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On a beam equation in Banach spaces

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Abstract. This paper is concerned with the existence and asymptotic behavior of solutions of the Cauchy problem for an abstract model for vertical vibrations of a viscous beam in Banach spaces. First is obtained a local solution of the problem by using the method of successive approximations, a characterization of the derivative of the nonlinear term of the equation defined in a Banach space and the Ascoli–Arzelà theorem. Then the global solution is found by the method of prolongation of solutions. The exponential decay of solutions is derived by considering a Lyapunov functional.

Keywords: beam equation, global solution, asymptotic behavior.

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1 Introduction

The small transverse vibrations due to flexion of an extensible beam, of length L, whose ends are held at fixed distance apart can be described by the following equation

$$u''(x,t) + \sigma\frac{\partial^4 u(x,t)}{\partial x^4} + \left[m_0 + m_1\int_0^L \left(\frac{\partial u(x,t)}{\partial x}\right)^2 dx\right]\left(-\frac{\partial^2 u(x,t)}{\partial x^2}\right) = 0, \tag{1.1}$$

where $0 < x < L$ and $t > 0$. Here $u(x,t)$ denotes the displacement of the point x of the beam at the instant t and σ, m_0 and m_1 are positive constants. The nonlinear term indicates the change in the tension of the beam due to its extensibility. Equation (1.1) was introduced by Woinowsky-Krieger [28].

Equation (1.1) with $\sigma = 0$ describes the small transverse vibrations of an elastic stretched string of length L. This equation was introduced by Kirchhoff [16]. Analyzing the same phenomenon, Carrier [7] obtained the following model:

$$u''(x,t) + \left[m_0 + m_1\int_0^L |u(x,t)|^2\, dx\right]\left(-\frac{\partial^2 u(x,t)}{\partial x^2}\right) = 0. \tag{1.2}$$

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Let $\Omega \subset \mathbb{R}^n$ be a bounded open set of $\mathbb{R}^n$. A generalization of (1.1) and (1.2) is the following equation:

$$u''(x,t) + \sigma(-\Delta)^2 u(x,t) + \left[m_0 + m_1 \int_\Omega |(-\Delta)^\alpha u(x,t)|^2 \, dx \right] (-\Delta u(x,t)) = 0, \tag{1.3}$$

where $x \in \Omega$, $t > 0$ and $0 \leq \alpha \leq 1$.

An abstract formulation for a mixed problem of equation (1.3) is the following:

$$\left| \begin{array}{ll} u''(t) + \sigma A^2 u(t) + M(|A^\alpha u(t)|^2)\, Au(t) = 0 & \text{in } H,\ t > 0 \\ u(0) = u^0, \qquad u'(0) = u^1, & \end{array} \right. \tag{1.4}$$

where $M(\xi)$ is a smooth function satisfying $M(\xi) \geq m_0 > 0$, A is an unbounded self-adjoint operator of a real separable Hilbert space H with A coercive and A^{-1} compact. Here σ and α are real numbers such that $\sigma \geq 0$ and $0 \leq \alpha \leq 1$.

The existence of a global solution of (1.4) was obtained by Medeiros [22]. The decay of solution with a dissipation in the equation of (1.4) was studied by [3–5, 9, 25].

There are many papers that analyze the equation (1.4) with $\sigma = 0$. Among of then we can mention [2, 6, 8, 10, 11, 19, 21, 23, 26]. In Medeiros et al. [24] there are an extensive list of references on problem (1.4) when $\sigma = 0$.

In the above papers the Faedo–Galerkin method is used. The study of hyperbolic problems using the theory of semigroups can be seen in J. A. Goldstein [13] and [12] for the linear and nonlinear case, respectively.

In Izaguirre et al. [14] is formulated problem (1.4), with $\sigma = 0$, in the context of Banach space. More precisely, they consider the problem

$$\left| \begin{array}{ll} Bu''(t) + M(\|u(t)\|_W^\beta) Au(t) = 0 & \text{in } V',\ t > 0 \\ u(0) = u^0, \qquad u'(0) = u^1, \qquad u^0 \neq 0, & \end{array} \right. \tag{1.5}$$

where V is a real separable Hilbert space with dual V'; $A, B : V \to V'$ are two positive linear symmetric operators with A^{-1} and B^{-1} not neccessarily compact; W is a real Banach space such that V is continuously embedded in W and β is a real number with $\beta > 1$. They obtain a local solution for (1.5).

Also, with the introduction of the damping $\delta Bu'(t), \delta > 0$, in the equation of (1.5), Izaguirre et al. [15] obtain a global solution and exponential decay of the energy for (1.5).

Considering $B \equiv I$ and introducing the expression $F(u) + (1 + \alpha \|u\|^\beta) Au'$ in the problem (1.5), where F is an operator and $\alpha > 0$, $\beta \geq 2$, Araruna and Carvalho [1] studied the existence of the global solution, uniqueness and exponential decay.

Motivated by (1.4) and (1.5), we formulate the following problem:

$$\left| \begin{array}{ll} u''(t) + M(\|u(t)\|_W^\beta) Au(t) + A^2 u(t) = 0 & \text{in } V',\ t > 0 \\ u(0) = u^0, \qquad u'(0) = u^1. & \end{array} \right. \tag{1.6}$$

Note that the nonlinear term $M(\|u\|^2_{D(A^\alpha)}) Au$ of (1.4) is a particular case of the nonlinear term $M(\|u\|_W^\beta) Au$ of (1.6) since the Hilbert space $D(A^\alpha)$ is a particular case of the Banach space W. Thus (1.6) generalizes (1.4).

The results of [22] are obtained in the framework of Hilbert spaces and under the hypothesis A^{-1} a compact operator. We want to work in the framework of Banach spaces and where A^{-1} is not necessarily compact, therefore the results of [22] do not apply in our case.

In our approach, we need to obtain two a priori estimates but we cannot differentiate two times with respect to t the term $\|u(t)\|_W^\beta$, $\beta > 1$. To overcome this difficulty we introduce a strong dissipation in equation (1.6), more precisely, we consider

$$\left| \begin{array}{l} u''(t) + M(\|u(t)\|_W^\beta)Au(t) + A^2u(t) + \left[1 + K(t)\left|A^{\frac{3}{2}}u(t)\right|^\beta\right] Au'(t) = 0, \qquad t > 0 \\ u(0) = u^0, \qquad u'(0) = u^1, \end{array} \right. \tag{1.7}$$

where M and K are functions satisfying suitable conditions.

It is possible to solve problem (1.6) with a weak internal dissipation $\delta u'$, $\delta > 0$, but in this case we obtain only solutions under the condition that the initial data belong to a ball whose radius depends on δ. We are interested in obtaining global solutions of (1.6) without restrictions on the norms of the initial data. For this purpose, we consider the dissipation of (1.7).

The objective of this paper is to investigate the existence and asymptotic behavior of solutions of problem (1.7). The plan is as follows: first, with general functions $M(\xi)$ and $K(t)$, we obtain a local solution of (1.7). Then for particular $M(\xi)$ and $K(t)$ increasing in t, we get a global solution of (1.7). Finally, with $K(t) = K$ positive constant and particular $M(t,\xi)$ with $M(t,\xi)$ decreasing in t, we derive a global solution of (1.7). This last solution decays exponentially in t.

To obtain a solution of (1.7), we proceed in the following way. First, by the successive approximation method, the characterization of the derivative of the nonlinear term $M(\|u(t)\|_W^\beta)$ and the Ascoli–Arzelà theorem, we obtain a local solution of (1.7). Then by the method of prolongation of solutions, we deduce the existence of a global solutions of (1.7). The exponential decay of the energy is derived by considering a Lyapunov functional (see V. Komornik and E. Zuazua [17] and V. Komornik [18]). In the last section, we give some examples.

2 Notations and results

Let V and H be two real Hilbert spaces whose scalar product and norm are represented, respectively, by $((u,v))$, $\|u\|$ and (u,v), $|u|$. Here H is separable.

Let us represent by A the unbounded self-adjoint operator of H defined by the triplet $\{V, H; ((u,v))\}$. We have

$$(Au, u) \geq \gamma_0 |u|^2,$$

$\forall u \in D(A)$, where γ_0 is a positive constant (see Lions [20]).

We consider the following hypotheses:

$$V \text{ is densely and continuously embedded in } H, \tag{2.1}$$

$$W \text{ is a real Banach space with dual } W' \text{ strictly convex}, \tag{2.2}$$

$$D(A) \text{ is continuously embedded in } W. \tag{2.3}$$

Consider the functions $M(\xi)$ and $K(t)$ satisfying

$$M \in C^1([0,\infty)),\ M(\xi) \geq m_0 > 0,\ \forall \xi \geq 0 \ \ (m_0 \text{ constant}) \tag{2.4}$$

and

$$K \in L^\infty_{\text{loc}}(0,\infty) \ \text{ and } K(t) \geq 0, \text{ a.e. in } (0,\infty). \tag{2.5}$$

Under the above considerations, we have the following result.

Theorem 2.1 (Local solution)**.** *Assume hypotheses (2.1)–(2.5). Consider a real number β with $\beta > 1$ and*

$$u^0 \in D(A^{\frac{5}{2}}), \qquad u^1 \in D(A^{\frac{3}{2}}). \tag{2.6}$$

Then for $T_0 = \frac{m_0(\ln 2)}{N_1} > 0$, where N_1 will be defined in (2.15), there exists a unique function $u : [0, T_0] \to \mathbb{R}$ in the class

$$\left| \begin{array}{l} u \in L^\infty(0, T_0; D(A^{\frac{5}{2}})) \\ u' \in L^\infty(0, T_0; D(A^{\frac{3}{2}})) \cap L^2(0, T_0; D(A^2)) \\ u'' \in L^\infty(0, T_0; D(A^{\frac{1}{2}})), \end{array} \right. \tag{2.7}$$

satisfying

$$(P1) \qquad \left| \begin{array}{l} u'' + M(\|u\|_W^\beta)Au + A^2u + \left[1 + K\left|A^{\frac{3}{2}}u\right|^\beta\right] Au' = 0, \quad \text{in } L^\infty(0, T_0; D(A^{\frac{1}{2}})) \\ u(0) = u^0, \qquad u'(0) = u^1. \end{array} \right.$$

Remark 2.2. By hypotheses (2.1) and (2.3), we obtain, respectively, two positive constants k_0 and k_1 such that

$$|u| \le k_0 \|u\|, \qquad \forall u \in V \tag{2.8}$$

and

$$\|u\|_W \le k_1 \|u\|_{D(A)}, \qquad \forall u \in D(A). \tag{2.9}$$

Remark 2.3. Let $\theta_1 \ge \theta_2 \ge 0$ be real numbers. Then $D(A^{\theta_1})$ is continuously embedded in $D(A^{\theta_2})$ and

$$\left|A^{\theta_2}u\right|^2 \le \frac{1}{\gamma_0^{2(\theta_1-\theta_2)}} \left|A^{\theta_1}u\right|^2, \qquad \forall u \in D(A^{\theta_1}). \tag{2.10}$$

Remark 2.4. As a consequence of (2.9) and (2.10), we obtain the following:

$$\|u\|_W \le k_2 \|u\|_{D(A^{\frac{3}{2}})}, \qquad \forall u \in D(A^{\frac{3}{2}}), \tag{2.11}$$

$$\|u\|_W \le k_3 \|u\|_{D(A^{\frac{5}{2}})}, \qquad \forall u \in D(A^{\frac{5}{2}}), \tag{2.12}$$

$$\|u\|_W \le k_4 \|u\|_{D(A^2)}, \qquad \forall u \in D(A^2), \tag{2.13}$$

where k_i, $i = 2, 3, 4$, are positive constants.

