Electronic Journal of Qualitative Theory of Differential Equations
2010, No. **48**, 1-19; **http://www.math.u-szeged.hu/ejqtde/**

OSCILLATORY BEHAVIOUR OF A CLASS OF NONLINEAR SECOND ORDER MIXED DIFFERENCE EQUATIONS

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Abstract

In this paper oscillatory and asymptotic behaviour of solutions of a class of nonlinear second order neutral difference equations with positive and negative coefficients of the form

$$\Delta(r(n)\Delta(y(n)+p(n)y(n-m)))+f(n)H_1(y(n-k_1))-g(n)H_2(y(n-k_2)) = q(n) \tag{E}$$

and

$$\Delta(r(n)\Delta(y(n)+p(n)y(n-m)))+f(n)H_1(y(n-k_1))-g(n)H_2(y(n-k_2)) = 0$$

are studied under the assumptions

$$\sum_{n=0}^{\infty}\frac{1}{r(n)} < \infty$$

and

$$\sum_{n=0}^{\infty}\frac{1}{r(n)} = \infty$$

for various ranges of $p(n)$. Using discrete Krasnoselskii's fixed point theorem sufficient conditions are obtained for existence of positive bounded solutions of (E).

Mathematics Subject classification (2000): 39 A 10, 39 A 12
Keywords: Oscillation, Nonoscillation, Neutral Difference Equations, Existence of Positive Solutions.

1 Introduction

In recent years there has been extensive research activity concerning the oscillation and nonoscillation of solutions of differential and difference equations. In particular, much attention has been given to nonlinear neutral delay equations with positive and negative coefficients for existence of positive bounded solutions. We refer the papers [2, 3, 6, 7, 4, 13, 14] for comprehensive treatment of this theory. But very little work [11, 12] is available on the study of oscillatory and asymptotic behaviour of solutions of such equations which is due to the technical difficulties arising in its analysis.

The object of this work is to study the oscillation properties of a class of non-linear neutral difference equations with positive and negative coefficients of the form

$$\Delta(r(n)\Delta(y(n)+p(n)y(n-m)))+f(n)H_1(y(n-k_1))-g(n)H_2(y(n-k_2))=q(n), \quad (1)$$

where Δ is the forward difference operator defined by $\Delta y(n)=y(n+1)-y(n),\ r,\ p,\ f,\ g$ and q are real valued functions such that $r(n)>0, f(n)>0$ and $g(n)\geq 0$ for all $n, m>0, k_i\geq 0$ are integers and $H_i\in C(R,\ R)$ is a nondecreasing function such that $xH_i(x)>0,\ x\neq 0$ for $i=1,2$ under the assumptions

(A_0) $\quad \sum\limits_{s=0}^{\infty}\frac{1}{r(s)}\sum\limits_{t=s}^{\infty}g(t)<\infty$

(A_1) $\quad \sum\limits_{n=0}^{\infty}\frac{1}{r(n)}<\infty$

and

(A_2) $\quad \sum\limits_{n=0}^{\infty}\frac{1}{r(n)}=\infty$

The corresponding unforced equation

$$\Delta(r(n)\Delta(y(n)+p(n)y(n-m)))+f(n)H_1(y(n-k_1))-g(n)H_2(y(n-k_2))=0 \quad (2)$$

is also studied under the assumptions (A_1) and (A_2) for various ranges of p(n).

This work is motivated by the recent paper [5], where the authors Li et al. have studied the existence of nonoscillatory solution of

$$\Delta^m(y(n)+p(n)y(\tau(n)))+f_1(n,\ y(\sigma_1(n)))-f_2(n,\ y(\sigma_2(n)))=g(n) \quad (3)$$

and its associated unforced equation

$$\Delta^m(y(n)+p(n)y(\tau(n)))+f_1(n,\ y(\sigma_1(n)))-f_2(n,\ y(\sigma_2(n)))=0 \quad (4)$$

under various ranges of $p(n)$. If $r(n)\equiv 1$ and $m=2$, then Eqns. (1) and (2) are particular cases of Eqns. (3) and (4) respectively. However, for $m=2$, Eqns. (1) and (2) can not be treated as the particular cases of (3) and (4) in view of (A_1) and (A_2). Hence study of

(1) and (2) is very much interesting. Necessary and sufficient conditions for oscillation of (1)/(2) are investigated in this paper.

By a solution of Eqn.(1) (see for e.g [9], [10]) we mean a real valued function y(n) defined on $N(-\rho) = \{-\rho, -\rho+1, \cdots\}$ which satisfies (1) for $n \geq 0$, where $\rho = max\{m,\ k_1,\ k_2\}$. If

$$y(n) = A_n,\ n = -\rho, -\rho+1, \cdots 0, \tag{5}$$

are given, then (1) admits a unique solution satisfying the initial condition (5). Recall that a solution y(n) of (1) is oscillatory, if for any given integer $N > 0$, there exists an $n \geq N$ such that $y(n)y(n+1) \leq 0$ for $n \geq N$; otherwise it is called nonoscillatory.

2 Preliminary Results

This section deals with the results which play an important role in establishing the present work.

Lemma 2.1 Assume that (A_1) hold. Let $u(n)$ be an eventually positive real valued function such that $\Delta(r(n)\Delta u(n) \leq 0$ but $\not\equiv 0$ for all large n and $r(n) > 0$. Then the following hold :

i) if $\Delta u(n) > 0$, then there exists a constant $C > 0$ such that $u(n) \geq CR(n)$, for all large n;

ii) if $\Delta u(n) < 0$, then $u(n) \geq -r(n)\Delta u(n)\ R(n)$, where $R(n) = \sum\limits_{s=n}^{\infty} \frac{1}{r(s)}$.

Proof (i) Since $R(0) < \infty$, $R(n) \to 0$ as $n \to \infty$ and $u(n)$ is nondecreasing, we can find a constant $C > 0$ such that $u(n) \geq C\ R(n)$ for all large n.

(ii) For $s \geq n+1 > n$, $r(s)\Delta u(s) \leq r(n)\Delta u(n)$ and hence

$$\sum_{t=n}^{s-1} \Delta u(t) < r(n)\Delta u(n) \sum_{t=n}^{s-1} \frac{1}{r(t)}$$

implies that

$$u(s) < u(n) + r(n)\Delta u(n) \sum_{t=n}^{s-1} \frac{1}{r(t)}.$$

Thus $0 < u(s) < u(n) + r(n)\ \Delta u(n) \sum\limits_{t=n}^{s-1} \frac{1}{r(t)}$ implies that $u(n) \geq -r(n)\ \Delta u(n)\ R(n)$ for all large n.

Lemma 2.2 Assume that (A_2) hold. Let $u(n)$ and $\Delta u(n)$ be eventually positive real valued functions for $n \geq M+1 > M \geq 0$. Then $u(n) \geq (n-M-1)\Delta u(n) = \beta(n)\Delta u(n)$ for $n \geq M+1$, where $\beta(n) = \frac{(n-M-1)}{r(n)}$ and $M > 0$ is an integer.

