 2.3 gives the following statement.

Theorem 2.4. *Let*

$$\tau(t) < \tau(t - \tau(t)/2) \leq M\tau(t) \quad (2.16)$$

for all $t \geq t_0$ and a constant M and

$$\int_{t-\tau(t)}^{t-\tau(t)/2} p(s) ds \leq \frac{1}{2e} + \beta(t), \quad \int_{t-\tau(t)/2}^t p(s) ds \leq \frac{1}{2e} + \beta(t)$$

for $t \in [t_0, \infty)$, where $\beta: [t_0, \infty) \rightarrow \mathbb{R}_+$. If, moreover, there exists a $\mu \in (0, 1)$ such that

$$\beta(t) \leq \frac{\mu(\tau(t - \tau(t)/2) - \tau(t))}{8te}, \quad (2.17)$$

then there exists a $t_0^* \in [t_0, \infty)$ and a positive solution $x = x(t)$ of (1.1) on $[t_0^*, \infty)$.

Proof. Modifying inequality (2.14) (where α is replaced by β) in the proof of Theorem 2.3, we get

$$\begin{aligned}
 & \frac{\tau(t) - \tau(t - \tau(t)/2)}{4te} \left(1 + \frac{\tau(t) - \tau(t - \tau(t)/2)}{4t} - \frac{6\tau(t - \tau(t)/2)\tau(t)}{24t^2} \right) \\
 & - \frac{\tau^3(t - \tau(t)/2) + \tau^3(t)}{96t^3} + 2\beta(t) + O\left(\frac{\beta(t)\tau(t)}{t}\right) + O\left(\frac{\tau^4(t)}{t^4}\right) \\
 & \leq \frac{\tau(t) - \tau(t - \tau(t)/2)}{4te} \left(1 - \mu + \frac{\tau(t) - \tau(t - \tau(t)/2)}{4t} - \frac{6\tau(t - \tau(t)/2)\tau(t)}{24t^2} \right) \\
 & - \frac{\tau^3(t - \tau(t)/2) + \tau^3(t)}{96t^3} + O\left(\frac{(\tau(t - \tau(t)/2) - \tau(t))\tau(t)}{t^2}\right) + O\left(\frac{\tau^4(t)}{t^4}\right) < 0. \quad (2.18)
 \end{aligned}$$

It is easy to see that the asymptotically leading terms in (2.18) are

$$\frac{\tau(t) - \tau(t - \tau(t)/2)}{4te}(1 - \mu) \quad \text{and} \quad -\frac{\tau^3(t - \tau(t)/2) + \tau^3(t)}{96t^3}$$

because all the remaining terms are asymptotically smaller than the first or the second one. Since $1 - \mu > 0$, inequality (2.18) is valid and $\mathcal{L}_1 < \mathcal{R}_1$. Further, we can proceed as in the proof of Theorem 2.3. \square

Remark 2.5. Comparing inequality (2.6) in Theorem 2.3 with inequality (2.17) in Theorem 2.4, we conclude that these theorems are independent. Let us illustrate this remark by two examples with different delays (and note that neither point-wise criteria mentioned in 1.2.1 nor integral criterion (1.3) are applicable).

Example 2.6. Let $t_0 > 0$ and $\tau(t) = e^{-t}$. Then,

$$\tau(t - \tau(t)/2) = e^{-(t - e^{-t}/2)}$$

and inequality (2.4) turns into inequality

$$1 \leq e^{e^{-t}/2} \leq M,$$

which holds with $M := \sqrt{e}$. Theorem 2.3 is applicable if α satisfies inequality (2.6), i.e.,

$$\alpha(t) \leq \frac{\mu(\tau^3(t - \tau(t)/2) + \tau^3(t))}{192t^3e} = \frac{\mu(e^{-3(t - e^{-t}/2)} + e^{-3t})}{192t^3e} = \frac{\mu e^{-3t}(e^{3e^{-t}/2} + 1)}{192t^3e}. \quad (2.19)$$

Inequality (2.16) in Theorem 2.4 holds as well. Theorem 2.4 is applicable if β satisfies inequality (2.17), i.e.,

$$\begin{aligned} \beta(t) &\leq \frac{\mu(\tau(t - \tau(t)/2) - \tau(t))}{8te} = \frac{\mu(e^{-(t - e^{-t}/2)} - e^{-t})}{8te} = \frac{\mu e^{-t}(e^{e^{-t}/2} - 1)}{8te} \\ &= \frac{\mu e^{-t}}{8te} \left(1 + \frac{1}{2}e^{-t}(1 + o(1)) - 1\right) = \frac{\mu e^{-2t}}{16te} (1 + o(1)). \end{aligned} \quad (2.20)$$

Comparing estimates (2.19) and (2.20), we conclude that (2.20) is less restrictive and therefore Theorem 2.4 is preferable.