In what follows, we introduce the real number $T_0 > 0$ mentioned in Theorem 2.1. In fact, consider u^0 and u^1 satisfying hypothesis (2.6). Take a real number $N^2 > 0$ such that

$$\left|A^{\frac{3}{2}}u^1\right|^2 + M\left(\|u^0\|_W^\beta\right)\left|A^2u^0\right|^2 + \left|A^{\frac{5}{2}}u^0\right|^2 < \frac{N^2}{2}. \tag{2.14}$$

Consider also the constant

$$N_1 = \beta R k_2 k_3^{\beta-1} N^\beta, \tag{2.15}$$

where $R = \max_{\xi \in [0, (k_3N)^\beta]} |M'(\xi)|$ and k_2 and k_3 were defined in Remark 2.4. Then T_0 is given by

$$0 < T_0 = \frac{m_0(\ln 2)}{N_1}. \tag{2.16}$$

In order to obtain the global solution of (1.7), we introduce the following hypotheses:

$$M(\xi) = m_0 + m_1\xi, \qquad \forall \xi \geq 0, \tag{2.17}$$

where m_0 and m_1 are constants such that $m_0 > 0$ and $m_1 \geq 0$ and

$$K \in L^{\infty}_{\text{loc}}(0,\infty) \ \text{ with } K(t) > 0 \text{ a.e. in } (0,\infty) \text{ and } \frac{1}{K} \in L^1(0,\infty). \tag{2.18}$$

Also we consider the exponent 2β instead β in the last term of the equation. For the justification, see Remark 3.6.

Theorem 2.5 (Global solution)**.** *Assume hypotheses* (2.1)–(2.3)*,* (2.17) *and* (2.18)*. Consider β a real number with $\beta > 1$ and*

$$u^0 \in D(A^{\frac{5}{2}}), \qquad u^1 \in D(A^{\frac{3}{2}}).$$

Then there exists a unique function $u : (0,\infty) \to \mathbb{R}$ in the class

$$\left| \begin{array}{l} u \in L^{\infty}(0,\infty; D(A^{\frac{5}{2}})) \\ u' \in L^{\infty}(0,\infty; D(A^{\frac{3}{2}})) \cap L^2(0,\infty; D(A^2)) \\ u'' \in L^{\infty}(0,\infty; D(A^{\frac{1}{2}})), \end{array} \right. \tag{2.19}$$

satisfying

$$(P2) \qquad \left| \begin{array}{l} u'' + M(\|u\|_W^{\beta})Au + A^2u + \left[1 + K\left|A^{\frac{3}{2}}u\right|^{2\beta}\right] Au' = 0, \quad \text{in } L^{\infty}_{\text{loc}}(0,\infty; D(A^{\frac{1}{2}})) \\ u(0) = u^0, \qquad u'(0) = u^1. \end{array} \right.$$

The asymptotic behavior of solutions of (1.7) is obtained under the following hypothesis:

$$M(t,\xi) = m_0 + m_1(t)\xi, \tag{2.20}$$

where

$m_0 > 0$ (m_0 constant);

$m_1(t) = \dfrac{1}{z(t)}$, $z(t) > 0$, $m_1(t) \leq m_2$, $t \geq 0$ (m_2 constant);

$z \in C^1([0,\infty))$, $z'(t) \geq C_0 > 0$, $|m'(t)| \leq C_1$, $t \geq 0$ (C_0 and C_1 constants);

$m_1 \in L^1(0,\infty)$.

The energy associated to problem $(P2)$ with the above $M(t,\xi)$ is the following:

$$E(t) = \|u'(t)\|^2 + M(t, \|u\|_W^{\beta})\,|Au(t)|^2 + \left|A^{\frac{3}{2}}u(t)\right|^2, \qquad \forall t \geq 0. \tag{2.21}$$

Theorem 2.6 (Decay of the energy)**.** *Assume hypotheses* (2.1)–(2.3) *and* (2.20)*. Consider real numbers β, K with $\beta \geq 2$, $K \geq \frac{\beta^2(k_1k_7)^{\beta}}{2C_0} > 0$ and*

$$u^0 \in D(A^{\frac{5}{2}}), \qquad u^1 \in D(A^{\frac{3}{2}}).$$

Then there exists a unique function u in the class (2.19)*, u solution of $(P2)$ with $M(t,\xi)$ given by* (2.20) *and $K(t)$ is the function constant K. Furthermore, there exists a positive constant τ_0 such that*

$$E(t) \leq 3E(0)\exp(-\tfrac{2}{3}\tau_0 t), \qquad \forall t \geq 0. \tag{2.22}$$

Note that k_1 was defined in (2.9) and k_7 denotes the immersion constant of $D(A^{\frac{3}{2}})$ into $D(A)$, see Remark 2.3.

Remark 2.7. To obtain the uniqueness of solutions of the above theorem it suffices to consider K a positive constant.

3 Proof of the results

In order to prove the results we need some previous propositions.

Proposition 3.1. *Let $M(\xi)$ be a function $M : [0,\infty[\to \mathbb{R}$ of class C^1 and u be a vectorial function such that $u \in C^1([0,\infty[;W)$, $u(t) \neq 0$, $\forall t \geq 0$. Consider hypothesis (2.2) and β a real number with $\beta \geq 1$. Then the Leibniz derivative of $M(\|u\|_W^\beta)$ is given by*

$$\frac{d}{dt}\left\{M(\|u\|_W^\beta)\right\} = \beta M'(\|u\|_W^\beta)\,\|u\|_W^{\beta-1}\left\langle \frac{Ju(t)}{\|u\|_W}, u'(t)\right\rangle_{W'\times W}, \; t \geq 0,$$

where $J : W \to W'$ is the duality application defined by

$$\langle Ju, u\rangle_{W'\times W} = \|u\|_W^2, \qquad \|Ju\|_{W'} = \|u\|_W, \; \forall u \in W.$$

Furthermore, if $\beta > 1$ and $u(t_0) = 0$, then $\frac{d}{dt}\left\{M(\|u(t_0)\|_W^\beta)\right\} = 0$.

For the proof of the Proposition 3.1, see [14] and [15].

Now consider the real functions μ_1 and μ_2 satisfying the following:

$$\mu_1 \in W^{1,\infty}_{\text{loc}}(0,\infty) \text{ with } \mu_1(t) \geq C^* > 0, \text{ a.e. in } (0,\infty) \qquad (C^* \text{ constant}) \tag{3.1}$$

and

$$\mu_2 \in L^\infty_{\text{loc}}(0,\infty) \text{ with } \mu_2(t) \geq C^{**} > 0, \text{ a.e. in } (0,\infty) \qquad (C^{**} \text{ constant}). \tag{3.2}$$

Proposition 3.2. *Assume hypotheses (3.1) and (3.2). Consider α and δ two real numbers such that $\alpha \geq 0$ and $\delta \geq 0$. If $u^0 \in D(A^{\alpha+2})$ and $u^1 \in D(A^{\alpha+1})$, then there exists a unique function u in the class*

$$\left|\begin{array}{l} u \in L^\infty_{\text{loc}}(0,\infty;D(A^{\alpha+2})) \\ u' \in L^\infty_{\text{loc}}(0,\infty;D(A^{\alpha+1})) \\ u'' \in L^\infty_{\text{loc}}(0,\infty;D(A^{\alpha})) \end{array}\right.$$

such that u is a solution of the problem

$$\left|\begin{array}{l} u'' + \mu_1 Au + A^2u + \delta\mu_2 Au' = 0, \quad \text{in } L^\infty_{\text{loc}}(0,\infty;D(A^\alpha)) \\ u(0) = u^0, \qquad u'(0) = u^1. \end{array}\right.$$

Proof. We apply the Faedo–Galerkin method. Let $\{w_1, w_2, \dots\}$ be a Hilbert basis of H. Consider the basis $\{A^{-2\alpha-2}w_1, A^{-2\alpha-2}, w_2, \dots\}$ of $D(A^{\alpha+2})$. Use the notation $z_j = A^{-2\alpha-2}w_j$, $j = 1,2,\dots$ and denote by $V_m = [z_1, z_2, \dots, z_m]$ the subspace of $D(A^{\alpha+2})$ generated by $z_1, z_2, \dots, z_m$. Consider the approximate solution

$$u_m(t) = \sum_{j=1}^m g_{jm}(t) z_j$$

defined by the system

$$(PA)\quad \left|\begin{array}{l} (u''_m(t), A^{2\alpha+2}z_j) + \mu_1(t)(Au_m(t), A^{2\alpha+2}z_j) + (A^2u_m(t), A^{2\alpha+2}z_j) \\ \qquad + \delta\mu_2(t)(Au'_m(t), A^{2\alpha+2}z_j) = 0, \qquad j = 1,2,\dots,m \\ u_m(0) = u^0_m \to u^0 \text{ in } D(A^{\alpha+2}),\; u^0_m \in V_m \\ u'_m(0) = u^1_m \to u^1 \text{in } D(A^{\alpha+1}),\; u^1_m \in V_m. \end{array}\right.$$

System (PA) has a solution on a certain interval $[0, t_m)$, which can be extended by the next priori estimates, over the interval $[0, T]$ for all real number $T > 0$.

Estimates

Taking $z_j = 2u'_m(t)$ in $(PA)_1$, we obtain

$$\frac{d}{dt}\left[\left|A^{\alpha+1}u'_m(t)\right|^2 + \mu_1(t)\left|A^{\alpha+\frac{3}{2}}u_m(t)\right|^2 + \left|A^{\alpha+2}u_m(t)\right|^2\right] + 2\delta\mu_2(t)\left|A^{\alpha+\frac{3}{2}}u'_m(t)\right|^2$$
$$= \mu'_1(t)\left|A^{\alpha+\frac{3}{2}}u_m(t)\right|^2.$$

Integrating the above equality from 0 to t, $t \leq t_m$ and using (3.1) and (3.2), we get

$$\begin{aligned}\left|A^{\alpha+1}u'_m(t)\right|^2 &+ \mu_1(t)\left|A^{\alpha+\frac{3}{2}}u_m(t)\right|^2 + \left|A^{\alpha+2}u_m(t)\right|^2 + 2\delta C^{**}\int_0^t \left|A^{\alpha+\frac{3}{2}}u'_m(s)\right|^2 ds \\ &\leq C + \int_0^t \mu_1(s)\frac{\mu'_1(s)}{\mu_1(s)}\left|A^{\alpha+\frac{3}{2}}u_m(s)\right|^2 ds,\end{aligned} \tag{3.3}$$

where $C > 0$ is a constant independent of m and t.

