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Floquet Boundary Value Problem of Fractional Functional Differential Equations*

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Abstract: In this paper, we prove the existence of positive solutions for Floquet boundary value problem concerning fractional functional differential equations with bounded delay. The results are obtained by using two fixed point theorems on appropriate cones.

Keywords: Positive solutions, Boundary value problems, Fractional functional differential equations.

1. Introduction

Fractional differential equations have gained considerable importance due to their application in various sciences, such as physics, chemistry, mechanics, engineering, etc. For details, see [1-4] and references therein. Naturally, such equations need to be solved. Recently, there are some papers focused on initial value problem of fractional functional differential equations[5-12], and boundary value problems of fractional ordinary differential equations [13-20]. But the results dealing with the boundary value problems of fractional functional differential equations are relatively scarce.

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In this paper, we consider the existence of positive solutions for the following fractional functional differential equation

$$^{C}D^{\alpha}x(t) = f(t, x_t), \quad t \in [0, T], \tag{1}$$

with the boundary condition

$$Ax_0 - x_T = \phi, \tag{2}$$

where α, A, T are real numbers with $0 < \alpha \leq 1$, $A > 1$ and $T > 0$. $^{C}D^{\alpha}$ denote Caputo's fractional derivative. $f : [0, T] \times C[-r, 0] \to R$ is a given function satisfying some assumptions that will be specified later, and $\phi \in C[-r, 0]$, where $0 \leq r < T$. As usual, $C[-r, 0]$ is the space of continuous functions on $[-r, 0]$, equipped with $\|\phi\| = \max_{-r\leq\theta\leq 0} |\phi(\theta)|$. For any $t \in [0, T]$ and $x \in C[-r, T]$, the function x_t is defined by $x_t(\theta) = x(t + \theta), \ -r \leq \theta \leq 0$.

The boundary value problem $(1) - (2)$ belongs to a class of problems knows as "Floquet problems" which arise from physics (see [21]). The existence of positive solutions of the first order functional differential equations concerned with this problem was discussed by Mavridis and Tsamatos in [22].

In this paper, we firstly deduced the problem $(1) - (2)$ to an equivalent operator equation. Next, using two fixed-point theorems, we get that the equivalent operator has (at least) a fixed point, it means that the boundary value problem $(1) - (2)$ has (at least) one positive solution, which is upper and lower bounded by specific real numbers .

2. Preliminaries

In this section, we introduce definitions and preliminary facts which are used throughout this paper.

Let Ω be a finite or infinite interval of the real axis $R = (-\infty, \infty)$. We denote by $L^p\Omega$ $(1 \leq p \leq \infty)$ the set of those Lebesgue measurable functions f on Ω for which $\|f\|_{L^p\Omega} < \infty$, where

$$\|f\|_{L^p\Omega} = \Big(\int\limits_{\Omega} |f(t)|^p dt \Big)^{\frac{1}{p}} \quad (1 \leq p < \infty)$$

and

$$\|f\|_{L^\infty\Omega} = \text{ess sup}_{x\in\Omega} |f(t)|.$$

Definition 2.1. [2,3] The fractional integral of order α with the lower limit t_0 for a function f is defined as

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int\limits_{t_0}^{t} \frac{f(s)}{(t - s)^{1-\alpha}} ds, \quad t > t_0, \quad \alpha > 0, \tag{3}$$

provided the right side is point-wise defined on $[t_0, \infty)$, where Γ is the gamma function.

Definition 2.2. [2,3] Riemann-Liouville derivative of order α with the lower limit t_0 for a function $f : [t_0, \infty) \to R$ can be written as

$$^{L}D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t_0}^{t} \frac{f(s)}{(t-s)^{\alpha+1-n}} ds, \quad t > t_0, \quad n-1 < q < n.$$

The first−and maybe the most important−property of Riemann-Liouville fractional derivative is that Riemann-Liouville fractional differentiation operator is a left inverse to the Riemann-Liouville fractional integration operator of the same order α.