Example 2.7. Let $t_0 > 0$ and $\tau(t) = 1 + e^{-t}$. Then,

$$\tau(t - \tau(t)/2) = 1 + e^{-(t - (1 + e^{-t})/2)}$$

and inequality (2.4) turns into inequality

$$1 \leq \left(1 + e^{-t}e^{(1 + e^{-t})/2}\right) (1 + e^{-t})^{-1} \leq M,$$

which holds with $M := 1 + e$. Theorem 2.3 is applicable if α satisfies inequality (2.6), i.e.,

$$\begin{aligned} \alpha(t) &\leq e \frac{\mu(\tau^3(t - \tau(t)/2) + \tau^3(t))}{192t^3e} = \frac{\mu((1 + e^{-(t - (1 + e^{-t})/2)})^3 + ((1 + e^{-t})^3))}{192t^3e} \\ &= \frac{\mu(2 + o(1))}{192t^3e}. \end{aligned} \quad (2.21)$$

Inequality (2.16) in Theorem 2.4 holds as well. Theorem 2.4 is applicable if β satisfies inequality (2.17), i.e.,

$$\begin{aligned} \beta(t) &\leq \frac{\mu(\tau(t - \tau(t)/2) - \tau(t))}{8te} = \frac{\mu(1 + e^{-(t - (1 + e^{-t})/2)} - 1 - e^{-t})}{8te} \\ &= \frac{\mu e^{-t}(e^{(1 + e^{-t})/2} - 1)}{8te}. \end{aligned} \quad (2.22)$$

Comparing estimates (2.21) and (2.22), we conclude that (2.21) is less restrictive and therefore Theorem 2.3 is preferable.

3 Further positivity criteria

In this part, some further positivity criteria are derived. First we generalize Theorem 2.2 in the case that the interval $[t - \tau(t), t]$ is divided by several points.

Theorem 3.1. *Let $\omega: [t_0 - r, \infty) \rightarrow \mathbb{R}$ be a nonincreasing locally integrable function, let $\theta_i: [t_0, \infty) \rightarrow [0, 1]$, $i = 1, 2, \dots, n - 1$ be functions satisfying inequalities $\theta_1(t) > \theta_2(t) > \dots > \theta_{n-1}(t)$ and let the inequality*

$$\begin{aligned} & e^{-\omega(t-\theta_1(t)\tau(t))} \int_{t-\tau(t)}^{t-\theta_1(t)\tau(t)} p(s) \, ds + e^{-\omega(t-\theta_2(t)\tau(t))} \int_{t-\theta_1(t)\tau(t)}^{t-\theta_2(t)\tau(t)} p(s) \, ds \\ & + \dots + e^{-\omega(t-\theta_{n-1}(t)\tau(t))} \int_{t-\theta_{n-2}(t)\tau(t)}^{t-\theta_{n-1}(t)\tau(t)} p(s) \, ds \\ & + e^{-\omega(t)} \int_{t-\theta_{n-1}(t)\tau(t)}^t p(s) \, ds \leq \frac{1}{e} [1 - \omega(t)] \end{aligned} \quad (3.1)$$

hold for $t \in [t_0, \infty)$. Then there exists a positive solution $x = x(t)$ of (1.1) on $[t_0, \infty)$.

Proof. In the proof, we apply Theorem 2.1 again. For the left-hand side \mathcal{L} of inequality (2.1) we get, using (3.1),

$$\begin{aligned} \mathcal{L} &= \int_{t-\tau(t)}^t p(s) e^{-\omega(s)} \, ds = \int_{t-\tau(t)}^{t-\theta_1(t)\tau(t)} p(s) e^{-\omega(s)} \, ds + \int_{t-\theta_1(t)\tau(t)}^{t-\theta_2(t)\tau(t)} p(s) e^{-\omega(s)} \, ds \\ &+ \dots + \int_{t-\theta_{n-2}(t)\tau(t)}^{t-\theta_{n-1}(t)\tau(t)} p(s) e^{-\omega(s)} \, ds + \int_{t-\theta_{n-1}(t)\tau(t)}^t p(s) e^{-\omega(s)} \, ds \\ &\leq e^{-\omega(t-\theta_1(t)\tau(t))} \int_{t-\tau(t)}^{t-\theta_1(t)\tau(t)} p(s) \, ds + e^{-\omega(t-\theta_2(t)\tau(t))} \int_{t-\theta_1(t)\tau(t)}^{t-\theta_2(t)\tau(t)} p(s) \, ds \\ &+ \dots + e^{-\omega(t-\theta_{n-1}(t)\tau(t))} \int_{t-\theta_{n-2}(t)\tau(t)}^{t-\theta_{n-1}(t)\tau(t)} p(s) \, ds + e^{-\omega(t)} \int_{t-\theta_{n-1}(t)\tau(t)}^t p(s) \, ds. \end{aligned}$$

By (3.1), an estimate of the right-hand side \mathcal{R} of inequality (2.1) is

$$\begin{aligned} \mathcal{R} &= \frac{1}{e} [1 - \omega(t)] \\ &\geq e^{-\omega(t-\theta_1(t)\tau(t))} \int_{t-\tau(t)}^{t-\theta_1(t)\tau(t)} p(s) \, ds + e^{-\omega(t-\theta_2(t)\tau(t))} \int_{t-\theta_1(t)\tau(t)}^{t-\theta_2(t)\tau(t)} p(s) \, ds \\ &+ \dots + e^{-\omega(t-\theta_{n-1}(t)\tau(t))} \int_{t-\theta_{n-2}(t)\tau(t)}^{t-\theta_{n-1}(t)\tau(t)} p(s) \, ds + e^{-\omega(t)} \int_{t-\theta_{n-1}(t)\tau(t)}^t p(s) \, ds \\ &\geq \int_{t-\tau(t)}^t p(s) e^{-\omega(s)} \, ds = \mathcal{L}, \end{aligned}$$

inequality (2.1) holds, and from Theorem 2.1 the proof of Theorem 3.1 is complete. \square