Lemma 2.3 [15, 8] Let $p,\ y,\ z$ be real valued functions such that $z(n) = y(n) + p(n)y(n-m),\ n \geq m \geq 0,\ y(n) > 0$ for $n \geq n_1 > m, \liminf_{n\to\infty} y(n) = 0$ and $\lim_{n\to\infty} z(n) = L$ exists. Let $p(n)$ satisfy one of the following conditions

(i)	$0 \leq p(n) \leq b_1 < 1$
(ii)	$1 < b_2 \leq p(n) \leq b_3,$
(iii)	$b_4 \leq p(n) \leq 0,$

where b_i is a constant, $1 \leq i \leq 4$. Then $L = 0$.

Lemma 2.4 [1] Let K be a closed bounded and convex subset of ℓ^∞, the Banach space consisting of all bounded real sequences. Suppose Γ is a continuous map such that $\Gamma(K) \subset K$ and suppose further that $\Gamma(K)$ is uniformly Cauchy. Then Γ has a fixed point in K.

3 Oscillation Results

This section deals with the sufficient conditions for oscillation of solutions of Eq.(1) and Eq.(2) under the assumptions (A_0), (A_1) and (A_2). We need the following conditions for our use in the sequel.

(A_3) For $u > 0$and $v > 0$, there exists $\lambda > 0$ such that $H_1(u) + H_1(v) \geq \lambda\ H_1(u+v)$,

(A_4) $H_1(uv) = H_1(u)H_1(v),\ H_2(uv) = H_2(u)H_2(v)$ for $u,\ v \in\ R$.

(A_5) $F(n) = min\{f(n),\ \ f(n-m)\},\ n \geq m.$

(A_6) $H_1(x)$ is sublinear and $\int_0^{\pm C} \frac{dx}{H_1(x)} < \infty.$

(A_7) There exists a real valued function $Q(n)$ such that $Q(n)$ changes sign with $-\infty < \liminf_{n\to\infty} Q(n) < 0 < \limsup_{n\to\infty} Q(n) < \infty$ and $\Delta(r(n)\Delta Q(n)) = q(n)$.

(A_8) $Q^+(n) = max\{Q(n),\ 0\}$ and $Q^-(n) = max\{-Q(n),\ 0\}$.

Remark The prototype of H_1 satisfying (A_3) and (A_4) is

$$H_1(u) = \left(a + b\ |u|^\lambda\right)\ |u|^\mu\ sgn\ u,$$

where $a \geq 0,\ b > 0,\ \lambda \geq 0$ and $\mu \geq 0$ such that $a + b = 1$.

Remark Indeed, $H_1(1)\ H_1(1) = H_1(1)$ due to (A$_4$). Hence $H_1(1) = 1$. Further, $H_1(-1)\ H_1(-1) = H_1(1) = 1$ gives to $(H_1(-1))^2 = 1$. Because $H_1(-1) < 0$, then $H_1(-1) = -1$. Consequently, $H_1(-u) = H_1(-1)\ H_1(u) = -H_1(u)$ for every $u \in R$.

Remark If $y(n)$ is a solution of (1)/(2), then $x(n) = -y(n)$ is also a solution of (1)/(2) provided that H_1 satisfies (A$_4$).

Theorem 3.1 Let $0 \leq p(n) \leq p < \infty$. If (A$_0$), (A$_1$), (A$_3$), (A$_4$), (A$_5$), (A$_7$), (A$_8$), and

(A$_9$) $\qquad \sum\limits_{n=k_1}^{\infty} F(n)\ H_1(R(n-k_1)) = \infty$

hold, then (1) is oscillatory.

Proof Suppose on the contrary that $y(n)$ is a nonoscillatory solution of (1) such that $y(n) > 0$ for $n \geq n_0$. Setting

$$K(n) = \sum_{s=n}^{\infty} \frac{1}{r(s)} \sum_{t=s}^{\infty} g(t) H_2(y(t-k_2))$$

$$w(n) = y(n) + p(n)\ y(n-m) - K(n) = z(n) - K(n), \tag{6}$$

let

$$U(n) = w(n) - Q(n) \tag{7}$$

for $n \geq n_0$. We note that $K(n) > 0,\ \Delta K(n) < 0$ and $\lim\limits_{n\to\infty} K(n) = 0$. Using (6) and (7), Eq.(1) becomes

$$\Delta(r(n)\Delta U(n)) = -f(n)\ H_1(y(n-k_1)) \leq 0, \not\equiv 0 \tag{8}$$

for $n \geq n_1 > n_0 + \rho$. Accordingly, $\Delta U(n)$ and $U(n)$ are monotonic functions. Assume that $\Delta U(n) < 0$ for $n \geq n_1$. If $U(n) < 0$ for $n \geq n_2 > n_1$, then $z(n) < K(n) + Q(n)$ and hence

$$\begin{aligned}
\liminf_{n\to\infty} z(n) &\leq \liminf_{n\to\infty}(k(n) + Q(n)) \\
&\leq \limsup_{n\to\infty} k(n) + \liminf_{n\to\infty} Q(n) \\
&= \lim_{n\to\infty} k(n) + \liminf_{n\to\infty} Q(n) \\
&< 0,
\end{aligned}$$

a contradiction to the fact that $z(n) > 0$. Thus $U(n) > 0$ for $n \geq n_2$. Using Lemma 2.1 (ii) with $u(n)$ is replaced by $U(n)$, we get $U(n) \geq -r(n)(\Delta U(n))R(n)$ and hence

$$\begin{aligned}
z(n) &\geq -r(n)R(n)\Delta U(n) + K(n) + Q^{+}(n) \\
&> -r(n)R(n)\Delta U(n)
\end{aligned}$$

for $n \geq n_2$. Further, $r(n)\Delta U(n)$ is non-increasing. So we can find a constant $-\alpha > 0$ and $n_3 > n_2$ such that $-r(n)\Delta U(n) \geq -\alpha$ for $n \geq n_3$. Using Eq.(1) and (7) we obtain

$$\begin{aligned} 0 = \Delta(r(n)\Delta U(n)) &+ H_1(p)\Delta(r(n-m)\Delta U(n-m)) \\ &+ f(n)H_1(y(n-k_1)) + H_1(p)f(n-m)H_1(y(n-m-k_1)), \end{aligned}$$

that is,

$$\begin{aligned} 0 \geq \Delta(r(n)\Delta U(n)) &+ H_1(p)\Delta(r(n-m)\Delta U(n-m)) \\ &+ \lambda F(n)H_1(z(n-k_1)) \end{aligned} \tag{9}$$

due to (A_3), (A_4) and (A_5). Hence (9) becomes

$$\lambda H_1(-\alpha)F(n)H_1(R(n-k_1)) \leq -\Delta(r(n)\Delta U(n)) - H_1(p)\Delta(r(n-m)\Delta U(n-m)),$$

for $n \geq n_4 > n_3 + k_1$. Since $\lim\limits_{n\to\infty} U(n)$ exists, we claim that $y(n)$ is bounded. If not, there exists $\{n^j\} \subset \{n\}$ such that $U(n) = z(n) - K(n) - Q(n) > y(n) - K(n) - Q(n)$ implies that $U(n^j) - K(n^j) - Q(n^j) \to \infty$ as $j \to \infty$ and $n^j \to \infty$, a contradiction. Consequently, $\lim\limits_{n\to\infty}(r(n)\Delta U(n))$ exists. Summing the last inequality from n_4 to ∞, we obtain

$$\sum_{n=n_4}^{\infty} F(n)H_1(R(n-k_1)) < \infty,$$

a contradiction to our hypothesis (A_9).