Applying Gronwall's inequality in (3.3) and using (3.1), we obtain

$$\left|A^{\alpha+1}u'_m(t)\right|^2 + C^*\left|A^{\alpha+\frac{3}{2}}u_m(t)\right|^2 + \left|A^{\alpha+2}u_m(t)\right|^2 + 2\delta C^{**}\int_0^t \left|A^{\alpha+\frac{3}{2}}u'_m(s)\right|^2 ds \leq C_T, \tag{3.4}$$

$\forall t \in [0, t_m)$, $t_m \leq T$, where

$$C_T = C\exp\left[\int_0^T \frac{\mu'_1(s)}{\mu_1(s)}ds\right].$$

As a consequence of estimates (3.4), we deduce, respectively, the existence of a subsequence of $(u_m)_{m\in\mathbb{N}}$, still denoted by $(u_m)_{m\in\mathbb{N}}$, such that

$$\left|\begin{array}{l} u_m \to u \text{ weak star in } L^\infty(0,T;D(A^{\alpha+2})) \\ u'_m \to u' \text{ weak star in } L^\infty(0,T;D(A^{\alpha+1})) \\ u'_m \to u' \text{ weak in } L^2(0,T;D(A^{\alpha+\frac{3}{2}})). \end{array}\right. \tag{3.5}$$

Now, multiplying the approximate equation $(PA)_1$ by $\theta \in D(0,T)$, integrating the result of 0 to T and using the convergences (3.5), we get

$$u'' + \mu_1 Au + A^2u + \delta\mu_2 Au' = 0 \quad \text{in } L^\infty(0,T;D(A^\alpha)). \tag{3.6}$$

Finally, using the diagonal process we obtain equality (3.6) in $L^\infty_{\text{loc}}(0,\infty;D(A^\alpha))$. By standard arguments, we verify the initial conditions and the uniqueness of the solutions. This concludes the proof of Proposition 3.2. □

3.1 Proof of Theorem 2.1

A sketch of the proof of Theorem 2.1 is as follows. First, we approximate u^0 and u^1 by functions u^0_l and u^1_l belonging to $D(A^4)$ and $D(A^3)$, respectively. Then by Proposition 3.1 and 3.2 and the method of successive approximations, we determine the solution u_l of the problem

$$(P_l) \qquad \left|\begin{array}{l} u''_l + M(\|u_l\|^\beta_W)Au_l + A^2u_l + \left[1 + K\left|A^{\frac{3}{2}}u_l\right|^\beta\right]Au'_l = 0 \\ u_l(0) = u^0_l, \qquad u'_l(0) = u^1_l. \end{array}\right.$$

Estimates obtained for the solution u_l allow us to pass to the limit in the equation in (P_l). The limit of the nonlinear terms follows by applying Proposition 3.1 and the Ascoli–Arzelà theorem for real functions.

We begin the proof. As a consequence of (2.14), we can choose $\eta > 0$ such that

$$\left[\left|A^{\frac{3}{2}}u^1\right|^2 + \eta\right] + \left[M(\|u^0\|_W^\beta) + \eta\right]\left[\left|A^2 u^0\right|^2 + \eta\right] + \left[\left|A^{\frac{5}{2}}u^0\right|^2 + \eta\right] < \frac{N^2}{2}. \tag{3.7}$$

Consider sequences $(u_l^0)_{l\in\mathbb{N}}$ and $(u_l^1)_{l\in\mathbb{N}}$ of vectors of $D(A^4)$ and $D(A^3)$, respectively, such that

$$u_l^0 \to u^0 \quad \text{in } D(A^{\frac{5}{2}}) \tag{3.8}$$

and

$$u_l^1 \to u^1 \quad \text{in } D(A^{\frac{3}{2}}). \tag{3.9}$$

Therefore it follows from (2.4), (2.12) and (3.8) the convergence

$$M(\|u_l^0\|_W^\beta) \to M(\|u^0\|_W^\beta). \tag{3.10}$$

As a consequence of (2.10) and (3.8)–(3.10), there exists $l_0(\eta)$ such that for $l \geq l_0(\eta)$, we have

$$\left|\begin{array}{ll} \left|A^{\frac{5}{2}}u_l^0\right|^2 \leq \left|A^{\frac{5}{2}}u^0\right|^2 + \eta, & \left|A^2 u_l^0\right|^2 \leq \left|A^2 u^0\right|^2 + \eta \\ M(\|u_l^0\|_W^\beta) \leq M(\|u^0\|_W^\beta) + \eta, & \left|A^{\frac{3}{2}}u_l^1\right|^2 \leq \left|A^{\frac{3}{2}}u^1\right|^2 + \eta. \end{array}\right. \tag{3.11}$$

Then inequalities (3.7) and (3.11) provide

$$\left|A^{\frac{3}{2}}u_l^1\right|^2 + M(\|u_l^0\|_W^\beta)\left|A^2 u_l^0\right|^2 + \left|A^{\frac{5}{2}}u_l^0\right|^2 < \frac{N^2}{2}, \tag{3.12}$$

$\forall l \geq l_0(\eta)$.

Let v be a function satisfying

$$v \in L^\infty(0, T_0; D(A^4)), \qquad v' \in L^\infty(0, T_0; D(A^3)), \qquad v'' \in L^\infty(0, T_0; D(A^2)) \tag{3.13}$$

and

$$\max_{0 \leq t < \leq T_0}\left[\left|A^{\frac{3}{2}}v'(t)\right|^2 + m_0\left|A^2 v(t)\right|^2 + \left|A^{\frac{5}{2}}v(t)\right|^2 + 2\int_0^t \left|A^2 v'(s)\right|^2 ds\right] \leq N^2. \tag{3.14}$$

Now we consider following technical lemma.

Lemma 3.3. *Suppose that v satisfies (3.13) and (3.14). Then*

$$\left|\frac{d}{dt}\left\{M(\|v(t)\|_W^\beta)\right\}\right| \leq N_1, \tag{3.15}$$

$\forall t \in [0, T_0]$, where N_1 was defined in (2.15).

The above Lemma follows by using (2.4), (2.11), (2.12), (3.14) and Proposition 3.1.

Remark 3.4. We note that inequality (3.15) remains valid even when $v(t) = 0$, for some $t \in [0, T]$, by virtue of Proposition 3.1.

In the sequel we will use the method of successive approximations to obtain the solution of problem (P_l). Thus, we consider the following problem:

$$(P_{l,1})\qquad \left|\begin{array}{l} u_{l,1}''(t)+M(\|u_l^0\|_W^\beta)Au_{l,1}(t)+A^2u_{l,1}(t)+\left[1+K(t)\left|A^{\frac{3}{2}}u_l^0\right|^\beta\right]Au_{l,1}'(t)=0, \quad t\in[0,T_0]\\ u_{l,1}(0)=u_l^0, \qquad u_{l,1}'(0)=u_l^1. \end{array}\right.$$

It follows from hypotheses (2.4), (2.5) and Proposition 3.2 that $u_{l,1}$ belongs to class (3.13). Now taking the scalar product of H of both sides of the equation in $(P_{l,1})$ with $2A^3u_{l,1}'$, integrating this result on $[0,t]$, $0<t\leq T_0$, using (3.12) and the hypothesis (2.4), we obtain that $u_{l,1}$ satisfies (3.14).

Define the sequence $(u_{l,\nu})_{\nu\geq 2}$, where $u_{l,\nu}$ is the solution of the problem

$$(P_{l,\nu})\qquad \left|\begin{array}{l} u_{l,\nu}''(t)+M(\|u_{l,\nu-1}(t)\|_W^\beta)Au_{l,\nu}(t)+A^2u_{l,\nu}(t)+\left[1+K(t)\left|A^{\frac{3}{2}}u_{l,\nu-1}(t)\right|^\beta\right]Au_{l,\nu}'(t)=0\\ u_{l,\nu}(0)=u_l^0, \qquad u_{l,\nu}'(0)=u_l^1. \end{array}\right.$$

Using induction we shall prove that $u_{l,\nu}$ satisfies the (3.13) and (3.14). In fact, assume that $u_{l,\nu-1}$ satisfies (3.13) and (3.14). Then, by Lemma 3.3, we have

$$\left|\frac{d}{dt}\left\{M(\|u_{l,\nu-1}\|_W^\beta)\right\}\right|\leq N_1,$$

$\forall t\in[0,T_0]$.

Also by Proposition 3.2, we derive that $u_{l,\nu}$ belongs to class (3.13).

Taking the scalar product of H of both sides of equation $(P_{l,\nu})_1$ with $2A^3u_{l,\nu}'(t)$, applying similar arguments used to prove that $u_{l,1}$ satisfies (3.14) and using the last inequality, we obtain

$$\begin{aligned} &\left|A^{\frac{3}{2}}u_{l,\nu}'(t)\right|^2+M(\|u_{l,\nu-1}\|_W^\beta)\left|A^2u_{l,\nu}(t)\right|^2+\left|A^{\frac{5}{2}}u_{l,\nu}(t)\right|^2+2\int_0^t\left|A^2u_{l,\nu}'(s)\right|^2ds\\ &\quad\leq\left|A^{\frac{3}{2}}u_l^1\right|^2+M(\|u_l^0\|_W^\beta)\left|A^2u_l^0\right|^2+\left|A^{\frac{5}{2}}u_l^0\right|^2+N_1\int_0^t\left|A^2u_{l,\nu}(s)\right|^2ds. \end{aligned}$$

Then by (3.12) we find

$$\begin{aligned} &\left|A^{\frac{3}{2}}u_{l,\nu}'(t)\right|^2+m_0\left|A^2u_{l,\nu}(t)\right|^2+\left|A^{\frac{5}{2}}u_{l,\nu}(t)\right|^2+2\int_0^t\left|A^2u_{l,\nu}'(s)\right|^2ds\\ &\quad\leq\frac{N^2}{2}+\frac{N_1}{m_0}\int_0^t m_0\left|A^2u_{l,\nu}(s)\right|^2ds, \end{aligned}$$

$\forall l\geq l_0(\eta)$, $t\in[0,T_0]$.

Hence Gronwall's inequality implies

$$\left|A^{\frac{3}{2}}u_{l,\nu}'(t)\right|^2+m_0\left|A^2u_{l,\nu}(t)\right|^2+\left|A^{\frac{5}{2}}u_{l,\nu}(t)\right|^2+2\int_0^t\left|A^2u_{l,\nu}'(s)\right|^2ds\leq\left(\frac{N^2}{2}\right)^2\exp\left(\frac{N_1}{m_0}t\right),$$

$\forall l\geq l_0(\eta)$, $t\in[0,T_0]$.

Then thanks to the choice of T_0, this inequality provides

$$\left|A^{\frac{3}{2}}u_{l,\nu}'(t)\right|^2+m_0\left|A^2u_{l,\nu}(t)\right|^2+\left|A^{\frac{5}{2}}u_{l,\nu}(t)\right|^2+2\int_0^t\left|A^2u_{l,\nu}'(s)\right|^2ds\leq N^2,$$

$\forall l \geq l_0(\eta)$, $t \in [0, T_0]$.

Thus $u_{l,\nu}$ satisfies (3.13) and (3.14).