Now we use Theorem 3.1 to get an easily verifiable explicit criterion when the interval $[t - \tau, t]$, $t \geq t_0$ is divided into n subintervals. It is necessary to underline that it is assumed that $n > 2$, i.e. Theorem 3.2 below cannot be reduced to Theorem 2.3 and both theorems are independent. It is a surprising fact that the proof of Theorem 3.2 is even simpler than that of proof of Theorem 2.3 (because the terms of the third order of accuracy in the asymptotic decomposition are not necessary) and, simultaneously, the function α satisfies an estimation (3.3) below which is weaker than (2.6) in Theorem 2.3.

Theorem 3.2. Let $n > 2$ be an integer and

$$\tau(t - ((n-i)/n)\tau(t)) \leq \tau(t - ((n-i+1)/n)\tau(t)) \leq M\tau(t) \quad (3.2)$$

for all $t \geq t_0 > 0$ and $i = 1, \dots, n$, and a constant M . If

$$\int_{t - ((n-i+1)/n)\tau(t)}^{t - ((n-i)/n)\tau(t)} p(s) \, ds \leq \frac{1}{ne} + \alpha(t)$$

for $t \in [t_0, \infty)$, $i = 1, \dots, n$ where $\alpha: [t_0, \infty) \rightarrow \mathbb{R}_+$ and there exist a $\mu \in (0, 1)$ such that

$$\alpha(t) \leq \frac{\mu(n-2)\tau^2(t)}{8n^2t^2e}, \quad t \in [t_0, \infty), \quad (3.3)$$

then there exists a $t_0^* \in [t_0, \infty)$ and a positive solution $x = x(t)$ of (1.1) on $[t_0^*, \infty)$.

Proof. Put in (3.1) $\omega(t) = \tau(t)/(2t)$ and $\theta_i(t) = (n-i)/n$, $i = 1, \dots, n$. Then, (3.1) equals (below by \mathcal{L}_1^* and \mathcal{R}_1^* the left-hand and the right-hand sides of (3.1) are denoted)

$$\begin{aligned} \mathcal{L}_1^* &:= \exp\left(\frac{-\tau(t - ((n-1)/n)\tau(t))}{2(t - ((n-1)/n)\tau(t))}\right) \int_{t - (n/n)\tau(t)}^{t - ((n-1)/n)\tau(t)} p(s) \, ds \\ &+ \exp\left(\frac{-\tau(t - ((n-2)/n)\tau(t))}{2(t - ((n-2)/n)\tau(t))}\right) \int_{t - ((n-1)/n)\tau(t)}^{t - ((n-2)/n)\tau(t)} p(s) \, ds \\ &+ \dots + \exp\left(\frac{-\tau(t - ((n-n)/n)\tau(t))}{2(t - ((n-n)/n)\tau(t))}\right) \int_{t - ((n-n+1)/n)\tau(t)}^{t - ((n-n)/n)\tau(t)} p(s) \, ds \\ &= \sum_{i=1}^n \exp\left(\frac{-\tau(t - ((n-i)/n)\tau(t))}{2(t - ((n-i)/n)\tau(t))}\right) \int_{t - ((n-i+1)/n)\tau(t)}^{t - ((n-i)/n)\tau(t)} p(s) \, ds \\ &\leq \mathcal{R}_1^* := \frac{1}{e} \left[1 - \frac{\tau(t)}{2t}\right]. \end{aligned} \quad (3.4)$$

Obviously, due to the boundedness of $\tau(t)$,

$$\lim_{t \rightarrow \infty} \frac{\tau(t - ((n-i)/n)\tau(t))}{t - ((n-i)/n)\tau(t)} = 0, \quad i = 1, \dots, n, \quad (3.5)$$

due to (3.2)

$$\tau(t - ((n-i)/n)\tau(t)) = O(\tau(t)), \quad i = 1, \dots, n-1 \quad (3.6)$$

and, combining both properties (3.5), (3.6), we have

$$\frac{\tau^m(t - ((n-i)/n)\tau(t))}{t^s} = O\left(\frac{\tau^m(t)}{t^s}\right), \quad \frac{\tau^m(t - ((n-i)/n)\tau(t))}{(t - ((n-i)/n)\tau(t))^s} = O\left(\frac{\tau^m(t)}{t^s}\right) \quad (3.7)$$

for positive integers m and s and $i = 1, \dots, n-1$. By (3.5)–(3.7), it is possible to asymptotically

decompose exponential functions in (3.4). This is the next step. For $i = 1, \dots, n$, we get