Let $\Delta U(n) > 0$ for $n \geq n_1$. The argument for the case $U(n) < 0$ is the same. Consider, $U(n) > 0$ for $n \geq n_2 > n_1$. By Lemma 2.1(i), it follows that $U(n) \geq C\ R(n)$, that is,

$$z(n) \geq C\ R(n) + k(n) + Q^+(n) > C\ R(n),$$

for $n \geq n_2$. Using (9) and proceeding as above we get a contradiction to our hypothesis (A_9).

If $y(n) < 0$ for $n \geq n_0$, then we set $x(n) = -y(n)$ to obtain $x(n) > 0$ for $n \geq n_0$ and

$$\Delta(r(n)\Delta(x(n) + p(n)x(n-m))) + f(n)H_1(x(n-k_1)) - g(n)H_2(x(n-k_2)) = \tilde{q}(n),$$

where $\tilde{q}(n) = -q(n)$. If $\tilde{Q}(n) = -Q(n)$, then $\Delta(r(n)\Delta\tilde{Q}(n)) = -q(n) = \tilde{q}(n)$ and $\tilde{Q}(n)$ changes sign. Further, $\tilde{Q}^+(n) = Q^-(n)$ and $\tilde{Q}^-(n) = Q^+(n)$. Proceeding as above we obtain a contradiction. Thus the proof of the theorem is complete.

Theorem 3.2 Let $-1 < p \le p(n) \le 0$. If (A_0), (A_1), (A_4), (A_7). (A_8) and

(A_{10}) $\qquad \sum_{n=0}^{\infty} f(n)H_1(R(n-k_1)) = \infty$

hold, then every solution of (1) oscillates.

Proof Let $y(n)$ be a nonoscillatory solution of (1) such that $y(n) > 0$ for $n \ge n_0$. The case $y(n) < 0$ for $n \ge n_0$ is similar. Setting as in (6) and (7), we get (8). Hence $\Delta U(n)$ is a monotonic function on $[n_1, \infty), n_1 > n_0 + \rho$. Let $\Delta U(n) < 0$ for $n \ge n_1$. Accordingly, $U(n)$ is a monotonic function and $\lim_{n\to\infty} U(n) = \lim_{n\to\infty}(z(n) - Q(n))$ implies that $z(n) - Q(n) < 0$ when $U(n) < 0$, that is, $z(n) < Q(n)$ for $n \ge n_2 > n_1$. If $z(n) > 0$, then $0 < z(n) < Q(n)$, which is absurd. Hence $z(n) < 0$ for $n \ge n_2$, that is, $y(n) < y(n-m), n \ge n_2$ implies that $y(n)$ is bounded on $[n_3, \infty), n_3 > n_2 + m$. Consequently, $U(n)$ is bounded and $\lim_{n\to\infty}(r(n)\Delta U(n))$ exists. Using the fact that $py(n-m) < z(n) < -Q^-(n)$ and $Q(n)$ is bounded, we may conclude that $\liminf_{n\to\infty} y(n) \neq 0$. On the other hand when (A_{10}) holds and since $R(n) \to 0$ as $n \to \infty$, then it follows that $\liminf_{n\to\infty} y(n) = 0$, a contradiction. Thus $U(n) > 0$ for $n \ge n_2 > n_1$. Using Lemma 2.1(ii), we have $U(n) \ge -r(n)R(n)\Delta U(n)$ and hence for $n \ge n_2$

$$z(n) \ge -r(n)R(n)\Delta U(n) + K(n) + Q^+(n),$$

that is,

$$\begin{aligned} y(n) &\ge -r(n)R(n)\Delta U(n) + K(n) + Q^+(n) \\ &> -r(n)R(n)\Delta U(n). \end{aligned}$$

Further, $r(n)\Delta U(n)$ is non-increasing. So we can find a constant $C_1 > 0$ and $n_3 > n_2$ such that $-r(n)\Delta U(n) \ge C_1$ for $n \ge n_3$. Hence $y(n) > -C_1 R(n)$ for $n \ge n_3$. Summing (8) from n_4 to ∞, we get

$$H_1(-C_1)\sum_{n=n_4}^{\infty} f(n)H_1(R(n-k_1)) < -\sum_{n=n_4}^{\infty} \Delta(r(n)\Delta U(n)),$$

$n_4 > n_3 + k_1$. Since $\lim_{n\to\infty} U(n)$ exists, then it follows from Theorem 3.1 that $y(n)$ is bounded. Consequently, $\lim_{n\to\infty}(r(n)\Delta U(n))$ exists and the last inequality becomes

$$\sum_{n=n_4}^{\infty} f(n)H_1(R(n-k_1)) < \infty,$$

a contradiction to (A_{10}).

Let $\Delta U(n) > 0$ for $n \geq n_1$. Then $\lim_{n\to\infty}(r(n)\Delta U(n))$ exists. Similar contradictions can be obtained for $U(n) > 0$ and $U(n) < 0$ for $n \geq n_2 > n_1$. The case $y(n) < 0$ for $n \geq n_0$ is similar. Hence the proof of the theorem is complete.

Theorem 3.3 Let $-\infty < p \leq p(n) \leq -1$. If all the conditions of Theorem 3.2 are satisfied, then every bounded solution of (1) oscillates.

Proof The proof follows from Theorem 3.2.

Theorem 3.4 If $0 \leq p(n) \leq p < \infty$ and (A_0), (A_2) - (A_5), (A_7), (A_8) and

(A_{11}) $$\sum_{n=0}^{\infty} F(n) = \infty$$

hold, then (1) is oscillatory.

Proof Proceeding as in the proof of Theorem 3.1, we assume that $\Delta U(n) < 0$ for $n \geq n_1$. Accordingly, $U(n) < 0$ for $n \geq n_2 > n_1$ due to (A_2). Using the same type of reasoning as in the proof of Theorem 3.1, $U(n) < 0$ is a contradiction. Hence $\Delta U(n) > 0$ for $n \geq n_1$. Ultimately, $U(n) > 0$, $n \geq n_2 > n_1$. Since $U(n)$ is nondecreasing, there exists a constant $\alpha > 0$ and $n_3 > n_2$ such that $U(n) \geq \alpha$, $n \geq n_3$. Therefore,

$$z(n) \geq \alpha + K(n) + F(n) \geq \alpha + K(n) + F^+(n) > \alpha,$$

for $n \geq n_3$. Summing (9) from n_4 to ∞ and using the last inequality, we obtain

$$\sum_{n=n_4}^{\infty} F(n) < \infty, \qquad n_4 > n_3 + k_1,$$

a contradiction to our assumption (A_{11}). This completes the proof of the theorem.