The last inequality implies that there exists a subsequence of $(u_{l,\nu})_{\nu\in\mathbb{N}}$, still denoted by $(u_{l,\nu})_{\nu\in\mathbb{N}}$, such that

$$\left| \begin{array}{l} u_{l,\nu} \to u_l \text{ weak star in } L^\infty(0, T_0; D(A^{\frac{5}{2}})) \\ u'_{l,\nu} \to u'_l \text{ weak star in } L^\infty(0, T_0; D(A^{\frac{3}{2}})) \\ u'_{l,\nu} \to u'_l \text{ weak in } L^2(0, T_0; D(A^2)). \end{array} \right. \tag{3.16}$$

Convergences (3.16) are not sufficient to pass to the limit in problem $(P_{l,\nu})$ due to the nonlinear terms. Next we will prove that

$$M(\|u_{l,\nu-1}\|_W^\beta) \to M(\|u_l\|_W^\beta) \quad \text{in } C^0([0, T_0]) \tag{3.17}$$

and

$$\left|A^{\frac{3}{2}} u_{l,\nu-1}\right|^\beta \to \left|A^{\frac{3}{2}} u_l\right|^\beta \quad \text{in } C^0([0, T_0]). \tag{3.18}$$

Let us begin considering the sequence $(\varphi_{l,\nu})_{\nu\in\mathbb{N}}$, where $\varphi_{l,\nu}(t) = \|u_{l,\nu-1}(t)\|_W^\beta$. As a consequence of (2.12) and (3.14) it follows that

$$\|u_{l,\nu-1}(t)\|_W^\beta \leq k_3^\beta \left|A^{\frac{5}{2}} u_{l,\nu-1}(t)\right|^\beta \leq (k_3 N)^\beta. \tag{3.19}$$

Now using the mean value theorem, Proposition 3.1, (2.11), (2.12) and (3.14), we have

$$\left| \|u_{l,\nu-1}(t_2)\|_W^\beta - \|u_{l,\nu-1}(t_1)\|_W^\beta \right| \leq \beta k_3^{\beta-1} k_2 N^\beta \left|t_2 - t_1\right|. \tag{3.20}$$

Therefore from (3.19), (3.20) and the Ascoli–Arzelà theorem it follows that there exists $\varphi_l \in C^0([0, T_0])$ such that

$$\|u_{l,\nu-1}\|_W^\beta \to \varphi_l \quad \text{in } C^0([0, T_0]). \tag{3.21}$$

Consequently we obtain from (3.21) and (2.4) the convergence

$$M(\|u_{l,\nu-1}\|_W^\beta) \to M(\varphi_l). \tag{3.22}$$

Now let us consider the sequence $(\psi_{l,\nu})_{\nu\in\mathbb{N}}$, where $\psi_{l,\nu}(t) = \left|A^{\frac{3}{2}} u_{l,\nu-1}(t)\right|^\beta$. In a similar way as in (3.22), we conclude that there exists a sequence $\psi_l \in C^0([0, T_0])$ such that

$$\left|A^{\frac{3}{2}} u_{l,\nu-1}\right|^\beta \to \psi_l \quad \text{in } C^0([0, T_0]). \tag{3.23}$$

Below we will show that $M(\varphi_l) = M(\|u_l\|_W^\beta)$ and $\psi_l = \left|A^{\frac{3}{2}} u_l\right|^\beta$. For that, one proceeds as follows. Let $u_{l,\nu}$ and $u_{l,\sigma}$ be the solutions of problems $(P_{l,\nu})$ and $(P_{l,\sigma})$, respectively. Consider $w_{\sigma\nu} = u_{l,\sigma} - u_{l,\nu}$. So $w_{\sigma\nu}$ is the solution of the problem

$$(P_{\sigma\nu}) \quad \left| \begin{array}{l} w''_{\sigma\nu}(t) + M(\|u_{l,\sigma-1}(t)\|_W^\beta) A w_{\sigma\nu}(t) + A^2 w_{\sigma\nu}(t) + \left[1 + K(t) \left|A^{\frac{3}{2}} u_{l,\sigma-1}(t)\right|^\beta\right] A w'_{\sigma\nu}(t) \\ \quad = \left[M(\|u_{l,\nu-1}(t)\|_W^\beta) - M(\|u_{l,\sigma-1}(t)\|_W^\beta)\right] A u_{l,\nu}(t) \\ \quad\quad + K(t) \left[\left|A^{\frac{3}{2}} u_{l,\nu-1}(t)\right|^\beta - \left|A^{\frac{3}{2}} u_{l,\sigma-1}(t)\right|^\beta\right] A u'_{l,\nu}(t), \qquad t \in [0, T_0] \\ w_{\sigma\nu}(0) = 0, \qquad w'_{\sigma\nu}(0) = 0. \end{array} \right.$$

Taking the scalar product of H of both sides of the equation $(P_{\sigma\nu})$ with $2A^2 w'_{\sigma\nu}(t)$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left[\left|Aw'_{\sigma\nu}(t)\right|^2 + M(\|u_{l,\sigma-1}(t)\|_W^\beta)\left|A^{\frac{3}{2}}w_{\sigma\nu}(t)\right|^2 + \left|A^2 w_{\sigma\nu}(t)\right|^2\right] \\
&\quad +2\left[1+K(t)\left|A^{\frac{3}{2}}u_{l,\sigma-1}(t)\right|^\beta\right]\left|A^{\frac{3}{2}}w'_{\sigma\nu}(t)\right|^2 \\
&= \left[\frac{d}{dt}M(\|u_{l,\sigma-1}(t)\|_W^\beta)\right]\left|A^{\frac{3}{2}}w_{\sigma\nu}(t)\right|^2 \\
&\quad +2\left[M(\|u_{l,\nu-1}(t)\|_W^\beta) - M(\|u_{l,\sigma-1}(t)\|_W^\beta)\right](A^2u_{l,\nu}(t), Aw'_{\sigma\nu}(t)) \\
&\quad +2K(t)\left[\left|A^{\frac{3}{2}}u_{l,\nu-1}(t)\right|^\beta - \left|A^{\frac{3}{2}}u_{l,\sigma-1}(t)\right|^\beta\right](A^2u'_{l,\nu}(t), Aw'_{\sigma\nu}(t)),
\end{aligned} \tag{3.24}$$

$t \in [0, T_0]$.

By Lemma 3.3, the first term of the second member of (3.24) can be bounded by

$$N_1\left|A^{\frac{3}{2}}w_{\sigma\nu}(t)\right|^2.$$

As $(M(\|u_{l,\nu-1}(t)\|_W^\beta))$ is convergent in $C^0([0,T_0])$, it follows that for $\varepsilon > 0$, there exists ν_0 such that

$$\left|M(\|u_{l,\nu-1}(t)\|_W^\beta) - M(\|u_{l,\sigma-1}(t)\|_W^\beta)\right| \le \varepsilon,$$

$\forall \sigma, \nu \ge \nu_0$, $t \in [0, T_0]$.

This inequality and (3.14) imply that the second term of the second member of (3.24) can be bounded by

$$2\varepsilon\frac{N}{m_0^{\frac{1}{2}}}\left|Aw'_{\sigma\nu}(t)\right|,$$

$\forall \sigma, \nu \ge \nu_0$.

In a similar way, the third term of the second member of (3.24)can be bounded by

$$2\varepsilon k^*(T_0)\frac{N}{m_0^{\frac{1}{2}}}\left|Aw'_{\sigma\nu}(t)\right|,$$

$\forall \sigma, \nu \ge \nu_0$, where $k^*(T_0) = \|K\|_{L^\infty(0,T_0)}$.

Integrating both members of (3.24) on $[0,t]$, $0 < t \le T_0$, and taking into account the last four results, we obtain

$$\left|Aw'_{\sigma\nu}(t)\right|^2 + m_0\left|A^{\frac{3}{2}}w_{\sigma\nu}(t)\right|^2 \le C\varepsilon^2 + \int_0^t \left|Aw'_{\sigma\nu}(s)\right|^2 ds + \frac{N_1}{m_0}\int_0^t m_0\left|A^{\frac{3}{2}}w_{\sigma\nu}(s)\right|^2 ds,$$

$\forall \sigma, \nu \ge \nu_0$, where $C > 0$ is a generic constant which is independent of σ and ν.

The last inequality and the Gronwall inequality imply that $(A^{\frac{3}{2}}u_{l,\nu})_{\nu\in\mathbb{N}}$ is a Cauchy sequence in $C^0([0,T_0];H)$. Consequently we have

$$u_{l,\nu} \to u_l \quad \text{in } C^0([0,T_0];D(A^{\frac{3}{2}})) \tag{3.25}$$

which provides convergence (3.18).

Using (2.11) and the convergence (3.25) it follows that $(u_{l,\nu})_{\nu\in\mathbb{N}}$ is a Cauchy sequence in $C^0([0,T_0];W)$. Therefore

$$u_{l,\nu} \to u_l \quad \text{in } C^0([0,T_0];W)$$

which implies the convergence

$$\|u_{l,\nu}\|_W^{\beta} \to \|u_l\|_W^{\beta} \quad \text{in } C^0([0,T_0]). \tag{3.26}$$

Convergences (3.21), (3.22) and (3.26) provide convergence (3.17).

Due to convergences (3.16), (3.17) and (3.18), we can pass to the limit in $(P_{l,\nu})$. The limit u_l is a solution of problem (P_l).

Our next goal is to take the limit in problem (P_l).

Write (3.14) with $u_{l,\nu}$ and take the limit inf of both sides of this inequality. Then convergences (3.16) provide

$$\operatorname*{ess\,sup}_{0<t<T_0}\left[\left|A^{\frac{3}{2}}u_l'(t)\right|^2 + m_0\left|A^2u_l(t)\right|^2 + \left|A^{\frac{5}{2}}u_l(t)\right|^2 + \int_0^t \left|A^2u_l'(s)\right|^2 ds\right] \leq N^2,$$

$\forall l \in \mathbb{N}$.