$$\begin{aligned}
 & \exp\left(\frac{-\tau(t - ((n-i)/n)\tau(t))}{2(t - ((n-i)/n)\tau(t))}\right) \\
 &= 1 - \frac{\tau(t - ((n-i)/n)\tau(t))}{2(t - ((n-i)/n)\tau(t))} + \frac{1}{2} \left(\frac{\tau(t - ((n-i)/n)\tau(t))}{2(t - ((n-i)/n)\tau(t))}\right)^2 + O\left(\frac{\tau^3(t)}{t^3}\right) \\
 &= 1 - \frac{1}{2t} \tau(t - ((n-i)/n)\tau(t)) \left(1 + \frac{n-i}{n} \frac{\tau(t)}{t} + O\left(\frac{\tau^2(t)}{t^2}\right)\right) \\
 &\quad + \frac{1}{2} \frac{\tau^2(t - ((n-i)/n)\tau(t))}{4t^2} \left(1 + O\left(\frac{\tau(t)}{t}\right)\right) + O\left(\frac{\tau^3(t)}{t^3}\right) \\
 &= 1 - \frac{\tau(t - ((n-i)/n)\tau(t))}{2t} - \frac{(n-i)\tau(t - ((n-i)/n)\tau(t))\tau(t)}{2nt^2} \\
 &\quad + \frac{\tau^2(t - ((n-i)/n)\tau(t))}{8t^2} + O\left(\frac{\tau^3(t)}{t^3}\right). \tag{3.8}
 \end{aligned}$$

Then, utilizing (3.3) and (3.8) we can estimate the left-hand side of (3.4),

$$\begin{aligned}
 \mathcal{L}_1^* &:= \sum_{i=1}^n \exp\left(\frac{-\tau(t - ((n-i)/n)\tau(t))}{2(t - ((n-i)/n)\tau(t))}\right) \int_{t - ((n-i+1)/n)\tau(t)}^{t - ((n-i)/n)\tau(t)} p(s) ds \\
 &\leq \sum_{i=1}^n \exp\left(\frac{-\tau(t - ((n-i)/n)\tau(t))}{2(t - ((n-i)/n)\tau(t))}\right) \left(\frac{1}{ne} + \alpha(t)\right) \\
 &\leq \sum_{i=1}^n \left(1 - \frac{\tau(t - ((n-i)/n)\tau(t))}{2t} - \frac{(n-i)\tau(t - ((n-i)/n)\tau(t))\tau(t)}{2nt^2}\right. \\
 &\quad \left. + \frac{\tau^2(t - ((n-i)/n)\tau(t))}{8t^2} + O\left(\frac{\tau^3(t)}{t^3}\right)\right) \left(\frac{1}{ne} + \alpha(t)\right) \\
 &= \frac{1}{e} - \sum_{i=1}^n \frac{\tau(t - ((n-i)/n)\tau(t))}{2net} \\
 &\quad + \sum_{i=1}^n \left(-\frac{(n-i)\tau(t - ((n-i)/n)\tau(t))\tau(t)}{2n^2et^2} + \frac{\tau^2(t - ((n-i)/n)\tau(t))}{8net^2}\right) \\
 &\quad + \alpha(t) \left(n + O\left(\frac{\tau(t)}{t}\right)\right) + O\left(\frac{\tau^3(t)}{t^3}\right).
 \end{aligned}$$

Then, for $\mathcal{L}_1^* \leq \mathcal{R}_1^*$,

$$\begin{aligned}
 & \frac{1}{e} - \sum_{i=1}^n \frac{\tau(t - ((n-i)/n)\tau(t))}{2net} \\
 & \quad + \sum_{i=1}^n \left(-\frac{(n-i)\tau(t - ((n-i)/n)\tau(t))\tau(t)}{2n^2et^2} + \frac{\tau^2(t - ((n-i)/n)\tau(t))}{8net^2}\right) \\
 & \quad + \alpha(t) \left(n + O\left(\frac{\tau(t)}{t}\right)\right) + O\left(\frac{\tau^3(t)}{t^3}\right) < \frac{1}{e} \left[1 - \frac{\tau(t)}{2t}\right]
 \end{aligned}$$

or

$$\begin{aligned}
& \sum_{i=1}^{n-1} \frac{\tau(t) - \tau(t - ((n-i)/n)\tau(t))}{2net} \\
& + \sum_{i=1}^n \left(-\frac{4(n-i)\tau(t - ((n-i)/n)\tau(t))\tau(t) + n\tau^2(t - ((n-i)/n)\tau(t))}{8n^2et^2} \right) \\
& + n\alpha(t) + O\left(\frac{\alpha(t)\tau(t)}{t}\right) + O\left(\frac{\tau^3(t)}{t^3}\right) < 0
\end{aligned} \tag{3.9}$$

is sufficient. Let us transform the second sum in (3.9). We get

$$\begin{aligned}
\sum_2 &= \frac{1}{8n^2et^2} \sum_{i=1}^n (n\tau^2(t - ((n-i)/n)\tau(t)) - 4(n-i)\tau(t)\tau(t - ((n-i)/n)\tau(t))) \\
&= \frac{1}{8n^2et^2} \sum_{i=1}^n \left(n(\tau(t) - \tau(t - ((n-i)/n)\tau(t)))^2 - n\tau^2(t) \right. \\
&\quad \left. + (2n - 4(n-i))\tau(t)\tau(t - ((n-i)/n)\tau(t)) \right) \\
&= [\text{we take into account a reduction in the sum if } i = n \text{ and } i = n - 1] \\
&= \frac{1}{8n^2et^2} \sum_{i=1}^{n-1} n(\tau(t) - \tau(t - ((n-i)/n)\tau(t)))^2 - \frac{n(n-2)\tau^2(t)}{8n^2et^2} \\
&\quad - \sum_{i=1}^{n-1} \frac{1}{8n^2et^2} (2n - 4i)\tau(t)\tau(t - ((n-i)/n)\tau(t)) \\
&= \frac{1}{8n^2et^2} \sum_{i=1}^{n-1} n(\tau(t) - \tau(t - ((n-i)/n)\tau(t)))^2 - \frac{n(n-2)\tau^2(t)}{8n^2et^2} \\
&\quad - \sum_{i=1}^{\lfloor n/2 \rfloor} \frac{1}{8n^2et^2} (2n - 4i)\tau(t) [\tau(t - ((n-i)/n)\tau(t)) - \tau(t - (i/n)\tau(t))]
\end{aligned}$$