Lemma 2.1.[2] Let $f(t) \in L^1[t_0, \infty)$. Then

$$^{L}D^{\alpha}(I^{\alpha}f(t)) = f(t), \qquad t > t_0 \ \text{ and } \ 0 < \alpha < 1$$

Definition 2.3. [2] Caputo's derivative of order α for a function $f : [t_0, \infty) \to R$ can be written as

$$^{C}D^{\alpha}f(t) = \ ^{L}D^{\alpha}\Big(f(t) - \sum_{k=0}^{n-1} \frac{(t-t_0)^k}{k!} f^{(k)}(t_0)\Big), \quad t > t_0, \quad n-1 < \alpha < n.$$

One can show that if $f(t) \in C^n[t_0, \infty)$, then

$$^{C}D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds.$$

Obviously, Caputo's derivative of a constant is equal to zero.

Definition 2.4. Let X be a real Banach space. A cone in X is a nonempty, closed set $P \subset X$ such that

(i) $\lambda u + \mu v \in P$ for all $u, v \in P$ and all $\lambda, \mu \geq 0$,

(ii) $u, -u \in P$ implies $u = 0$.

Let P be a cone in a Banach space X. Then, for any $b > 0$, we denote by P_b the set

$P_b = \{x \in P : \|x\| < b\}$,

and by ∂P_b the boundary of P_b in P, i.e, the set

$\partial P_b = \{x \in P : \|x\| = b\}$.

In order to prove our results, and since we are looking for positive solutions, we will use the following two lemmas, which are applications of the fixed point theory in a cone. Their proofs can be found in [23,24].

Lemma 2.2. Let $g : \overline{P}_b \to P$ be a completely continuous map such that $g(x) \neq \lambda x$ for all $x \in \partial P_b$ and $\lambda \geq 1$, then g has a fixed point in P_b.

Lemma 2.3. Let $E = (E, \|\cdot\|)$ be a Banach space, $P \subset E$ be a cone, and $\|\cdot\|$ be increasing (strictly) with respect to P. Also, σ, τ are positive constants with $\sigma \neq \tau$, suppose $g : \overline{P}_{\max\{\sigma,\tau\}} \to P$ is a completely continuous map and assume the conditions

(i) $g(x) \neq \lambda x$ for every $x \in \partial P_{\sigma}$ and $\lambda \geq 1$,

(ii) $\|g(x)\| \geq \|x\|$ for $x \in \partial P_\tau$,

hold, then g has at least a fixed point x with $\min\{\sigma, \tau\} \leq \|x\| \leq \max\{\sigma, \tau\}$.

3. Main results

Let the intervals $I := [0, T]$ and $J := [-r, 0]$ and set $C(J \cup I)$ be endowed with the ordering $x \leq y$ if $x(t) \leq y(t)$ for all $t \in (J \cup I)$, and the maximum norm, $\|x\|_{J\cup I} = \max_{-r\leq t\leq T} |x(t)|$. Define the cone $P \subset C(J \cup I)$ by $P = \{x \in C(J \cup I) \mid x(t) > 0\}$ and set $P_l = \{x \in P|\ \|x\|_{J\cup I} \leq l, l > 0\}$ and $C^+(J) = \{x \in C(J)|\ x(t) \geq 0,\ t \in J\}$.

The following assumptions are adopted throughout this section.

(H_1) for any $\varphi \in C(J)$, $f(t, \varphi)$ is measurable with respect to t on I,

(H_2) for any given $l > 0, x \in P_l$, there exist $\alpha_1 \in (0, \alpha)$ and a function $m_l(t) \in L^{\frac{1}{\alpha_1}} I$ such that

$|f(t, x_t)| \leq m_l(t), t \in I,$

(H_3) $f(t, \varphi)$ is continuous with respect to φ on $C(J)$,

(H_4) $f : I \times C^+(J) \to R^+$, $\phi(0) > 0$ and $\phi(t) > -\frac{\phi(0)}{A-1}$, for $t \in J$,

(H_5) there exists a $\rho > 0$ such that

$$\rho > \frac{\phi(0)}{A-1} + \frac{\|\phi\|}{A} + \frac{AM_\rho}{(A-1)\Gamma(\alpha)(1+\beta)^{1-\alpha_1}} T^{(1+\beta)(1-\alpha_1)},$$

where $M_\rho = \|m_\rho(t)\|_{L^{\frac{1}{\alpha_1}} I}$.