Theorem 3.5 Let $0 \leq p(n) \leq p < \infty$ and $m \leq k_1$. If (A_0), (A_2) - (A_8) and

(A_{12}) $$\sum_{n=k_1}^{\infty} F(n)\beta(n-k_1) = \infty$$

hold, then every solution of (1) oscillates.

Proof Following the proof of the Theorem 3.4, we may consider the case $\Delta U(n) > 0$ and $U(n) > 0$ for $n \geq n_1$. Applying Lemma 2.2, inequality (9) yields

$$\begin{aligned}\lambda F(n)H_1(\beta(n-k_1)) \leq \; & - \; [H_1(r(n-k_1)\Delta U(n-k_1))]^{-1}\Delta(r(n)\Delta U(n)) \\ & - \; H_1(p)[H_1(r(n-k_1)\Delta U(n-k_1))]^{-1}\Delta(r(n-m)\Delta U(n-m))\end{aligned}$$

for $n \geq n_2 > n_1$. Consequently,

$$\lambda F(n)H_1(\beta(n-k_1)) \leq \frac{-\Delta(r(n)\Delta U(n))}{H_1(r(n)\Delta U(n))} - \frac{H_1(p)\Delta(r(n-m)\Delta U(n-m))}{H_1(r(n-m)\Delta U(n-m))}$$

for $n \geq n_2$. Hence for $r(n)\Delta U(n) < x < r(n+1)\Delta U(n+1)$ and $r(n-m)\Delta U(n-m) < y < r(n+1-m)\Delta U(n+1-m)$, the last inequality becomes

$$\begin{aligned}
\lambda F(n) H_1(\beta(n-k_1)) \leq & - \int_{a(n)}^{a(n+1)} \frac{dx}{H_1(r(n)\Delta U(n))} \\
& - H_1(p) \int_{a(n-m)}^{a(n+1-m)} \frac{dy}{H_1(r(n-m)\Delta U(n-m))} \\
< & - \int_{a(n)}^{a(n+1)} \frac{dx}{H_1(x)} - H_1(p) \int_{a(n-m)}^{a(n+1-m)} \frac{dy}{H_1(y)}
\end{aligned}$$

where $a(n) = r(n)\Delta U(n)$. Because $\lim_{n\to\infty}(r(n)\Delta U(n))$ exits, then for $n \geq n_3 > n_2$,

$$\begin{aligned}
\lambda \lim_{t\to\infty} \sum_{n=n_3}^{t} F(n) H_1(\beta(n-k_1)) < & - \lim_{t\to\infty} \sum_{n=n_3}^{t} \int_{a(n)}^{a(n+1)} \frac{dx}{H_1(x)} \\
& - H_1(p) \lim_{t\to\infty} \sum_{n=n_3}^{t} \int_{a(n-m)}^{a(n+1-m)} \frac{dy}{H_1(y)} \\
= & - \lim_{t\to\infty} \left[\int_{a(n_3)}^{a(t+1)} \frac{dx}{H_1(x)} + H_1(p) \int_{a(n_3-m)}^{a(t+1-m)} \frac{dy}{H_1(y)} \right] \\
< & \ \infty \ ,
\end{aligned}$$

a contradiction to (A_{12}). Thus the proof of the theorem is complete.

Theorem 3.6 If $-1 < p \leq p(n) \leq 0$ and (A_0), (A_2), (A_4), (A_7), (A_8) and

(A_{13}) $$\sum_{n=0}^{\infty} f(n) = \infty$$

hold, then (1) is oscillatory.

Proof Let $y(n)$ be a nonoscillatory solution of (1) such that $y(n) > 0$ for $n \geq n_0$. The case $y(n) < 0$ for $n \geq n_0$ is similar. Proceeding as in the proof of Theorem 3.4, $U(n) < 0$ for $n \geq n_2 > n_1$ when $\Delta U(n) < 0$ for $n \geq n_1$. Using the same type of reasoning as in the proof of Theorem 3.2, $U(n) < 0$ for $n \geq n_2$ yields a contradiction. Hence $\Delta U(n) > 0$ for $n \geq n_1$. Consequently, $\lim_{n\to\infty}(r(n)\Delta U(n))$ exists. Since $U(n)$ is monotonic, then either $U(n) > 0$ or $U(n) < 0$ for $n \geq n_2 > n_1$. The contrdiction is similar, if $U(n) < 0$ for $n \geq n_2$. Assume that $U(n) > 0$ for $n \geq n_2$. Then $z(n) - K(n) > Q(n)$ for $n \geq n_2$. If $z(n) < 0$ then $Q(n) < 0$ for $n \geq n_2$, which is absurd. Let $z(n) > 0$ for $n \geq n_3 > n_2$. Following to the proof of the Theorem 3.4, we have a contradiction due to (A_{13}). This completes the proof of the theorem.

Theorem 3.7 Suppose that $-\infty < p \leq p(n) \leq -1$. Let all the conditions of Theorem 3.6 hold. Then every bounded solution of (1) is oscillatory.

Proof The proof of the theorem follows from the proof of the Theorem 3.6 and hence the detail is omitted.

Theorem 3.8 Let $0 \leq p(n) \leq p < \infty$. Assume that (A_0), (A_1), (A_3) - (A_5) and (A_9) hold. Then every solution of (2) either oscillates or tends to zero as $n \to \infty$.

Proof Let $y(n)$ be a nonoscillatory solution of (2) such that $y(n) > 0$ for $n \geq n_0$. Setting as in (6), we write Eq.(2) as follows :

$$\Delta(r(n)\Delta w(n)) = -f(n)H_1(y(n-k_1)) \leq 0, \not\equiv 0 \tag{10}$$

for $n \geq n_1 > n_0 + \rho$. Hence $(r(n)\Delta w(n))$ is a monotonic function on $[n_1, \infty)$. We may suppose that $\Delta w(n) < 0$ for $n \geq n_1$. If $w(n) < 0$, then $y(n) \leq z(n) \leq k(n)$, $n \geq n_1$. Because $k(n)$ is bounded, there exists a constant $\gamma > 0$ such that $y(n) \leq \gamma$ for $n \geq n_2 > n_1$. Ultimately, $w(n)$ is bounded and $\lim_{n\to\infty} w(n)$ exists. This is a contradiction to the fact that $\lim_{n\to\infty} w(n) = \lim_{n\to\infty} z(n) \neq 0$ implies that $z(n) < 0$ for $n \geq n_2 > n_1$. Assume that $w(n) > 0$ for $n \geq n_1$. Successive summations of the inequality $\Delta(r(n)\Delta w(n)) \leq 0$ from n_1 to n, we can find a constant $\eta > 0$ such that $w(n) \leq \eta$ for $n \geq n_2 > n_1$. Thus $y(n) \leq z(n) \leq K(n) + \eta$ implies $y(n)$ is bounded. Therefore $\lim_{n\to\infty}(r(n)\Delta w(n))$ exists. Using Lemma 2.1(ii) with $U(n)$ replaced by $w(n)$, we get $w(n) \geq -r(n)R(n)\Delta w(n)$ and hence $z(n) \geq -r(n)R(n)\Delta w(n)$ for $n \geq n_2$. Repeated application of Eq.(2) and use of (A_3) and (A_4) yields