where $\lfloor \cdot \rfloor$ is the floor function. Since, by (3.2),

$$\tau(t - ((n-i)/n)\tau(t)) - \tau(t - (i/n)\tau(t)) \geq 0,$$

we have

$$\sum_2 \leq \frac{1}{8n^2et^2} \sum_{i=1}^{n-1} n(\tau(t) - \tau(t - ((n-i)/n)\tau(t)))^2 - \frac{n(n-2)\tau^2(t)}{8n^2et^2}$$

and (3.9) will hold if

$$\begin{aligned}
& \sum_{i=1}^{n-1} \frac{\tau(t) - \tau(t - ((n-i)/n)\tau(t))}{2net} \\
& + \frac{1}{8net^2} \sum_{i=1}^{n-1} (\tau(t) - \tau(t - ((n-i)/n)\tau(t)))^2 - \frac{(n-2)\tau^2(t)}{8net^2} \\
& + n\alpha(t) + O\left(\frac{\alpha(t)\tau(t)}{t}\right) + O\left(\frac{\tau^3(t)}{t^3}\right) < 0
\end{aligned}$$

or if

$$\begin{aligned} \sum_{i=1}^{n-1} \frac{\tau(t) - \tau(t - ((n-i)/n)\tau(t))}{2net} \left(1 + \frac{\tau(t) - \tau(t - ((n-i)/n)\tau(t))}{4t} \right) \\ - \frac{(n-2)\tau^2(t)}{8net^2} + n\alpha(t) + O\left(\frac{\alpha(t)\tau(t)}{t}\right) + O\left(\frac{\tau^3(t)}{t^3}\right) < 0. \end{aligned} \quad (3.10)$$

Now we apply inequality (3.3). Then, inequality (3.10) will hold if

$$\begin{aligned} \sum_{i=1}^{n-1} \frac{\tau(t) - \tau(t - ((n-i)/n)\tau(t))}{2net} \left(1 + \frac{\tau(t) - \tau(t - ((n-i)/n)\tau(t))}{4t} \right) \\ - \frac{(n-2)(1-\mu)\tau^2(t)}{8net^2} + O\left(\frac{\tau^3(t)}{t^3}\right) < 0. \end{aligned} \quad (3.11)$$

Since, by (3.2),

$$\tau(t) - \tau(t - ((n-i)/n)\tau(t)) \leq 0, \quad i = 1, \dots, n-1,$$

inequality (3.11) will be valid if

$$- \frac{(n-2)(1-\mu)\tau^2(t)}{8net^2} + O\left(\frac{\tau^3(t)}{t^3}\right) = - \frac{(n-2)(1-\mu)\tau^2(t)}{8net^2} \left(1 + O\left(\frac{\tau(t)}{t}\right) \right) < 0. \quad (3.12)$$

We have $n > 2$ and $\mu \in (0, 1)$ so that the last inequality is obvious and $\mathcal{L}_1^* < \mathcal{R}_1^*$ on $[t_0^*, \infty)$, where $t_0^* \geq t_0$ is sufficiently large. Inequality (3.1) holds, Theorem 3.1 is applicable, and a positive solution $x = x(t)$ of (1.1) on $[t_0^*, \infty)$ exists. \square

Remark 3.3. Let us note that if delay $\tau(t)$ is a non increasing function, condition (3.2) holds. As noted above, Theorems 2.3, 3.2 are independent. The reason why Theorem 3.2 does not cover the case considered by Theorem 2.3 is the following. In the proof of Theorem 3.2, the crucial term determining the sign of the final estimate is

$$- \frac{(n-2)\tau^2(t)}{8net^2}.$$

If $n = 2$ (which is not allowed in Theorem 3.2), this term disappears and the sign will be determined by expressions of order higher than $\tau^2(t)/t^2$. Such an approach and detailed analysis is carried out in the proof of Theorem 2.3.

A minor modification in the proof of Theorem 3.2 results in the following statement.

Theorem 3.4. Let $n > 2$ be an integer and

$$\tau(t - ((n-i)/n)\tau(t)) < \tau(t - ((n-i+1)/n)\tau(t)) \leq M\tau(t) \quad (3.13)$$

for all $t \geq t_0 > 0$ and $i = 1, \dots, n$, and a constant M . If

$$\int_{t - ((n-i+1)/n)\tau(t)}^{t - ((n-i)/n)\tau(t)} p(s) ds \leq \frac{1}{ne} + \beta(t)$$

for $t \in [t_0, \infty)$, $i = 1, \dots, n$, where $\beta: [t_0, \infty) \rightarrow \mathbb{R}_+$ and there exist a $\mu \in (0, 1)$ such that

$$\beta(t) \leq \sum_{i=1}^{n-1} \frac{\mu(\tau(t - ((n-i)/n)\tau(t)) - \tau(t))}{2n^2te}, \quad t \in [t_0, \infty), \quad (3.14)$$

then there exists a $t_0^* \in [t_0, \infty)$ and a positive solution $x = x(t)$ of (1.1) on $[t_0^*, \infty)$.