In order to gain our results, firstly, we must reformulate our boundary value problem $(1) - (2)$ into an abstract operator equation. This is done in the following lemma.

Lemma 3.1. Assume that $(H_1) - (H_3)$ hold. Then a function $x \in P_l$ is a solution of the boundary value problem $(1) - (2)$ if and only if $x(t) = Fx(t), t \in J \cup I$, where $F : P_l \to C(J \cup I)$ is given by the formula

$$Fx(t) = \begin{cases} \frac{\phi(0)}{A-1} + \frac{1}{(A-1)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, x_s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_s) ds, & t \in I, \\ \frac{\phi(0)}{A(A-1)} + \frac{1}{A(A-1)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, x_s) ds & \\ + \frac{1}{A\Gamma(\alpha)} \int_0^{T+t} (T+t-s)^{\alpha-1} f(s, x_s) ds + \frac{\phi(t)}{A}, & t \in J. \end{cases}$$

Proof. Firstly, it is easy to obtain that $f(t, x_t)$ is Lebesgue measurable in I according to conditions (H_1) and (H_3). The direct calculation gives that $(t-s)^{\alpha-1} \in L^{\frac{1}{1-\alpha_1}}[0, t]$, for $t \in I$. In light of Hölder inequality and the condition (H_2), we obtain that $(t-s)^{\alpha-1} f(s, x_s)$ is Lebesgue integrable with respect to $s \in [0, t]$ for all $t \in I$, and

$$\int_0^t |(t-s)^{\alpha-1} f(s, x_s)| ds \leq \|(t-s)^{\alpha-1}\|_{L^{\frac{1}{1-\alpha_1}}[0,t]} \|m_l(t)\|_{L^{\frac{1}{\alpha_1}} I}. \tag{4}$$

Hence, Fx exists. From the formula of Fx, we have

$$Fx(0^-) = Fx(0^+), \quad x \in P_l.$$

So it is clear that Fx is a continuous function for every $x \in P_l$. It is to say that $F : P_l \to C(J \cup I)$. Moreover, from (1), we have

$$I^\alpha \, {}^C D^\alpha x(t) = I^\alpha f(t, x_t), \quad t \in I,$$

i.e.,

$$x(t) = x(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_s) ds, \quad t \in I. \tag{5}$$

Since $r < T$, if $\theta \in J$, then $T + \theta \in I$. Thus from (2) and (6) we get

$$Ax(\theta) - \left(x(0) + \frac{1}{\Gamma(\alpha)} \int_0^{T+\theta} (T+\theta-s)^{\alpha-1} f(s, x_s) ds \right) = \phi(\theta). \tag{6}$$

Therefore, for $\theta = 0$, we get

$$x(0) = \frac{\phi(0)}{A-1} + \frac{1}{(A-1)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, x_s) ds. \tag{7}$$

Using (6) and (8) we have

$$x(t) = \frac{\phi(0)}{A-1} + \frac{1}{(A-1)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, x_s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_s) ds, \quad t \in I.$$

Also, by (7) and (8), for $t \in J$ we get

$$x(t) = \frac{\phi(0)}{A(A-1)} + \frac{1}{A(A-1)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, x_s) ds$$
$$+ \frac{1}{A\Gamma(\alpha)} \int_0^{T+t} (T+t-s)^{\alpha-1} f(s, x_s) ds + \frac{\phi(t)}{A}.$$

So,

$$x(t) = Fx(t), \quad t \in J \cup I.$$

On the other hand, if $x \in P_l$ is such that $x(t) = Fx(t), t \in J \cup I$, then, by Definition 2.3 and Lemma 2.1, for every $t \in I$ we have

$${}^C D^\alpha x(t) = {}^L D^\alpha Fx(t) = f(t, x_t).$$

Also for any $\theta \in J$, it is clear that

$$\begin{aligned} Ax_0(\theta) - x_T(\theta) &= Ax(\theta) - x(T+\theta) \\ &= \phi(\theta). \end{aligned}$$

The proof is complete.

Lemma 3.2. Assume that $(H_1) - (H_4)$ hold. Then $Fx(t) > 0$, for $t \in J \cup I$, $x \in P_l$, and $F : P_l \to P$ is completely continuous operator.