$$\begin{aligned} 0 = \Delta(r(n)\Delta w(n)) &+ H_1(p)\Delta(r(n-m)\Delta w(n-m)) \\ &+ f(n)H_1(y(n-k_1)) + H_1(p)f(n-m)H_1(y(n-m-k_1)), \end{aligned}$$

that is,

$$\begin{aligned} 0 \geq \Delta(r(n)\Delta w(n)) &+ H_1(p)\Delta(r(n-m)\Delta w(n-m)) \\ &\quad + \lambda F(n)H_1(z(n-k_1)) \\ \geq \Delta(r(n)\Delta w(n)) &+ H_1(p)\Delta(r(n-m)\Delta w(n-m)) \\ &+ \lambda F(n)H_1(-r(n-k_1)R(n-k_1)\Delta w(n-k_1))) \\ = \Delta(r(n)\Delta w(n)) &+ H_1(p)\Delta(r(n-m)\Delta w(n-m)) \\ &+ \lambda F(n)H_1(R(n-k_1))H_1(-r(n-k_1)\Delta w(n-k_1)) \end{aligned} \tag{11}$$

due to (A_5) for $n \geq n_3 > n_2 + k_1$. Since $-r(n)\Delta w(n)$ is nondecreasing, there exists a constant $C_1 > 0$ and $n_4 > n_3$ such that $-r(n)\Delta w(n) \geq C_1$ for $n \geq n_4$. Accordingly, the last inequality becomes

$$\lambda F(n)H_1(C_1)H_1(R(n-k_1)) \leq -\Delta(r(n)\Delta w(n)) - H_1(p)\Delta(r(n-m)\Delta w(n-m))$$

for $n \geq n_5 > n_4 + k_1$ which on summation from n_5 to ∞, we get

$$\sum_{n=n_5}^{\infty} F(n)H_1(R(n-k_1)) < \infty,$$

a contradiction to (A_9).

Next, we suppose that $\Delta w(n) > 0$ for $n \geq n_1$. If $w(n) < 0$, then $\lim_{n\to\infty} w(n)$ exists and $0 \neq \lim_{n\to\infty} w(n) = \lim_{n\to\infty} z(n)$ will imply that $z(n) < 0$, a contradiction to the fact that $z(n) > 0$. Let $\lim_{n\to\infty} w(n) = 0$. Accordingly, $\lim_{n\to\infty} z(n) = 0$ which provides $\lim_{n\to\infty} y(n) = 0$ due to $y(n) \leq z(n)$ for $n \geq n_2 > n_1$. Consider, $w(n) > 0$ for $n \geq n_2 > n_1$. Lemma 2.1(i) can be applied here and it follows that $w(n) \geq C\ R(n)$, that is, $z(n) \geq w(n) \geq C\ R(n)$ for $n \geq n_2$. Consequently, (11) yields

$$\lambda F(n)H_1(C)H_1(R(n-k_1)) \leq -\Delta(r(n)\Delta w(n)) - H_1(p)\Delta(r(n-m)\Delta w(n-m))$$

for $n \geq n_2 + k_1$. Summing the above inequality from n_3 to ∞, we get

$$\sum_{n=n_3}^{\infty} F(n)H_1(R(n-k_1)) < \infty, \qquad\qquad n_3 > n_2 + 2k_1$$

a contradiction.

If $y(n) < 0$ for $n \geq n_0 > 0$, then we set $x(n) = -y(n)$ to obtain $x(n) > 0$ for $n \geq n_0$. Above procedure can be applied for $x(n) > 0$ and hence we have the contradiction to (A_9). This completes the proof of the theorem.

Theorem 3.9 Let$-1 < p \leq p(n) \leq 0$. If (A_0), (A_1), (A_4) and (A_{10}) hold, then every solution of (2) either oscillates or tends to zero as $n \to \infty$.

Proof Let $y(n)$ be a nonoscillatory solution of (2) such that $y(n) > 0$ for $n \geq n_0 \geq 0$. Setting as in (6), we get (10) for $n \geq n_0 + \rho$. Accordingly, $\Delta w(n)$ is a monotonic function on $[n_1, \infty)$ for which either $w(n) > 0$ or $w(n) < 0$ for $n \geq n_2 > n_1$. Consider $\Delta w(n) < 0$ and $w(n) < 0$ for $n \geq n_2$. Then $0 \neq \lim_{n\to\infty} w(n) = \lim_{n\to\infty} z(n)$ yields that $z(n) < 0$ for $n \geq n_2$. Hence $y(n) < y(n-m)$ for $n \geq n_3 > n_2$, that is, $y(n)$ is a bounded real valued function on $[n_3,\ \infty)$. Consequently, $w(n)$ is bounded and $\lim_{n\to\infty}(r(n)\Delta w(n))$ exists. Further, $w(n)$ is monotonic implies that $\lim_{n\to\infty} w(n) = L,\ L \in (-\infty,\ 0)$, that is, $\lim_{n\to\infty} z(n) = L$. We claim that $\liminf_{n\to\infty} y(n) = 0$. If not, there exists $\gamma > 0$ and $n_4 > n_3$ such that $y(n) \geq \gamma$ or $n \geq n_4$. Summing (10) from n_4 to ∞, we get

$$\sum_{n=n_4}^{\infty} F(n) < \infty,$$

a contradiction to the fact that $R(n) \to 0$ as $n \to \infty$ and (A_{10}) implies that (A_{13}) hold. So our claim holds and by the Lemma 2.3, $L = 0$. We note that

$$\begin{aligned} 0 = \lim_{n\to\infty} z(n) &= \limsup_{n\to\infty} \, (y(n) + p(n)\, y(n-m)) \\ &\geq \limsup_{n\to\infty} \, [y(n) + py(n-m)] \\ &\geq \limsup_{n\to\infty} \, y(n) + \liminf_{n\to\infty} \, (py(n-m)) \\ &= (1+p) \limsup_{n\to\infty} \, y(n) \end{aligned}$$

implies $\limsup_{n\to\infty} \, y(n) = 0$ and hence $\lim_{n\to\infty} y(n) = 0$. Next, we suppose that $w(n) > 0$ for $n \geq n_2$. Let $\lim_{n\to\infty} w(n) = a, \; a \in [0, \; \infty)$. If $y(n)$ is unbounded, then there exists $\{n_j^1\}_{j=1}^{\infty}$ such that $n_j^1 \to \infty$ and $y(n_j^1) \to \infty$ as $j \to \infty$ and