This implies that there exists a subsequence of (u_l), still denoted by (u_l), such that

$$\left|\begin{array}{l} u_l \to u \text{ weak star in } L^\infty(0,T_0;D(A^{\frac{5}{2}})) \\ u_l' \to u' \text{ weak star in } L^\infty(0,T_0;D(A^{\frac{3}{2}})) \\ u_l' \to u' \text{ weak in } L^2(0,T_0;D(A^2)). \end{array}\right. \tag{3.27}$$

In the sequel we will prove that

$$M(\|u_l\|_W^{\beta}) \to M(\|u\|_W^{\beta}) \quad \text{in } C^0([0,T_0]) \tag{3.28}$$

and

$$\left|A^{\frac{3}{2}}u_l\right|^{\beta} \to \left|A^{\frac{3}{2}}u\right|^{\beta} \quad \text{in } C^0([0,T_0]). \tag{3.29}$$

Let us consider two sequences $(\varphi_l)_{l\in\mathbb{N}}$ and $(\psi_l)_{l\in\mathbb{N}}$, such that $\varphi_l(t) = \|u_l(t)\|_W^{\beta}$ and $\psi_l(t) = \left|A^{\frac{3}{2}}u_l(t)\right|^{\beta}$. Then by applying arguments similar to those used to obtain (3.21) and (3.23), we get two functions $\varphi, \psi \in C^0([0,T_0])$ such that

$$\|u_l\|_W^{\beta} \to \varphi \quad \text{in } C^0([0,T_0]) \tag{3.30}$$

and

$$\left|A^{\frac{3}{2}}u_l\right|^{\beta} \to \psi \quad \text{in } C^0([0,T_0]). \tag{3.31}$$

Thus hypothesis (2.4) and convergence (3.30) provide

$$M(\|u_l\|_W^{\beta}) \to M(\varphi) \quad \text{in } C^0([0,T_0]). \tag{3.32}$$

In the sequel, we will show that $\varphi = \|u\|_W^{\beta}$ and $\psi = \left|A^{\frac{3}{2}}u\right|^{\beta}$. Let us begin considering u_l and u_k two solutions of problems (P_l) and (P_k), respectively. Consider still $w_{lk} = u_l - u_k$. So w_{lk} is the solution of the problem

$$(P_{lk}) \quad \left|\begin{array}{l} w_{lk}''(t) + M(\|u_l(t)\|_W^{\beta})Aw_{lk}(t) + A^2w_{lk}(t) + \left[1 + K(t)\left|A^{\frac{3}{2}}u_l(t)\right|^{\beta}\right]Aw_{lk}'(t) \\ \quad = \left[M(\|u_k(t)\|_W^{\beta}) - M(\|u_l(t)\|_W^{\beta})\right]Au_k(t) \\ \quad\quad + K(t)\left[\left|A^{\frac{3}{2}}u_k(t)\right|^{\beta} - \left|A^{\frac{3}{2}}u_l(t)\right|^{\beta}\right]Au_k'(t), \qquad t \in [0,T_0] \\ w_{lk}(0) = u_l^0 - u_k^0, \qquad w_{lk}'(0) = u_l^1 - u_k^1. \end{array}\right.$$

Taking the scalar product of H of both sides of the equation (P_{lk}) with $2A^3 w'_{lk}(t)$ and integrating the result into $[0,t]$, $0 < t \leq T_0$, we obtain

$$\begin{aligned}
&\left|A^{\frac{3}{2}} w'_{lk}(t)\right|^2 + \left[M(\|u_l(t)\|_W^\beta)\left|A^2 w_{lk}(t)\right|^2\right] + \left|A^{\frac{5}{2}} w_{lk}(t)\right|^2 + 2\int_0^t \left|A^2 w'_{lk}(s)\right|^2 ds \\
&\quad + 2\int_0^t K(s)\left|A^{\frac{3}{2}} u_l(s)\right|^\beta \left|A^2 w'_{lk}(s)\right|^2 ds \\
&= \left|A^{\frac{3}{2}} w'_{lk}(0)\right|^2 + \left[M(\|u_l(0)\|_W^\beta)\left|A^2 w_{lk}(0)\right|^2\right] + \left|A^{\frac{5}{2}} w_{lk}(0)\right|^2 \\
&\quad + \int_0^t \left[\frac{d}{ds}\left\{M(\|u_l(s)\|_W^\beta)\right\}\right]\left|A^2 w_{lk}(s)\right|^2 ds \\
&\quad + 2\int_0^t \left[M(\|u_k(s)\|_W^\beta) - M(\|u_l(s)\|_W^\beta)\right](A^2 u_k(s), A^2 w'_{lk}(s)) ds \\
&\quad + 2\int_0^t K(s)\left[\left|A^{\frac{3}{2}} u_k(s)\right|^\beta - \left|A^{\frac{3}{2}} u_l(s)\right|^\beta\right](A^2 u'_k(s), A^2 w'_{lk}(s)) ds.
\end{aligned} \tag{3.33}$$

By similar arguments used to bound the terms of the second member of (3.24) and using convergences (3.30) and (3.31), we obtain

$$\left|\frac{d}{ds}\left\{M(\|u_l(s)\|_W^\beta)\right\}\right| \leq N_1,$$

a.e. in $(0, T_0)$, and for $\varepsilon > 0$,

$$2\left|\left[M(\|u_k(s)\|_W^\beta) - M(\|u_l(s)\|_W^\beta)\right](A^2 u_k(s), A^2 w'_{lk}(s))\right| \leq \frac{2\varepsilon^2 N^2}{m_0} + \frac{1}{2}\left|A^2 w'_{lk}(s)\right|^2,$$

and

$$\begin{aligned}
&2\left|K(s)\left[\left|A^{\frac{3}{2}} u_k(s)\right|^\beta - \left|A^{\frac{3}{2}} u_l(s)\right|^\beta\right](A^2 u'_k(s), A^2 w'_{lk}(s))\right| \\
&\quad \leq 2\varepsilon^2\left[k^*(T_0)\right]^2\left|A^2 u'_k(s)\right|^2 + \frac{1}{2}\left|A^2 w'_{lk}(s)\right|^2,
\end{aligned}$$

$\forall k, l \geq l_0$.

Taking into account the last three inequalities in (3.33), noting that $\left|A^2 u'_k(s)\right|^2 \leq N^2$, a.e. in $(0, T_0)$ and that the first three terms of the second member of (3.33) can be bounded by ε^2, we find

$$\begin{aligned}
&\left|A^{\frac{3}{2}} w'_{lk}(t)\right|^2 + m_0\left|A^2 w_{lk}(t)\right|^2 + \left|A^{\frac{5}{2}} w_{lk}(t)\right|^2 + 2\int_0^t \left|A^2 w'_{lk}(s)\right|^2 ds \\
&\quad \leq 2\epsilon^2\left[\frac{N^2}{m_0}T_0 + \left[k^*(T_0)\right]^2 N^2 T_0 + 1\right] + \frac{N_1}{m_0}\int_0^t m_0\left|A^2 w_{lk}(s)\right|^2 ds,
\end{aligned}$$

$\forall k, l \geq l_0$.

The last inequality and Gronwall's inequality imply that

$$\left|A^2 w_{lk}(t)\right|^2 \leq C\varepsilon^2,$$

$t \in [0, T_0]$, $\forall k, l \geq l_0$, where $C > 0$ denotes a generic constant which is independent of l and k.

The last inequality implies that

$$u_l \to u \quad \text{in } C^0([0, T_0]; D(A^2)). \tag{3.34}$$

In view of Remark 2.3, it follows from (3.34) the following convergence:

$$u_l \to u \text{ in } C^0([0, T_0]\,; D(A^{\frac{3}{2}})) \tag{3.35}$$

which implies convergence (3.29).

Combining (3.35) with (2.11) it results that

$$u_l \to u \quad \text{in } C^0([0, T_0]\,; W)$$

which implies convergence

$$\|u_l\|_W^\beta \to \|u\|_W^\beta \quad \text{in } C^0([0, T_0]). \tag{3.36}$$

Combining (3.30), (3.32) and (3.36), we obtain the convergence (3.28).

Due the convergences (3.27), (3.28)and (3.29), we can pass to the limit in (P_l). The limit u is a solution of problem $(P1)_1$ and u verifies (2.7). Using a standard argument, we can verify the initial conditions $(P1)_2$.

The uniqueness of the solution is proved by the energy method. In fact, we consider u and v in the conditions of the Theorem 2.1. Then $w = u - v$ satisfies

$$(P)\quad \left|\begin{aligned} & w''(t) + M(\|u(t)\|_W^\beta) A w(t) + A^2 w(t) + \left[1 + K(t)\left|A^{\frac{3}{2}} u(t)\right|^\beta\right] A w'(t) \\ & \quad = \left[M(\|v(t)\|_W^\beta) - M(\|u(t)\|_W^\beta)\right] A v(t) \\ & \qquad + K(t)\left[\left|A^{\frac{3}{2}} v(t)\right|^\beta - \left|A^{\frac{3}{2}} u(t)\right|^\beta\right] A v'(t), \qquad t \in [0, T_0] \\ & w(0) = w'(0) = 0. \end{aligned}\right.$$

Taking the scalar product in H of both sides of equation of (P) with $2Aw'(t)$, we obtain

$$\begin{aligned} & \frac{d}{dt}\left[\left|A^{\frac{1}{2}} w'(t)\right|^2 + M(\|u(t)\|_W^\beta)\,|Aw(t)|^2 + \left|A^{\frac{3}{2}} w(t)\right|^2\right] \\ & \quad + 2\left[1 + K(t)\left|A^{\frac{3}{2}} u(t)\right|^\beta\right] |Aw'(t)|^2 \\ & = \left[\frac{d}{dt} M(\|u(t)\|_W^\beta)\right] |Aw(t)|^2 \\ & \quad + 2\left[M(\|v(t)\|_W^\beta) - M(\|u(t)\|_W^\beta)\right] (A^{\frac{3}{2}} v(t), A^{\frac{1}{2}} w'(t)) \\ & \quad + 2K(t)\left[\left|A^{\frac{3}{2}} v(t)\right|^\beta - \left|A^{\frac{3}{2}} u(t)\right|^\beta\right] (A^{\frac{3}{2}} v'(t), A^{\frac{1}{2}} w'(t)), \end{aligned} \tag{3.37}$$

$t \in [0, T_0]$.

By the mean value theorem, we get

$$M(\|v(t)\|_W^\beta) - M(\|u(t)\|_W^\beta) = \beta M'(\xi^*)(s^*)^{\beta-1}\left[\|v(t)\|_W - \|u(t)\|_W\right],$$

where ξ^* is between the real numbers $\|v(t)\|_W^\beta$ and $\|u(t)\|_W^\beta$ and s^* is between the real numbers $\|v(t)\|_W$ and $\|u(t)\|_W$. By R given in (2.15), inequality (3.14) and (2.9), we find

$$\begin{aligned} & 2\left|\left[M(\|v(t)\|_W^\beta) - M(\|u(t)\|_W^\beta)\right] (A^{\frac{3}{2}} v(t), A^{\frac{1}{2}} w'(t))\right| \\ & \quad \le 2\beta R (k_3 N)^{\beta-1} \|v(t) - u(t)\|_W \left|A^{\frac{3}{2}} v(t)\right| \left|A^{\frac{1}{2}} w'(t)\right| \\ & \quad \le 2\beta R k_3^{\beta-1} k_1 k_5 N^\beta\, |Aw(t)| \left|A^{\frac{1}{2}} w'(t)\right|, \end{aligned} \tag{3.38}$$

where k_5 denotes the immersion constant of $D(A^{\frac{5}{2}})$ into $D(A^{\frac{3}{2}})$ (see Remark 2.3).

In similar way we obtain that

$$\begin{aligned} 2\left|K(t)\left[\left|A^{\frac{3}{2}}v(t)\right|^{\beta}-\left|A^{\frac{3}{2}}u(t)\right|^{\beta}\right](A^{\frac{3}{2}}v'(t),A^{\frac{1}{2}}w'(t))\right| \\ \leq 2k^*(T_0)k_5^{\beta-1}N\left|A^{\frac{3}{2}}w(t)\right|\left|A^{\frac{1}{2}}w'(t)\right|, \end{aligned} \tag{3.39}$$

$t\in[0,T_0]$.