Proof. From Lemma 3.1, we get $F : P_l \to C(J \cup I)$, and by (H_4), we easily obtain $Fx(t) > 0$ for $x \in P_l$. Also, it is clear that $F : P_l \to P$ is continuous according to condition (H_3).

Let $\beta = \frac{\alpha-1}{1-\alpha_1} \in (-1, 0)$. For every $t \in I$, we have

$$\begin{aligned} |Fx(t)| &\le \frac{\phi(0)}{A-1} + \frac{M_l}{(A-1)\Gamma(\alpha)} \|(T-s)^{\alpha-1}\|_{L^{\frac{1}{1-\alpha_1}} I} + \frac{M_l}{\Gamma(\alpha)} \|(t-s)^{\alpha-1}\|_{L^{\frac{1}{1-\alpha_1}} [0,t]} \\ &\le \frac{\phi(0)}{A-1} + \frac{M_l}{(A-1)\Gamma(\alpha)(1+\beta)^{1-\alpha_1}} T^{(1+\beta)(1-\alpha_1)} + \frac{M_l}{\Gamma(\alpha)(1+\beta)^{1-\alpha_1}} T^{(1+\beta)(1-\alpha_1)}. \end{aligned}$$

Also, for every $t \in J$, we have

$$\begin{aligned} |Fx(t)| &\le \frac{\|\phi\|}{A-1} + \frac{M_l}{A(A-1)\Gamma(\alpha)} \|(T-s)^{\alpha-1}\|_{L^{\frac{1}{1-\alpha_1}} I} + \frac{M_l}{A\Gamma(\alpha)} \|(T+t-s)^{\alpha-1}\|_{L^{\frac{1}{1-\alpha_1}} [0,T+t]} \\ &\le \frac{\|\phi\|}{A-1} + \frac{M_l}{A(A-1)\Gamma(\alpha)(1+\beta)^{1-\alpha_1}} T^{(1+\beta)(1-\alpha_1)} + \frac{M_l}{A\Gamma(\alpha)(1+\beta)^{1-\alpha_1}} T^{(1+\beta)(1-\alpha_1)}. \end{aligned}$$

Hence, LP_l is bounded.

Now, we will prove that LP_l is equicontinuous.

In the following, we divide the proof into three cases.

Case 1. $x \in P_l, 0 \le t_1 < t_2 \le T$,

$$\begin{aligned} & |Fx(t_2) - Fx(t_1)| \\ &\le \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1} f(s, x_s) ds - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} f(s, x_s) ds \right| \\ &\le \frac{1}{\Gamma(\alpha)} \int_0^{t_1} |(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1})| f(s, x_s) ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} f(s, x_s) ds \\ &\le \frac{M_l}{\Gamma(\alpha)} \Big(\int_0^{t_1} [(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1})]^{\frac{1}{1-\alpha_1}} ds \Big)^{1-\alpha_1} + \frac{M_l}{\Gamma(\alpha)} \Big(\int_{t_1}^{t_2} [(t_2 - s)^{\alpha-1}]^{\frac{1}{1-\alpha_1}} ds \Big)^{1-\alpha_1} \\ &\le \frac{M_l}{\Gamma(\alpha)} \Big(\int_0^{t_1} (t_1 - s)^{\beta} - (t_2 - s)^{\beta} ds \Big)^{1-\alpha_1} + \frac{M_l}{\Gamma(\alpha)} \Big(\int_{t_1}^{t_2} (t_2 - s)^{\beta} ds \Big)^{1-\alpha_1} \\ &\le \frac{M_l}{\Gamma(\alpha)(1+\beta)^{1-\alpha_1}} \Big(t_1^{1+\beta} - t_2^{1+\beta} + (t_2 - t_1)^{1+\beta} \Big)^{1-\alpha_1} \end{aligned}$$

$$+\frac{M_l}{\Gamma(\alpha)(1+\beta)^{1-\alpha_1}}(t_2-t_1)^{(1+\beta)(1-\alpha_1)}$$
$$\leq \frac{2M_l}{\Gamma(\alpha)(1+\beta)^{1-\alpha_1}}(t_2-t_1)^{(1+\beta)(1-\alpha_1)}.$$