Proof. Repeating the proof of Theorem 3.2, we get inequality (3.10) (where α is replaced by β), i.e.

$$\begin{aligned} \sum_{i=1}^{n-1} \frac{\tau(t) - \tau(t - ((n-i)/n)\tau(t))}{2net} \left(1 + \frac{\tau(t) - \tau(t - ((n-i)/n)\tau(t))}{4t} \right) \\ - \frac{(n-2)\tau^2(t)}{8net^2} + n\beta(t) + O\left(\frac{\beta(t)\tau(t)}{t}\right) + O\left(\frac{\tau^3(t)}{t^3}\right) < 0. \end{aligned} \quad (3.15)$$

Utilizing (3.14), inequality (3.15) will hold if

$$\begin{aligned} \sum_{i=1}^{n-1} \frac{\tau(t) - \tau(t - ((n-i)/n)\tau(t))}{2net} \left((1-\mu) + \frac{\tau(t) - \tau(t - ((n-i)/n)\tau(t))}{4t} \right) \\ - \frac{(n-2)\tau^2(t)}{8net^2} \\ + O\left(\frac{\left(\sum_{i=1}^{n-1} (\tau(t - ((n-i)/n)\tau(t)) - \tau(t))\right) \tau(t)}{t^2}\right) + O\left(\frac{\tau^3(t)}{t^3}\right) < 0. \end{aligned} \quad (3.16)$$

Asymptotic analysis of all the terms on the left-hand side of (3.16) leads to a conclusion that its sign is determined by the sum of two negative terms (recall that $1 - \mu > 0$)

$$(1-\mu) \sum_{i=1}^{n-1} \frac{\tau(t) - \tau(t - ((n-i)/n)\tau(t))}{2net} - \frac{(n-2)\tau^2(t)}{8net^2}$$

because all the remaining terms are of an asymptotically higher order than at least one these two negative terms. Further, we can proceed as in the proof of Theorem 3.2. \square

Remark 3.5. Comparing Theorem 3.4 with Theorem 2.4, we see that the latter is not a particular case of Theorem 3.4 due to the same reason as described in Remark 3.3. Moreover, a comparison between Theorem 3.4 and Theorem 3.2 can be made in much the same way as between Theorem 2.4 and Theorem 2.3 (see Remark 2.5 where Example 2.6 and Example 2.7 were utilized) using suitable examples.

4 Concluding remarks and open problems

For a constant delay $\tau(t) \equiv \tau$, modifying slightly the proof of Theorem 2.3 and the proof of Theorem 3.2, we get the following theorems respectively:

Theorem 4.1. *Let*

$$\int_{t-\tau/2}^t p(s) ds \leq \frac{1}{2e} + \alpha(t) \quad (4.1)$$

for $t \in [t_0, \infty)$, where $\alpha: [t_0, \infty) \rightarrow \mathbb{R}_+$. If, moreover, there exists a $\mu \in (0, 1)$ such that

$$\alpha(t) \leq \frac{\mu\tau^3}{96t^3e}, \quad (4.2)$$

then there exists a $t_0^* \in [t_0, \infty)$ and a positive solution $x = x(t)$ of (1.1) on $[t_0^*, \infty)$.

Proof. From (4.1) we get

$$\int_{t-\tau}^{t-\tau/2} p(s) \, ds \leq \frac{1}{2e} + \alpha(t - \tau/2)$$

and (4.2) yields

$$\alpha(t - \tau/2) \leq \frac{\mu\tau^3}{96(t-\tau)^3e} = \frac{\mu\tau^3}{96t^3e} + O\left(\frac{1}{t^4}\right).$$

With this modification against the original proof of Theorem 2.3, we can repeat it (with small changes) up to inequality (2.13) which equals

$$\begin{aligned} \mathcal{L}_1 &\leq \frac{1}{e} - \frac{\tau}{2te} - \frac{\tau^3}{48t^3e} + \alpha(t - \tau/2) \left(1 + O\left(\frac{1}{t}\right)\right) + \alpha(t) \left(1 + O\left(\frac{1}{t}\right)\right) + O\left(\frac{1}{t^4}\right) \\ &\leq \frac{1}{e} - \frac{\tau}{2te} - \frac{(1-\mu)\tau^3}{48t^3e} + O\left(\frac{1}{t^4}\right). \end{aligned}$$

Then, $\mathcal{L}_1 \leq \mathcal{R}_1$ holds if

$$\frac{1}{e} - \frac{\tau}{2te} - \frac{(1-\mu)\tau^3}{48t^3e} + O\left(\frac{1}{t^4}\right) < \frac{1}{e} \left[1 - \frac{\tau}{2t}\right],$$

i.e., if

$$-\frac{(1-\mu)\tau^3}{48t^3e} + O\left(\frac{1}{t^4}\right) < 0.$$

The last inequality holds for all $t \in [t_0^*, \infty)$ where t_0^* is sufficiently large. \square

Theorem 4.2. Let $n > 2$ be an integer and

$$\int_{t-\tau/n}^t p(s) \, ds \leq \frac{1}{ne} + \alpha(t) \quad (4.3)$$

for $t \in [t_0, \infty)$, where $\alpha: [t_0, \infty) \rightarrow \mathbb{R}_+$. If, moreover, there exists a $\mu \in (0, 1)$ such that

$$\alpha(t) \leq \frac{\mu(n-2)\tau^2}{8n^2t^2e}, \quad t \in [t_0, \infty), \quad (4.4)$$

then there exists a $t_0^* \in [t_0, \infty)$ and a positive solution $x = x(t)$ of (1.1) on $[t_0^*, \infty)$.