Combining (3.37) with (3.38) and (3.39) and using the Lemma 3.3, we obtain

$$\begin{aligned} \frac{d}{dt}\left[\left|A^{\frac{1}{2}}w'(t)\right|^2+M(\|u(t)\|_W^{\beta})\,|Aw(t)|^2+\left|A^{\frac{3}{2}}w(t)\right|^2\right] \\ \leq C\left[|Aw(t)|^2+\left|A^{\frac{1}{2}}w'(t)\right|^2+\left|A^{\frac{3}{2}}w(t)\right|^2\right], \end{aligned} \tag{3.40}$$

$t\in[0,T_0]$, where C is a generic constant which is independent of u and v.

Integrating (3.40) from 0 to $t\leq T_0$ and noting that $M(\xi)\geq m_0>0$ and $w(0)=w'(0)=0$, we have

$$\begin{aligned} \left|A^{\frac{1}{2}}w'(t)\right|^2+m_0\,|Aw(t)|^2+\left|A^{\frac{3}{2}}w(t)\right|^2 \\ \leq \frac{C}{m_0}\int_0^t m_0\,|Aw(s)|^2\,ds+C\int_0^t\left|A^{\frac{3}{2}}w(s)\right|^2 ds+C\int_0^t\left|A^{\frac{1}{2}}w'(s)\right|^2 ds, \end{aligned} \tag{3.41}$$

$t\in[0,T_0]$.

Finally, applying Gronwall's inequality in (3.41), we obtain that $|Aw(t)|=0$, for all $t\in[0,T_0]$, that is, $u(t)=v(t)$, for all $t\in[0,T_0]$. This concludes the proof of Theorem 2.1.

3.2 Proof of Theorem 2.5

Initially we consider the following problem:

$$(P_l')\quad \left|\begin{array}{l} u_l''(t)+M(\|u_l(t)\|_W^{\beta})Au_l(t)+A^2u_l(t)+\left[1+K(t)\left|A^{\frac{3}{2}}u_l(t)\right|^{2\beta}\right]Au_l'(t)=0,\quad t>0 \\ u_l(0)=u_l^0,\qquad u_l'(0)=u_l^1, \end{array}\right.$$

where $(u_l^0)_{l\in\mathbb{N}}$ and $(u_l^1)_{l\in\mathbb{N}}$ are sequences of $D(A^4)$ and $D(A^3)$, respectively.

Consequently we have

$$u_l^0\to u^0\quad \text{in } D(A^{\frac{5}{2}}) \tag{3.42}$$

and

$$u_l^1\to u^1\quad \text{in } D(A^{\frac{3}{2}}). \tag{3.43}$$

By Theorem 2.1, there exists a unique solution u_l of (P_l') belonging to class (2.7), but Proposition 3.2 with $\mu_1(t)=M(\|u_l(t)\|_W^{\beta})$ and $\mu_2(t)=1+K(t)\left|A^{\frac{3}{2}}u_l(t)\right|^{2\beta}$ says us that u_l belong to class

$$\left|\begin{array}{l} u_l\in L^{\infty}(0,T_{0l};D(A^4)) \\ u_l'\in L^{\infty}(0,T_{0l};D(A^3)) \\ u_l''\in L^{\infty}(0,T_{0l};D(A^2)). \end{array}\right. \tag{3.44}$$

Fix $l \in \mathbb{N}$. Let $\mathcal{M}_l$ be the set constituted by the real numbers $T > 0$ such that there exists a unique solution u_l of (P_l') belongs to class (3.44) (changing T_{0l} with T). By the preceding arguments it follows that $\mathcal{M}_l \neq \varnothing$. We denote by $T_{\max,l}$ the supremum of the $T \in \mathcal{M}_l$.

Next we obtain estimates for the solution u_l. Taking the scalar product of the H of both sides of equation in (P_l') with $2A^3u_l'(t)$, $t \in [0, T_{\max,l})$, and using the Proposition 3.1, we obtain

$$\begin{aligned} \frac{d}{dt}&\left[\left|A^{\frac{3}{2}}u_l'(t)\right|^2 + M(\|u_l(t)\|_W^\beta)\left|A^2u_l(t)\right|^2 + \left|A^{\frac{5}{2}}u_l(t)\right|^2\right] \\ &+ 2\left[1 + K(t)\left|A^{\frac{3}{2}}u_l(t)\right|^{2\beta}\right]\left|A^2u_l'(t)\right|^2 \\ &\leq \beta m_1 \|u_l(t)\|_W^{\beta-1}\|u_l'(t)\|_W\left|A^2u_l(t)\right|^2, \end{aligned} \tag{3.45}$$

$t \in [0, T_{\max,l})$. Here we assume that $u_l(t) \neq 0$.

Remark 3.5. We note that

$$\left|A^2u\right|^2 = (A^2u, A^2u) = (A^{\frac{3}{2}}u, A^{\frac{5}{2}}u),$$

$\forall u \in D(A^{\frac{5}{2}})$. Consequently

$$\left|A^2u\right|^2 \leq \left|A^{\frac{3}{2}}u\right|\left|A^{\frac{5}{2}}u\right|, \tag{3.46}$$

$\forall u \in D(A^{\frac{5}{2}})$.

Combining (2.11), (2.13) and (3.46) with (3.45), we have

$$\begin{aligned} \frac{d}{dt}&\left[\left|A^{\frac{3}{2}}u_l'(t)\right|^2 + M(\|u_l(t)\|_W^\beta)\left|A^2u_l(t)\right|^2 + \left|A^{\frac{5}{2}}u_l(t)\right|^2\right] \\ &+ 2\left[1 + K(t)\left|A^{\frac{3}{2}}u_l(t)\right|^{2\beta}\right]\left|A^2u_l'(t)\right|^2 \\ &\leq \beta m_1 k_2^{\beta-1}k_4\left|A^{\frac{3}{2}}u_l(t)\right|^\beta\left|A^2u_l'(t)\right|\left|A^{\frac{5}{2}}u_l(t)\right|, \end{aligned} \tag{3.47}$$

$t \in [0, T_{\max,l})$.

Due to hypothesis (2.18), we can rewrite (3.47) in the form

$$\begin{aligned} \frac{d}{dt}&\left[\left|A^{\frac{3}{2}}u_l'(t)\right|^2 + M(\|u_l(t)\|_W^\beta)\left|A^2u_l(t)\right|^2 + \left|A^{\frac{5}{2}}u_l(t)\right|^2\right] \\ &+ 2\left[1 + K(t)\left|A^{\frac{3}{2}}u_l(t)\right|^{2\beta}\right]\left|A^2u_l'(t)\right|^2 \\ &\leq 2K(t)\left|A^{\frac{3}{2}}u_l(t)\right|^{2\beta}\left|A^2u_l'(t)\right|^2 + \frac{(\beta m_1 k_2^{\beta-1}k_4)^2}{8K(t)}\left|A^{\frac{5}{2}}u_l(t)\right|^2, \end{aligned} \tag{3.48}$$

$t \in [0, T_{\max,l})$.

Remark 3.6. In (3.48) is justified the introduction of the damping term $2K(t)\left|A^{\frac{3}{2}}u(t)\right|^{2\beta}$ in equation (1.6).

Integrating (3.48) from 0 to t, $t < T_{\max,l}$, using the convergences (3.42) and (3.43), hypothesis (2.17) and u_l with regularities (3.44), we obtain

$$\begin{aligned} &\left|A^{\frac{3}{2}}u_l'(t)\right|^2 + M(\|u_l(t)\|_W^\beta)\left|A^2u_l(t)\right|^2 + \left|A^{\frac{5}{2}}u_l(t)\right|^2 + 2\int_0^t\left|A^2u_l'(s)\right|^2 ds \\ &\quad\leq C + C\int_0^t\frac{1}{K(s)}\left|A^{\frac{5}{2}}u_l(s)\right|^2 ds, \end{aligned} \tag{3.49}$$

$t \in [0, T_{\max,l})$, where $C > 0$ denotes a generic constant which is independent of l and t.

Using hypotheses (2.17) and (2.18) and the Gronwall inequality in (3.49), we have

$$\left|A^{\frac{3}{2}}u_l'(t)\right|^2 + m_0\left|A^2u_l(t)\right|^2 + \left|A^{\frac{5}{2}}u_l(t)\right|^2 + 2\int_0^t \left|A^2u_l'(s)\right|^2 ds \leq C_1, \tag{3.50}$$

$\forall t \in [0, T_{\max,l})$, where

$$C_1 = C\exp\left[C\int_0^\infty \frac{1}{K(s)}ds\right].$$

Consequently, we obtain

$$\left|A^{\frac{5}{2}}u_l(t)\right|^2 \leq C_1 \tag{3.51}$$

and

$$\left|A^{\frac{3}{2}}u_l'(t)\right|^2 \leq C_1, \tag{3.52}$$

$t \in [0, T_{\max,l})$, where C_1 is a constant independent of l and t.

Remark 3.7. Due the Proposition 3.1 is possible to obtain the inequality (3.50) even when $u_l(t) = 0$.

Now we will prove that $T_{\max,l}$ is infinite $\forall l \in \mathbb{N}$. Let us suppose that $T_{\max,l} < \infty$. Now consider a sequence of the real numbers (t_ν) such that $0 < t_\nu < T_{\max,l}$ with $t_\nu \to T_{\max,l}$. By (3.51) and (3.52) we obtain, respectively, that there exists $\xi \in D(A^{\frac{5}{2}})$ and $\eta \in D(A^{\frac{3}{2}})$ such that

$$u_l(t_\nu) \to \xi \text{ weak in } D(A^{\frac{5}{2}}) \quad \text{and} \quad u_l'(t_\nu) \to \eta \text{ weak in } D(A^{\frac{3}{2}}).$$

With ξ and η we determine, by Theorem 2.1, the local solution of the problem

$$(P^*) \qquad \left| \begin{array}{l} v''(t) + M(\|v(t)\|_W^\beta)Av(t) + A^2v(t) + \left[1 + K(t)\left|A^{\frac{3}{2}}v(t)\right|^{2\beta}\right]Av'(t) = 0 \\ v(0) = \xi, \qquad v'(0) = \eta. \end{array} \right.$$

We note that the function

$$\tilde{u}(t) = \left| \begin{array}{ll} v(t), & 0 \leq t < T_{\max,l} \\ v(t - T_{\max,l}), & T_{\max,l} \leq t < T_{\max,l} + T_0 \end{array} \right.$$

is a solution of problem (P_l') in $[0, T_{\max,l} + T_0]$. This is a contradiction with the definition of $T_{\max,l}$. So $T_{\max,l}$ is infinite. Consequently we obtain from (3.50) that

$$\left|A^{\frac{5}{2}}u_l(t)\right|^2 \leq C_1, \tag{3.53}$$

$$\left|A^{\frac{3}{2}}u_l'(t)\right|^2 \leq C_1, \tag{3.54}$$

$$\int_0^t \left|A^2u_l'(s)\right|^2 ds \leq C_1, \tag{3.55}$$

$\forall t \in [0, \infty[$, where C_1 is a constant independent of l and t.