$$y(n_j^1) = max \left\{ y(n) : n_2 \leq n \leq n_j^1 \right\}.$$

Hence

$$\begin{aligned} w(n_j^1) &\geq y(n_j^1) + p\, y(n_j^1 - m) - k(n_j^1) \\ &\geq (1+p) y(n_j^1) - k(n_j^1) \end{aligned}$$

yields that $w(n_j^1) \to \infty$ as $j \to \infty$, a contradiction to the fact that $\lim_{n\to\infty} w(n)$ exists. Thus $y(n)$ is bounded and hence $\lim_{n\to\infty} (r(n)\Delta w(n)$ exists. Using Lemma 2.1(ii) with $u(n)$ replaced by $w(n)$, we get $w(n) \geq -r(n)R(n)\Delta w(n)$ and

$$y(n) \geq w(n) \geq -r(n)R(n)\Delta w(n), \quad n \geq n_3 > n_2.$$

Consequently, (10) becomes

$$f(n)H_1(R(n-k_1))H_1(-r(n-k_1)\Delta w(n-k_1)) \leq -\Delta(r(n)\Delta w(n))$$

for $n \geq n_4 > n_3 + k_1$. Due to nonincreasing $(r(n)\Delta w(n))$, we can find a constant $b > 0$ and $n_5 > n_4 + k_1$ such that $r(n-k_1)\Delta w(n-k_1) \leq -b$ for $n \geq n_5$. Summing the last inequality from n_5 to ∞ , we get

$$\sum_{n=n_5}^{\infty} f(n)H_1(R(n-k_1)) < \infty,$$

a contradiction to (A_{10}).

Assume that $\Delta w(n) > 0$ for $n \geq n_1$. We have two cases, $w(n) > 0$ and $w(n) < 0$. If the former holds, then by the Lemma 2.1(i)

$$y(n) \geq w(n) \geq C\, R(n), \qquad n \geq n_2 > n_1$$

and accordingly, Eq.(10) can be written as

$$f(n)H_1(C\ R(n-k_1)) \leq -\Delta(r(n)\Delta w(n)),$$

for $n \geq n_3 > n_2 + k_1$. Summing the above inequality from n_3 to ∞, we get

$$\sum_{n=n_3}^{\infty} f(n)H_1(R(n-k_1)) < \infty,$$

a contradiction. Suppose the later holds. Then $\lim_{n\to\infty} w(n)$ exists and $0 \neq \lim_{n\to\infty} w(n) = \lim_{n\to\infty} z(n)$ implies that $z(n) < 0$ for $n \geq n_2 > n_1$. Consquently, $y(n)$ is a bounded real valued function on $[n_3,\ \infty), n_3 > n_2 + \rho$. Using the same type of reasoning as above, we obtained $\lim_{n\to\infty} y(n) = 0$. If $0 = \lim_{n\to\infty} w(n) = \lim_{n\to\infty} z(n)$, we claim that $y(n)$ is bounded. Otherwise there is a contradiction to the fact that $w(n_j^1) \to \infty$ as $j \to \infty$. Proceeding as above, we obtained $\lim_{n\to\infty} y(n) = 0$.

The case $y(n) < 0$ for $n \geq n_0 \geq 0$ is similar. Hence the theorem is proved.

Theorem 3.10 Let $-\infty < p_1 \leq p(n) \leq p_2 < -1$. If (A_0), (A_1), (A_4) and (A_{10}) hold, then every bounded solution of (2) either oscillates or tends to zero as $n \to \infty$.

Proof Let $y(n)$ be a bounded nonoscillatory solution of (2) such that $y(n) > 0$ for $n \geq n_0 \geq 0$. Then setting as in (6), we get (10) for $n \geq n_1 > 0 + \rho$. From (10) it follows that $\Delta w(n) > 0$ or $\Delta w(n) < 0$ for $n \geq n_1$. Consider the case $\Delta w(n) < 0$ for $n \geq n_1$. Proceeding as in the proof of the Theorem 3.9, we obtain $L = 0$. Thus

$$\begin{aligned} 0 = \lim_{n\to\infty} z(n) &= \liminf_{n\to\infty}\ [y(n) + p(n)\ y(n-m)] \\ &\leq \liminf_{n\to\infty}\ [y(n) + p_2\ y(n-m)] \\ &\leq \limsup_{n\to\infty}\ y(n) + \liminf_{n\to\infty}\ (p_2\ y(n-m)) \\ &\leq \limsup_{n\to\infty}\ y(n) + p_2\ \limsup_{n\to\infty}\ y(n-m) \\ &= (1+p_2) \limsup_{n\to\infty} y(n) \end{aligned}$$

implies that $\lim_{n\to\infty} y(n) = 0$, since $(1+p_2) < 0$. Rest of the proof can be followed from the proof of the Theorem 3.9. Hence the proof of the theorem is complete.

Theorem 3.11 Let $0 \leq p(n) \leq p < \infty$. If (A_0), (A_2) - (A_5), and (A_{11}) hold, then a solution of (2) either oscillates or tends to zero as $n \to \infty$.

Proof Let $y(n)$ be a non-oscillatory solution of (2) such that $y(n) > 0$ for $n \geq n_0 \geq 0$. The case $y(n) < 0$ for $n \geq n_0$ can similarly dealt with. Proceeding as in the proof of the Theorem 3.8, we assume that $\Delta w(n) < 0$ for $n \geq n_1$. Accordingly, $w(n) < 0$ for $n \geq n_2 > n_1$

due to (A_2). Using the same type of reasoning as in the proof of Theorem 3.8, we obtain a contradiction. Hence $\Delta w(n) > 0$ for $n \geq n_1$. If $w(n) < 0$, then $\lim_{n\to\infty} w(n)$ exists for which there is a contradiction when $0 \neq \lim_{n\to\infty} w(n) = \lim_{n\to\infty} z(n)$. Let $\lim_{n\to\infty} w(n) = 0$. Following the proof of Theorem 3.8 we get, $\lim_{n\to\infty} y(n) = 0$. Assume that $w(n) > 0$ for $n \geq n_1$. Hence there exists a constant $\alpha > 0$ such that $w(n) \geq \alpha$ for $n \geq n_2 > n_1$, that is, $z(n) \geq w(n) \geq \alpha$ for $n \geq n_2$. Accordingly, (10) yields that

$$\sum_{n=n_3}^{\infty} F(n) < \infty, \qquad n_3 > n_2 + k_1,$$

a contradiction to our hypothesis. This completes the proof of the theorem.

Theorem 3.12 Let $0 \leq p(n) \leq p < \infty$ and $m \leq k_1$. If (A_0), (A_2)-(A_6) and (A_{12}) hold, then every solution of (2) either oscillates or tends to zero as $n \to \infty$.