Proof. From (4.3), we get

$$\int_{t-((n-i)/n)\tau}^{t-((n-i)/n)\tau} p(s) \, ds \leq \frac{1}{ne} + \alpha(t - ((n-i)/n)\tau), \quad i = 1, \dots, n$$

and (4.4) yields

$$\alpha(t - ((n-i)/n)\tau) \leq \frac{\mu(n-2)\tau^2}{8n^2(t - ((n-i)/n)\tau)^2e} = \frac{\mu(n-2)\tau^2}{8n^2t^2e} + O\left(\frac{1}{t^3}\right).$$

With this modification against the original proof of Theorem 3.2, we can repeat it without changes up to formula (3.11), which coincides (in the case of a constant delay) with (3.12). \square

Utilizing equation (1.1) with a constant delay $\tau(t) \equiv \tau$, it is easy to demonstrate the mutual independence of criteria (1.3), (4.1) and (4.3).

Remark 4.3. It is easy to see that the criterion of positivity (4.1) is independent of the classical criterion (1.3). To show this, assume that (4.1) holds with a positive function α satisfying (4.2). Then,

$$\int_{t-\tau}^t p(s) ds = \int_{t-\tau}^{t-\tau/2} p(s) ds + \int_{t-\tau/2}^t p(s) ds \leq \frac{1}{e} + \alpha(t - \tau/2) + \alpha(t) \not\leq \frac{1}{e}$$

and the classical criterion (1.3) does not hold.

The independence of criteria (4.1) and (4.3) is demonstrated in the following example.

Example 4.4. Consider the equation

$$\dot{x}(t) + \left(\frac{1}{\tau e} + \varepsilon \sin \frac{6\pi}{\tau} t + e^{-t} \right) x(t - \tau) = 0 \quad (4.5)$$

with $p(t) = 1/\tau e + \varepsilon \sin(6\pi/\tau)t + e^{-t}$ where $0 < \varepsilon < 1/\tau e$. Criterion (4.3) is applicable with $n = 3$ since

$$\int_{t-\tau/3}^t p(s) ds = \frac{1}{3e} + \frac{\varepsilon\tau}{6\pi} \left[-\cos \frac{6\pi}{\tau} s \right] \Big|_{t-\tau/3}^t - e^{-s} \Big|_{t-\tau/3}^t = \frac{1}{3e} + \alpha(t)$$

where

$$\alpha(t) := e^{-t} (e^{\tau/3} - 1).$$

Let us show that α satisfies inequality (4.4), i.e., that

$$e^{-t} (e^{\tau/3} - 1) \leq \frac{\mu\tau^2}{72t^2e}, \quad t \in [t_0, \infty)$$

or

$$t^2 e^{-t} \leq \frac{\mu\tau^2}{72(e^{\tau/3} - 1)e}, \quad t \in [t_0, \infty) \quad (4.6)$$

holds for a $\mu \in (0, 1)$ and a positive t_0 . Let μ be fixed. Since

$$t^2 e^{-t} = t^2 e^{-t/2} e^{-t/2} \leq 16e^{-2} e^{-t/2}, \quad t \in (0, \infty)$$

inequality (4.6) will be valid if simultaneously

$$e^{-t/2} \leq \frac{\mu e^2 \tau^2}{16 \cdot 72 (e^{\tau/3} - 1) e} \quad \text{and} \quad t > 0.$$

Then (4.4) holds if

$$t_0 > \max \left\{ -2 \ln \frac{\mu e \tau^2}{1152 (e^{\tau/3} - 1)}, 0 \right\}$$

and, by Theorem 4.2, equation (4.5) has a positive solution on $[t_0^*, \infty)$ where $t_0^* \geq t_0$ is sufficiently large. However, criterion (4.1) is not applicable. Indeed,

$$\int_{t-\tau/2}^t p(s) ds = \frac{1}{2e} + \frac{\varepsilon\tau}{6\pi} \left[-\cos \frac{6\pi}{\tau} s \right] \Big|_{t-\tau/2}^t - e^{-s} \Big|_{t-\tau/2}^t = \frac{1}{2e} + \alpha(t),$$

where

$$\alpha(t) := -\frac{\varepsilon\tau}{3\pi} \cos \frac{6\pi}{\tau} t + e^{-t} (e^{\tau/2} - 1)$$

and, for a sequence

$$t = t_k = \tau(1 + 2k)/6, \quad k = 0, 1, \dots \quad (4.7)$$

we have

$$\alpha(t_k) := \frac{\varepsilon\tau}{3\pi} + e^{-t_k} \left(e^{\tau/2} - 1 \right)$$

and $\lim_{k \rightarrow \infty} \alpha(t_k) = \varepsilon\tau/(3\pi) \neq 0$.