By arguments similar to those employed in the proof of Theorem 2.1, we obtain the convergences

$$M(\|u_l\|_W^\beta) \to M(\|u\|_W^\beta) \quad \text{in } C^0([0, T_0]) \tag{3.56}$$

and

$$\left|A^{\frac{3}{2}}u_l\right|^{\beta} \to \left|A^{\frac{3}{2}}u\right|^{\beta} \quad \text{in } C^0([0,T_0]). \tag{3.57}$$

Due to the estimates (3.53)–(3.55) and convergences (3.56) and (3.57), we can pass to the limit in (P_l'). The limit u is a solution of problem $(P2)_1$ and u verifies (2.19). Using a standard argument, we can verify the initial conditions $(P2)_2$. The uniqueness of solution is obtained as is the proof of Theorem 2.1. So Theorem 2.5 is proved.

3.3 Proof of Theorem 2.6

i) Existence of solutions

Let us begin by showing that problem $(P2)$ possesses solution in the class (2.19) when we consider the hypothesis (2.20) instead of the hypothesis (2.17) and when we consider a function $K(t)$ satisfying hypothesis $K(t) = K$. As the calculations are similar, we will obtain only the estimates.

Initially we consider the following problem:

$$(P_l'')\quad \left|\begin{array}{l} u_l''(t) + M(t, \|u_l(t)\|_W^{\beta})Au_l(t) + A^2u_l(t) + \left[1 + K\left|A^{\frac{3}{2}}u_l(t)\right|^{\beta}\right] Au_l'(t) = 0, \qquad t > 0 \\ u_l(0) = u_l^0, \qquad u_l'(0) = u_l^1, \end{array}\right.$$

where $(u_l^0)_{l\in\mathbb{N}} \subset D(A^4)$ and $(u_l^1)_{l\in\mathbb{N}} \subset D(A^3)$.

Consequently we obtain the convergences (3.42) and (3.43).

With the same arguments as were used in the proof of Theorem 2.5, we have

$$\begin{aligned} &\frac{d}{dt}\left[\left|A^{\frac{3}{2}}u_l'(t)\right|^2 + M(t, \|u_l(t)\|_W^{\beta})\left|A^2u_l(t)\right|^2 + \left|A^{\frac{5}{2}}u_l(t)\right|^2\right] \\ &\qquad + 2\left[1 + K\left|A^{\frac{3}{2}}u_l(t)\right|^{\beta}\right]\left|A^2u_l'(t)\right|^2 \\ &\leq m_1'(t)\,\|u_l(t)\|_W^{\beta}\left|A^2u_l(t)\right|^2 \\ &\qquad + \beta m_1(t)\,\|u_l(t)\|_W^{\beta-1}\,\|u_l'(t)\|_W\left|A^2u_l(t)\right|^2, \end{aligned} \tag{3.58}$$

$t \in [0, T_{\max,l})$.

It results from (2.11) and of the fact that $\beta \geq 2$ the following:

$$\|u_l(t)\|_W^{\beta-1} = \|u_l(t)\|_W^{\frac{\beta}{2}}\,\|u_l(t)\|_W^{\frac{\beta}{2}-1} \leq k_2^{\frac{\beta}{2}-1}\,\|u_l(t)\|_W^{\frac{\beta}{2}}\left|A^{\frac{3}{2}}u_l(t)\right|^{\frac{\beta}{2}-1}. \tag{3.59}$$

Now let us note that

$$\left|A^2u_l(t)\right|^2 = (A^{\frac{3}{2}}u_l(t), A^{\frac{1}{2}}u_l(t)) \leq k_6\left|A^{\frac{3}{2}}u_l(t)\right|\left|A^2u_l(t)\right|, \tag{3.60}$$

where k_6 is the immersion constant of $D(A^2)$ into $D(A^{\frac{1}{2}})$ (see Remark 2.3).

Substituting (3.59) and (3.60) into (3.58), we get

$$\begin{aligned}
&\frac{d}{dt}\left[\left|A^{\frac{3}{2}}u_l'(t)\right|^2 + M(t, \|u_l(t)\|_W^\beta)\left|A^2u_l(t)\right|^2 + \left|A^{\frac{5}{2}}u_l(t)\right|^2\right] \\
&\qquad + 2\left[1 + K\left|A^{\frac{3}{2}}u_l(t)\right|^\beta\right]\left|A^2u_l'(t)\right|^2 \\
&\qquad \le m_1'(t)\,\|u_l(t)\|_W^\beta\left|A^2u_l(t)\right|^2 + 2K\left|A^{\frac{3}{2}}u_l(t)\right|^\beta\left|A^2u_l'(t)\right|^2 \\
&\qquad + \frac{1}{8K}k_2^{(\frac{\beta}{2}-1)^2}k_4^2k_6^2\beta^2(m_1(t))^2\,\|u_l(t)\|_W^\beta\left|A^2u_l(t)\right|^2,
\end{aligned}$$

that is,

$$\begin{aligned}
&\frac{d}{dt}\left[\left|A^{\frac{3}{2}}u_l'(t)\right|^2 + M(t, \|u_l(t)\|_W^\beta)\left|A^2u_l(t)\right|^2 + \left|A^{\frac{5}{2}}u_l(t)\right|^2\right] + 2\left|A^2u_l'(t)\right|^2 \\
&\quad \le m_1'(t)\,\|u_l(t)\|_W^\beta\left|A^2u_l(t)\right|^2 + \frac{1}{8K}k_2^{(\frac{\beta}{2}-1)^2}k_4^2k_6^2\beta^2(m_1(t))^2\,\|u_l(t)\|_W^\beta\left|A^2u_l(t)\right|^2.
\end{aligned} \tag{3.61}$$

Integrating (3.61) from 0 to t, $t < T_{\max,l}$, using the hypothesis (2.20) and convergences (3.42) and (3.43), we derive

$$\begin{aligned}
&\left|A^{\frac{3}{2}}u_l'(t)\right|^2 + m_0\left|A^2u_l(t)\right|^2 + m_1(t)\,\|u_l(t)\|_W^\beta\left|A^2u_l(t)\right|^2 + \left|A^{\frac{5}{2}}u_l(t)\right|^2 \\
&\quad + 2\int_0^t\left|A^2u_l'(s)\right|^2 ds + \int_0^t(-m_1'(s))\,\|u_l(s)\|_W^\beta\left|A^2u_l(s)\right|^2 ds \\
&\quad \le C + \frac{1}{8K}k_2^{(\frac{\beta}{2}-1)^2}k_4^2k_6^2\beta^2\int_0^t m_1(s)\left[m_1(s)\,\|u_l(s)\|_W^\beta\left|A^2u_l(s)\right|^2\right]ds,
\end{aligned} \tag{3.62}$$

where $C > 0$ is a generic constant independent of l and t.

As $m_1' \le 0$ and $m_1 \in L^1(0,\infty)$ (see (2.20)), it results from (3.62) and Gronwall's inequality

$$\left|A^{\frac{3}{2}}u_l'(t)\right|^2 + m_0\left|A^2u_l(t)\right|^2 + \left|A^{\frac{5}{2}}u_l(t)\right|^2 + 2\int_0^t\left|A^2u_l'(s)\right|^2 ds \le C, \tag{3.63}$$

$t \in [0, T_{\max,l})$, where $C > 0$ is a generic constant independent of l and t. Note that with (3.63) we derive similar estimates to (3.51) and (3.52) for u_l.

With (3.63) and similar arguments used in the proof of Theorem 2.5 we obtain that $T_{\max,l}$ is infinite and that u is the solution of $(P2)$ in the class (2.19).

ii) Decay of solutions

Take the scalar product of the H of both sides of equation in $(P2)$ with $2Au'(t)$ and use (2.21), Proposition 3.1 and hypothesis (2.20). We obtain

$$\begin{aligned}
&\frac{d}{dt}E(t) + 2\left[1 + K\left|A^{\frac{3}{2}}u(t)\right|^\beta\right]\left|Au'(t)\right|^2 \\
&\quad \le -\frac{z'(t)}{[z(t)]^2}\,\|u(t)\|_W^\beta\left|Au(t)\right|^2 + \beta\frac{1}{z(t)}\,\|u(t)\|_W^{\beta-1}\left\|u'(t)\right\|_W\left|Au(t)\right|^2,
\end{aligned} \tag{3.64}$$

where $E(t)$ was defined in (2.21). Here $u(t) \neq 0$.

It follows from (2.9), (2.20) and the inequality $ab \leq \frac{C_0 a^2}{2} + \frac{b^2}{2C_0}$ ($a, b \geq 0$ and $C_0 > 0$) that

$$\begin{aligned}
&-\frac{z'(t)}{[z(t)]^2} \|u(t)\|_W^\beta |Au(t)|^2 + \beta \frac{1}{z(t)} \|u(t)\|_W^{\beta-1} \|u'(t)\|_W |Au(t)|^2 \\
&\leq -\frac{C_0}{[z(t)]^2} \|u(t)\|_W^\beta |Au(t)|^2 + \beta (k_1 k_7)^{\frac{\beta}{2}} \frac{1}{z(t)} \|u(t)\|_W^{\frac{\beta}{2}} |Au'(t)| \left| A^{\frac{3}{2}} u(t) \right|^{\frac{\beta}{2}} |Au(t)| \\
&\leq -\frac{C_0}{[z(t)]^2} \|u(t)\|_W^\beta |Au(t)|^2 + \frac{C_0}{2[z(t)]^2} \|u(t)\|_W^\beta |Au(t)|^2 \\
&\quad + \frac{\beta^2 (k_1 k_7)^\beta}{2C_0} \left| A^{\frac{3}{2}} u(t) \right|^\beta |Au'(t)|^2 \\
&\leq \frac{\beta^2 (k_1 k_7)^\beta}{2C_0} \left| A^{\frac{3}{2}} u(t) \right|^\beta |Au'(t)|^2,
\end{aligned} \tag{3.65}$$

where k_7 is the immersion constant of $D(A^{\frac{3}{2}})$ into $D(A)$ (see Remark 2.3).

Combining (3.64) and (3.65) and using the hypothesis $K \geq \frac{\beta^2 (k_1 k_7)^\beta}{2C_0}$, we obtain

$$\frac{d}{dt} E(t) + 2 |Au'(t)|^2 \leq 0. \tag{3.66}$$

Remark 3.8. Due to Proposition 3.1 it is possible to obtain the inequality (3.66) even when $u(t) = 0$.

Let $\varepsilon > 0$. Consider the functions

$$\rho(t) = (u'(t), Au(t)) + \frac{1}{2} |Au(t)|^2, \qquad \forall t \geq 0 \tag{3.67}$$

and

$$E_\varepsilon(t) = E(t) + \varepsilon \rho(t), \qquad \forall t \geq 0, \tag{3.68}$$

where $E(t)$ was defined in (2.21).

Therefore, it follows from (2.8) and (3.67) that

$$|\rho(t)| \leq \frac{1}{2} k_0^2 \|u'(t)\|^2 + \frac{k_7^2}{2} \left| A^{\frac{3}{2}} u(t) \right|^2 + \frac{1}{2} |Au(t)|^2 \leq P_0 E(t), \qquad \forall t \geq 0, \tag{3.69}$$

where $P_0 = \max\left\{\frac{1}{2} k_0^2, k_7^2\right\}$.