Proof Proceeding as in the proof of the Theorem 3.11, we only consider the case $\Delta w(n) > 0$ and $w(n) > 0$ for $n \geq n_1$. From (11) it follows that

$$\begin{aligned} 0 \geq \Delta(r(n)\Delta w(n)) &+ H_1(p)\Delta(r(n-m)\Delta w(n-m)) \\ &+ \lambda F(n) H_1(\beta(n-k_1)) H_1(r(n-k_1)\Delta w(n-k_1)) \end{aligned}$$

due to Lemma 2.2, for $n \geq n_2 > n_1$. Hence

$$\begin{aligned} \lambda F(n) H_1(\beta(n-k_1)) &\leq -\left[H_1(r(n-k_1)\Delta w(n-k_1))\right]^{-1} \Delta(r(n)\Delta w(n)) \\ &- H_1(p)\left[H_1(r(n-k_1)\Delta w(n-k_1))\right]^{-1} \Delta(r(n-m)\Delta w(n-m)). \end{aligned}$$

Rest of the proof follows from the Theorem 3.5 and hence the details are omitted.

Remark In Theorem 3.11, H_1 could be linear, sublinear or superlinear. However, if we restrict m and k_1, H_1 could be sublinear in Theorem 3.12 due to (A_6).

Theorem 3.13 Assume that $-1 < p \leq p(n) \leq 0$. If (A_0), (A_2), (A_4) and (A_{13}) hold, then a solution of (2) either oscillates or tends to zero as $n \to \infty$.

Proof Proceeding as in the proof of the Theorem 3.11. We consider the case $\Delta w(n) < 0$ and $w(n) < 0$ for $n \geq n_2 > n_1$. Accordingly, $w(n)$ is a monotonic function on $[n_2, \infty)$ and $0 \neq \lim_{n\to\infty} w(n) = \lim_{n\to\infty} z(n)$ exists. It follows from the Theorem 3.9 that $\lim_{n\to\infty} y(n) = 0$.

Let $\Delta w(n) > 0$ for $n \geq n_1$. If $w(n) < 0$ for $n \geq n_2 > n_1$, then use the same arguments as in Theorem 3.9 to obtain $\lim_{n\to\infty} y(n) = 0$. Suppose that $w(n) > 0$ for $n \geq n_2 > n_1$. Then there exists a constant $\gamma > 0$ and $n_3 > n_2$ such that $w(n) \geq \gamma$, $n \geq n_3$. Consequently, $y(n) \geq w(n) \geq \gamma$ for $n \geq n_3$. Summing (10) from $n_3 + k_1$ to ∞, we obtain a contradiction to (A_{13}). Hence the proof of the theorem is complete.

Theorem 3.14 Let $-\infty < p_1 \le p(t) \le p_2 < -1$. If (A_0), (A_2), (A_4) and (A_{13}) hold, then every bounded solution of (2) either oscillates or tends to zero as $n \to \infty$.

Proof The proof of the theorem can be followed from the Theorems 3.13 and 3.10.

Example Consider

$$\Delta[e^n \Delta(y(n) + p(n)\, y(n-1))] + f(n)\, y(n-2) - g(n)\, y^3(n-4) = (-1)^n e^n, \quad n \ge 0, \qquad (12)$$

where $p(n) = [2 + (-1)^n]$, $f(n) = (2e+3)e^n + e^{-n}$, $g(n) = e^{-n}$. Indeed, $F(n) = (2e+3)e^{n-1} + e^{-(n-1)}$ and $R(n) = \frac{e}{e-1}e^{-n}$. If we choose $Q(n) = [2(1+e)]^{-1}(-1)^n$, then $q(n) = \Delta(e^n \Delta Q(n))$. Clearly, (A_9) is satisfied. Theorem 3.1 can be applied to (12). Consequently, every solution of (12) oscillates. In particular, $y(n) = (-1)^n$ is one of such solution.

Example Consider

$$\Delta[e^{-n} \Delta(y(n) + p(n)\, y(n-1))] + f(n)\, y^3(n-1) - g(n)\, y(n-2) = 0, \qquad (13)$$

where $0 \ge p(n) = -\frac{1}{2}e^{-n} > -1$, $f(n) = \frac{(e+1)(e^2+1)}{e^6}e^n$ and $g(n) = \frac{(e^2+1)(e^3+1)}{2e^6}e^{-2n}$. Indeed, all the conditions of Theorem 3.13 are satisfied. Hence every solution of (13) either oscillates or tends to zero as $n \to \infty$. In particular, $y(n) = (-1)^n e^{-n}$ is such a solution of (13).

4 Existence Theorems

Theorem 4.1 Let $0 \le p(n) \le b_1 < 1$. Suppose that $q(n)$ satisfies (A_7). If

(A_{14}) $\qquad \sum_{n=0}^{\infty} A(n+1)f(n) < \infty\,; \qquad \sum_{n=0}^{\infty} A(n+1)g(n) < \infty,$

where $A(n) = \sum_{S=M_0}^{n-1} \frac{1}{r(s)}$, then Eq.(1) admits a positive bounded solution.

Proof It is possible to find M_0 large enough such that

$$\sum_{n=M_0}^{\infty} A(n+1)f(n) < \frac{1-b_1}{10\, H_1(1)}, \qquad \sum_{n=M_0}^{\infty} A(n+1)g(n) < \frac{1-b_1}{20\, H_1(1)}$$

Let $Q(n)$ be such that $-\frac{(1-b_1)}{20} \le Q(n) \le \frac{1-b_1}{10}$ for $n \ge M_0$. We can choose $m_1 > m_2 \ge M_2 > M_1 > M_0$ such that $|Q(m_1) - Q(m_2)| < \frac{1-b_1}{3}$ and

$$\sum_{n=m_2}^{m_1-1} A(n+1)f(n) < \frac{1-b_1}{6\, H_1(1)}, \qquad \sum_{n=m_2}^{m_1-1} A(n+1)g(n) < \frac{1-b_1}{6\, H_1(1)}$$

Let $X = \ell^{\infty}_{M_0}$ be the Banach space of all bounded real valued functions $x(n),\ n \geq M_0$ with the sup norm defined by

$$\|x\| = sup\{|x(n)| : n \geq M_0\}.$$

Define a set $S \subset X$ as follows:

$$S = \left\{ x \in X : \frac{1-b_1}{10} \leq x(n) \leq 1,\ n \geq M_0 \right\}.$$

Then S is a closed bounded and convex subset of X. Define two maps T_1 and T_2 on S as follows :

$$(T_1\, x)(n) = \begin{cases} (T_1\, x)(M_1) \quad , & M_0 \leq n \leq M_1 \\ -p(n)x(n-m) + \frac{1+4b_1}{5}, n \geq M_1 \end{cases}$$

$$(T_2\, x)(n) = \begin{cases} (T_2\, x)(M_1) \quad , & M_0 \leq n \leq M_1 \\ Q(n) + A(n) \sum\limits_{s=n}^{\infty} [f(s)H_1(x(s-k_1)) - g(s)H_2(x(s-k_2))] & \\ + \sum\limits_{s=M_0}^{n-1} A(s+1)[f(s)H_1(x(s-k_1)) - g(s)H_2(x(s-k_2))], & \\ & n \geq M_1. \end{cases}$$

Clearly,

$$\begin{aligned} (T_1\, x)(n) + (T_2\, x)(n) \leq \frac{1+4b_1}{5} + \frac{1-b_1}{10} &+ H_1(1) \sum_{s=n}^{\infty} A(s)f(s) \\ &+ H_1(1) \sum_{s=M_0}^{n-1} A(s+1)f(s) \\ \leq \frac{1+4b_1}{5} + \frac{1-b_1}{10} &+ H_1(1) \sum_{s=M_0}^{\infty} A(s+1)f(s) < 1 \end{aligned}$$

and

$$(T_1\, x)(n) + (T_2\, x)(n) \geq -b_1 + \frac{1+4b_1}{5} - \frac{1-b_1}{20} - \frac{1-b_1}{20} = \frac{1-b_1}{10}$$

implies that $T_1\, x + T_2\, x \in S$. Since $0 \leq p(n) \leq b_1 < 1$, it is easy to check that T_1 is a contraction mapping.