We will finish this discussion by verifying the fact that the original criterion (1.3) is not applicable either. Indeed,

$$\int_{t-\tau}^t p(s) \, ds = \frac{1}{e} + e^{-t} (e^\tau - 1) > \frac{1}{e} \quad (4.8)$$

for every $t \in [t_0, \infty)$.

Let us formulate some open problems for future research. Although, in the paper, we provided several new explicit integral criteria for the existence of a positive solution $x = x(t)$ of (1.1) and we demonstrated that our criteria are independent of the previously known results, unfortunately, our approach could not, in its present form, improve the classical criterion (1.3). Moreover, it is well-known that the equation

$$\dot{x}(t) + \left(\frac{1}{\tau e} + \frac{\tau}{8et^2} \right) x(t - \tau) = 0$$

has a positive solution (by, e.g., criterion (1.8)), and all solutions of the equation

$$\dot{x}(t) + \left(\frac{1}{\tau e} + \frac{\nu\tau}{8et^2} \right) x(t - \tau) = 0,$$

where $\nu > 1$ oscillate (e.g., by [10, Theorem 12]). For $n \geq 1$, we have

$$\int_{t-\tau}^t \left(\frac{1}{\tau e} + \frac{\tau}{8es^2} \right) \, ds = \frac{1}{e} + \frac{\tau^2}{8et^2} + O\left(\frac{1}{t^3}\right).$$

The result of this computation suggests the best possible expected form of an estimate of the integral of coefficient in (1.1). Based on our results we can formulate, e.g., the following open problem.

Open Problem 1. Prove or disprove the following conjecture. Let $\tau(t) \equiv \tau$ and

$$\int_{t-\tau}^t p(s) \, ds \leq \frac{1}{e} + \alpha(t)$$

for $t \in [t_0, \infty)$ where $\alpha: [t_0, \infty) \rightarrow \mathbb{R}_+$. If, moreover, there exists a constant $\mu \in (0, 1]$ such that

$$\alpha(t) \leq \frac{\mu\tau^2}{8et^2}, \quad t \in [t_0, \infty),$$

then there exists a $t_0^* \in [t_0, \infty)$ and a positive solution $x = x(t)$ of equation

$$\dot{x}(t) + p(t)x(t - \tau) = 0$$

on $[t_0^*, \infty)$.

Similarly, tracing carefully the present results, open problems connected with Theorems 2.3, 2.4, 3.2 and 3.4 can be formulated in the case of a variable delay.

In the paper we also pointed out the difference between Theorem 2.3 and Theorem 3.2. A question arises, how the method used can be improved to get a better estimate of the function α in Theorem 2.3. Therefore, the following problem described below is another challenge for future investigation.

Open Problem 2. Prove or disprove the following conjecture. Let

$$\tau(t) \leq \tau(t - \tau(t)/2) \leq M\tau(t)$$

for all $t \geq t_0 > 0$ and a constant M . If

$$\int_{t-\tau(t)}^{t-\tau(t)/2} p(s) ds \leq \frac{1}{2e} + \alpha(t), \quad \int_{t-\tau(t)/2}^t p(s) ds \leq \frac{1}{2e} + \alpha(t)$$

for $t \in [t_0, \infty)$ where $\alpha: [t_0, \infty) \rightarrow \mathbb{R}_+$ and there exists a $\mu \in (0, 1]$ such that

$$\alpha(t) \leq \frac{\mu\tau^2(t)}{16et^2}, \quad t \in [t_0, \infty),$$

then there exists a $t_0^* \in [t_0, \infty)$ and a positive solution $x = x(t)$ of (1.1) on $[t_0^*, \infty)$.

When solving Open Problem 2, perhaps another scheme of the division of interval $[t - \tau(t), t]$ should be developed, different from the one used in the paper.

The following open problem was suggested by an anonymous referee.

Open Problem 3. Is it possible to generalize the results of Theorems 4.1 and 4.2 if the expressions $\tau/2$ and τ/n in (4.1) and (4.3) are replaced by $\delta\tau$ with $\delta \in (0, 1)$? Is, e.g., the following conjecture true? Let

$$\int_{t-\delta\tau}^t p(s) ds \leq \frac{\delta}{e} + \alpha(t)$$

for $t \in [t_0, \infty)$ where $\alpha: [t_0, \infty) \rightarrow \mathbb{R}_+$. If, moreover, there exists a $\mu \in \mathbb{R}_+$ such that

$$\alpha(t) \leq \frac{\mu}{t^2},$$

then there exists a $t_0^* \in [t_0, \infty)$ and a positive solution $x = x(t)$ of (1.1) on $[t_0^*, \infty)$.

In Theorems 2.3, 2.4, 3.2, and 3.4, inequalities (2.4), (2.16), (3.2), and (3.13) were used. These inequalities are valid if delay is nonincreasing (the case of inequalities (2.4), (3.2)) or decreasing (such possibility is admitted in all four inequalities). The last but not least task is whether similar results on the existence of positive solutions can be derived if the delay is nondecreasing or increasing. Finally, we refer to papers [5, 6, 8, 39] where similar problems of the behavior of solutions of delayed equations are treated.

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