Combining (3.68) and (3.69) we have

$$(1 - \varepsilon P_0) E(t) \leq E_\varepsilon(t) \leq (1 + \varepsilon P_0) E(t), \qquad \forall \varepsilon > 0.$$

Considering $0 < \varepsilon < \frac{1}{2P_0}$, it follows from the preceding inequality that

$$\frac{1}{2} E(t) \leq E_\varepsilon(t) \leq \frac{3}{2} E(t), \qquad \forall t \geq 0. \tag{3.70}$$

Now taking the scalar product of the H of both sides of equation in $(P2)$ with $Au(t)$, we have:

$$\begin{aligned}
&\frac{d}{dt}(u'(t), Au(t)) + \frac{d}{dt}\left[\frac{1}{2} |Au(t)|^2\right] + M(t, \|u(t)\|_W^\beta)(Au(t), Au(t)) \\
&\quad + (A^2 u(t), Au(t)) + K \left| A^{\frac{3}{2}} u(t) \right|^\beta (Au'(t), Au(t)) \\
&= \|u'(t)\|^2.
\end{aligned} \tag{3.71}$$

Therefore, it results from (3.67) and (3.71) that

$$\frac{d}{dt}(\rho(t)) + M(t, \|u(t)\|_W^\beta) |Au(t)|^2 + \left|A^{\frac{3}{2}}u(t)\right|^2 + K\left|A^{\frac{3}{2}}u(t)\right|^\beta (A^{\frac{1}{2}}u'(t), A^{\frac{3}{2}}u(t)) \le \|u'(t)\|^2,$$

which implies,

$$\rho'(t) \le \|u'(t)\|^2 - M(t, \|u(t)\|_W^\beta) |Au(t)|^2 - \left|A^{\frac{3}{2}}u(t)\right|^2 + KC_2 \|u'(t)\| \left|A^{\frac{3}{2}}u(t)\right|,$$

where $C_2 > 0$ is a bound for $\left|A^{\frac{3}{2}}u(t)\right|^\beta$ (see (3.63) and that $\left|A^{\frac{3}{2}}u(t)\right| \le \left|A^2u(t)\right|$).

Consequently, we obtain by this last inequality that

$$\rho'(t) \le \left(1 + \frac{K^2C_2^2}{2}\right) \|u'(t)\|^2 - M(t, \|u(t)\|_W^\beta) |Au(t)|^2 - \frac{1}{2}\left|A^{\frac{3}{2}}u(t)\right|^2. \tag{3.72}$$

Combining (3.66), (3.68) and (3.72), we obtain

$$\begin{aligned} E_\varepsilon'(t) \le\; & -2\left|Au'(t)\right|^2 - \varepsilon M(t, \|u(t)\|_W^\beta) |Au(t)|^2 \\ & + \varepsilon\left(1 + \frac{K^2C_2^2}{2}\right) \|u'(t)\|^2 - \frac{\varepsilon}{2}\left|A^{\frac{3}{2}}u(t)\right|^2. \end{aligned} \tag{3.73}$$

Noting that $\|u'(t)\|^2 \le k_8^2 |Au'(t)|^2$, where k_8 is the immersion constant of $D(A)$ into $D(A^{\frac{1}{2}})$, it results from (3.73) that

$$E_\varepsilon'(t) \le -\left[\frac{2}{k_8^2} - \varepsilon\left(1 + \frac{K^2C_2^2}{2}\right)\right] \|u'(t)\|^2 - \varepsilon M(t, \|u(t)\|_W^\beta) |Au(t)|^2 - \frac{\varepsilon}{2}\left|A^{\frac{3}{2}}u(t)\right|^2.$$

Considering $\varepsilon = \min\left\{\frac{1}{2P_0}, \frac{4}{k_8^2(2+K^2C_2^2)}\right\}$, it follows from the preceding inequality that

$$E_\varepsilon'(t) \le -\tau_0 E(t), \qquad \forall t \ge 0, \tag{3.74}$$

where $\tau_0 = \min\left\{\frac{\varepsilon}{2}, \delta\right\}$, with $\delta = \frac{2}{k_8^2} - \varepsilon(1 + \frac{K^2C_2^2}{2})$.

Using (3.70) and (3.74), we have

$$E_\varepsilon'(t) \le -\frac{2}{3}\tau_0 E_\varepsilon(t), \qquad \forall t \ge 0,$$

which gives

$$E_\varepsilon(t) \le E_\varepsilon(0) \exp\left(-\frac{2}{3}\tau_0 t\right), \qquad \forall t \ge 0. \tag{3.75}$$

Finally we get (2.22) as a consequence of (3.70) and (3.75). So Theorem 2.6 is proved.

Remark 3.9. We observe that Theorems 2.1–2.6 are true if, instead of the hypothesis (2.3), we consider the hypothesis

$$V \text{ is continuously embedded in } W.$$

Remark 3.10. With the same technique and hypotheses as were used in the solution of problem $(P2)$, it is possible to solve

$$(P') \quad \left| \begin{aligned} & u''(t) + M_1(\|u(t)\|^2)Au(t) + M_2(\|u(t)\|_W^\beta)Au(t) + A^2u(t) \\ & \qquad + \sigma\left[1 + K(t)\left(\left|A^{\frac{3}{2}}u(t)\right|^4 + \left|A^{\frac{3}{2}}u(t)\right|^\beta\right)\right] Au'(t) = 0, \qquad t > 0 \\ & u(0) = u^0, \qquad u'(0) = u^1, \end{aligned} \right.$$

where $M_1(\xi)$ and $M_2(\xi)$ are similar to $M(\xi)$ of problem $(P2)$ and $\sigma \ge 0$ is a real number. Also with our techniques it is possible to solve problem (P') replacing $M_1(\|u(t)\|^2)$ by $M_1(|u(t)|^2)$, where $\|\cdot\|$ and $|\cdot|$ denote the norms of V and H, respectively. Remark 3.9 also remains valid in these cases.

4 Examples

1^o) In the deduction of the beam equation $(1+u_x^2)^{\frac{1}{2}}$ is approximated by $(1+\frac{1}{2}u_x)^2$. Utilizing Taylor's formula is possible to approximate $(1+u_x^2)^{\frac{1}{2}}$ by $1+\frac{3}{8}u_x^2+\frac{3}{128}u_x^6$. Motivated by this latter approach, we may consider the generalized equation

$$u_{tt}(x,t)+\left[m_0+m_1\int_0^L(u_x(x,t))^2dx+m_2\int_0^L(u_x(x,t))^6dx\right](-u_{xx}(x,t))+u_{xxxx}(x,t)=0, \quad (4.1)$$

where m_0, m_1, $m_2>0$ are constants.

In a similar manner, we have:

$$u_{tt}(x,t)+\left[m_0+m_1\int_0^L(u(x,t))^2dx+m_2\int_0^L(u(x,t))^6dx\right](-u_{xx}(x,t))+u_{xxxx}(x,t)=0, \quad (4.2)$$

where m_0, m_1, $m_2>0$ are constants.

Considering Remark 3.10 with $V=H_0^1(0,L)$, $H=L^2(0,L)$, $A=-\frac{\partial^2}{\partial x^2}$ the operator defined by the triplet $\{H_0^1(0,L),L^2(0,L);((u,v))_{H_0^1(0,L)}\}$, $W=W_0^{1,6}(0,L)$, $\beta=6$ and $\sigma=0$, we obtain a local solution of the mixed problem for equation (4.1). In this case we have the following embeddings

$$D(A)=H_0^1(0,L)\cap H^2(0,L)\hookrightarrow W=W_0^{1,6}(0,L)\hookrightarrow V=H_0^1(0,L),$$

where $\hookrightarrow$ denotes continuous immersion.

Analogously, Remark 3.10 with $V=H_0^1(0,L)$, $H=L^2(0,L)$, $A=-\frac{\partial^2}{\partial x^2}$ the operator defined by the triplet $\{H_0^1(0,L),L^2(0,L);((u,v))_{H_0^1(0,L)}\}$, $W=L^6(0,L)$, $\beta=6$ and $\sigma=0$, we obtain a local solution of the mixed problem for equation (4.2). In this case we have the following embeddings

$$D(A)=H_0^1(0,L)\cap H^2(0,L)\hookrightarrow V=H_0^1(0,L)\hookrightarrow W=L^6(0,L).$$

2^o) Let $\Omega\subset\mathbb{R}^n$ be a bounded open of $\mathbb{R}^n$ with smooth boundary Γ. If $\beta>1$, then the Sobolev embedding theorem allows to obtain the existence of solution to the following problems.

$$(E1)\qquad \left|\begin{array}{l} u''+M\left(\|u\|^{\beta}_{W_0^{1,p}(\Omega)}\right)(-\Delta u)+\Delta^2u=0,\ \text{in }\Omega\times]0,T_0[\\ u=0\text{ in }\Gamma\times]0,T_0[\\ u(x,0)=u^0(x),\qquad u'(x,0)=u^1(x)\text{ in }\Omega,\end{array}\right.$$

where $1<p<\infty$ if $1\le n\le[4\alpha+2]$ and $1<p\le\frac{2n}{n-[4\alpha+2]}$ if $n>[4\alpha+2]$. Here $[\gamma]$ denotes the integer part of the real number γ.

$$(E2)\qquad \left|\begin{array}{l} u''+M\left(\|u\|^{\beta}_{L^p(\Omega)}\right)(-\Delta u)+\Delta^2u=0,\ \text{in }\Omega\times]0,T_0[\\ u=0\text{ in }\Gamma\times]0,T_0[\\ u(x,0)=u^0(x),\qquad u'(x,0)=u^1(x)\text{ in }\Omega,\end{array}\right.$$

where n and p are in the same conditions of the problem $(E1)$.

$$(E3)\qquad \left|\begin{array}{l} u''+M\left(\left|(-\Delta)^{\theta}u\right|\right)(-\Delta u)+\Delta^2u=0,\ \text{in }\Omega\times]0,T_0[\\ u=0\text{ in }\Gamma\times]0,T_0[\\ u(x,0)=u^0(x),\qquad u'(x,0)=u^1(x)\text{ in }\Omega,\end{array}\right.$$

where $0 \leq \theta \leq \alpha + 1$.

$$(E4) \qquad \begin{vmatrix} u'' + M\left(\left|(-\Delta + I)^{\theta} u\right|\right)(-\Delta u) + \Delta^2 u = 0, \text{ in } \Omega \times]0, T_0[\\ \dfrac{\partial u}{\partial \eta} = 0 \text{ in } \Gamma \times]0, T_0[\\ u(x,0) = u^0(x), \qquad u'(x,0) = u^1(x) \text{ in } \Omega, \end{vmatrix}$$

where $\eta(x)$ is the exterior normal to x in Γ and $0 \leq \theta \leq \alpha + 1$.

3^o) As A^{-1} is not necessarily compact, we can consider problems defined on $\Omega \times]0, \infty[$ with Ω an unbounded smooth open set of $\mathbb{R}^n$.

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