Next, we show that T_2 is continuous. Let $\{x_j(n)\}$ be a sequence in S such that, $\|x_j - x\| = 0$ as $j \to \infty$. Since S is a closed set, then $x1 - b_1 S$ and

$$\begin{aligned}|(T_2\, x_j)(n) - (T_2 x)(n)| &\leq \sum_{s=M_0}^{\infty} A(s+1)|f(s)\{H_1(x_j(s-k_1))H_1(x(s-k_1))\} \\ &- g(s)\{H_2(x_j(S-k_2)) - H_2(x(s-k_2))\}|.\end{aligned}$$

As H_i is continuous, then $\lim_{j\to\infty} \|T_2\, x_j - T_2\, x\| = 0$. We know that T_2 is uniformly bounded, there exists $M_2 > 0$ such that $m_1 > m_2 \geq M_2$ and for all $x(n) \,\epsilon\, S$,

$$\begin{aligned}|T_2 x(m_1) - T_2 x(m_2)| &\leq |Q(m_1) - Q(m_2)| \\ &+ \left|\sum_{s=m_2}^{m_1-1} A(s+1)[f(s)\; H_1(x(s-k_1)) - g(s)H_2(x(s-k_2))]\right| \\ &+ \left|\sum_{s=M_2}^{m_1-1} A(s+1)\,[f(s)\; H_1(x(s-k_1)) - g(s)H_2(x(s-k_2))]\right| \\ &< \frac{1-b_1}{3} + 2\left(\frac{1-b_1}{6} + \frac{1-b_1}{6}\right) = 1 - b_1.\end{aligned}$$

Hence by the Lemma 2.4, T_2 has a fixed point. Consequently, it follows from the discrete Krasnoselskii's fixed point theorem that $T_2\, x + T_1\, x$ has a fixed point in S, that is

$$\begin{aligned}x(n) = \frac{1+4b_1}{5} &- p(n)x(n-m) + Q(n) \\ &+ A(n)\sum_{s=n}^{\infty}[f(s)\; H_1(x(s-k_1)) - g(s)H_2(x(s-k_2))] \\ &+ \sum_{s=M_0}^{n-1} A(s+1)\,[f(s)\; H_1(x(s-k_1)) - g(s)H_2(x(s-k_2))]\,.\end{aligned}$$

Theorem 4.2 Let $1 < b_1 \leq p(n) \leq b_2 < \frac{1}{2}\, b_1^2$. Suppose that $q(n)$ satisfies (A_7). If (A_{14}) holds, then Eq.(1) admits a positive bounded solution.

Proof It is possible to find M_0 large enough such that

$$H_1(1)\sum_{n=M_0}^{\infty} A(n+1)f(n) < \frac{b_1-1}{8b_1} + \frac{b_1-1}{16b_2}, H_1(1)\sum_{n=M_0}^{\infty} A(n+1)g(n) < \frac{b_1-1}{16b_2}.$$

Let $Q(n)$ be such that $-\frac{(b_1-1)}{16b_1\, b_2} \leq Q(n) \leq \frac{b_1-1}{8b_1^2} + \frac{b_1-1}{16b_1\, b_2}$ for $n \geq M_0$. We can choose $m_1 > m_2 \geq M_2 > M_1 > M_0$ such that $|Q(m_1) - Q(m_2)| < \frac{1-b_1}{3b_1}$ and

$$\sum_{n=m_2+m}^{m_1+m-1} A(n+1)f(n) < \frac{1-b_1}{6b_1\; H_1(1)}, \sum_{n=m_2+m}^{m_1+m-1} A(n+1)g(n) < \frac{1-b_1}{6b_1\; H_1(1)}.$$

Let $X = \ell^{\infty}_{M_0}$ be the Banach space of all bounded real valued functions $x(n),\ n \geq M_0$ with the sup norm defined by

$$\|x\| = sup\{|x(n)| : n \geq M_0\}.$$

Define a set $S \subset X$ as follows :

$$S = \left\{ x \in X \ : \ \frac{b_1 - 1}{8b_1\, b_2} \leq x(n) \leq 1, n \geq M_0 \right\}.$$

Clearly, S is a closed bounded and convex subset of X. Define two maps T_1 and T_2 on S as follows :

$$(T_1\, x)(n) = \begin{cases} (T_1\, x)(M_1) \quad , & M_0 \leq n \leq M_1 \\ -\frac{x(n+m)}{p(n+m)} + \frac{2b_1^2 + b_1 - 1}{4b_1\, p(n+m)}, & n \geq M_1 \end{cases}$$

$$(T_2\, x)(n) = \begin{cases} (T_2\, x)(M_1) \quad , & M_0 \leq n \leq M_1 \\ \frac{Q(n+m)}{p(n+m)} + \frac{A(n+m)}{p(n+m)} \sum\limits_{s=n+m}^{\infty} [f(s)H_1(x(s-k_1)) - g(s)H_2(x(s-k_2))] & \\ +\frac{1}{p(n+m)} \sum\limits_{s=M_0}^{n+m-1} A(s+1)[f(s)H_1(x(s-k_1)) - g(s)H_2(x(s-k_2))], & \\ & n \geq M_1. \end{cases}$$

It is easy to verify that T_1 is a contraction mapping and $T_1\, x + T_2\, x \in S$.

Rest of the analysis can be followed from the Theorem 4.1. Hence the proof of the theorem is complete.

Remark In other ranges of $p(n)$ except $p(n) = \pm 1$, the discrete Krasnoselskii's fixed point theorem can be applied for existence of positive solutions of (1) under the suitable mappings T_1 and T_2. The following theorems are stated without proof.

Theorem 4.3 Let$-1 < b_1 \leq p(n) \leq 0$. Suppose that $q(n)$ satisfies (A_7). If (A_{14}) holds, then Eq.(1) admits a positive bounded solution.

Theorem 4.4 Let $-\infty < b_1 \leq p(n) \leq b_2 < -1$. Assume that $q(n)$ satisfied (A_7). If (A_{14}) holds, then Eq.(1) admits a positive bounded solution.

Acknowledgment : The author is thankful to the Referee for his kind helpful suggestions and necessary correction.

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(Received October 23, 2009)