Case 2. $x \in P_l, -r \leq t_1 < t_2 \leq 0$,

$$|Fx(t_2) - Fx(t_1)|$$
$$\leq |\frac{\phi(t_2)}{A} - \frac{\phi(t_1)}{A}|$$
$$+\frac{1}{A\Gamma(\alpha)}\Big|\int_0^{T+t_2}(T+t_2-s)^{\alpha-1}f(s,x_s)ds - \int_0^{T+t_1}(T+t_1-s)^{\alpha-1}f(s,x_s)ds\Big|$$
$$\leq \frac{1}{A}|\phi(t_2)-\phi(t_1)| + \frac{M_l}{A\Gamma(\alpha)}\Big(\int_0^{T+t_1}[(T+t_1-s)^{\alpha-1}-(T+t_2-s)^{\alpha-1})]^{\frac{1}{1-\alpha_1}}ds\Big)^{1-\alpha_1}$$
$$+\frac{M_l}{A\Gamma(\alpha)}\Big(\int_{T+t_1}^{T+t_2}(T+t_2-s)^{\beta}ds\Big)^{1-\alpha_1}$$
$$\leq \frac{1}{A}|\phi(t_2)-\phi(t_1)| + \frac{M_l}{A\Gamma(\alpha)(1+\beta)^{1-\alpha_1}}[(T+t_1)^{1+\beta}-(T+t_2)^{1+\beta}+(t_2-t_1)^{1+\beta}]^{1-\alpha_1}$$
$$+\frac{M_l}{A\Gamma(\alpha)(1+\beta)^{1-\alpha_1}}(t_2-t_1)^{(1+\beta)(1-\alpha_1)}$$
$$\leq \frac{1}{A}|\phi(t_2)-\phi(t_1)| + \frac{2M_l}{A\Gamma(\alpha)(1+\beta)^{1-\alpha_1}}(t_2-t_1)^{(1+\beta)(1-\alpha_1)}.$$

Case 3. $x \in P_l, -r \leq t_1 < 0 \leq t_2 \leq T$,

$$|Fx(t_2) - Fx(t_1)|$$
$$\leq |Fx(t_2) - Fx(0)| + |Fx(0) - Fx(t_1)|$$
$$\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}f(s,x_s)ds + \frac{|\phi(0)-\phi(t_1)|}{A}$$
$$\frac{1}{A\Gamma(\alpha)}\Big|\int_0^{T}(T-s)^{\alpha-1}f(s,x_s)ds - \int_0^{T+t_1}(T+t_1-s)^{\alpha-1}f(s,x_s)ds\Big|$$
$$\leq \frac{M_l}{\Gamma(\alpha)(1+\beta)^{1-\alpha_1}}t_2^{(1+\beta)(1-\alpha_1)} + \frac{|\phi(0)-\phi(t_1)|}{A} + \frac{2M_l}{A\Gamma(\alpha)(1+\beta)^{1-\alpha_1}}(-t_1)^{(1+\beta)(1-\alpha_1)}.$$

According to the continuity of ϕ and $t^{(1+\beta)(1-\alpha_1)}$, we can easily obtain LP_l is equicontinuous. Then $F : P_l \to P$ is a completely continuous operator by Arzela-Ascoli theorem. The proof is complete.

Theorem 3.1. Assume that the conditions $(H_1) - (H_5)$ hold. Then the boundary value problem $(1) - (2)$ has at least one positive solution x, such that

$$\frac{\phi(0)}{A-1} \leq \|x\|_{J\cup I} < \rho.$$

Proof. From Lemma 3.2, $F : \overline{P}_\rho \to P$ is a completely continuous operator.

Furthermore, we will show that $\lambda x \neq Fx$ for every $\lambda \geq 1$ and $x \in \partial P_\rho$. Otherwise, let $x \in \partial P_\rho$ and $\lambda \geq 1$ such that $\lambda x = Fx$. Then for every $t \in I$, we have

$$\begin{aligned}
|x(t)| &\leq \lambda |x(t)| \\
&= \frac{\phi(0)}{A-1} + \frac{1}{(A-1)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, x_s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_s) ds \\
&\leq \frac{\phi(0)}{A-1} + \frac{M_\rho}{(A-1)\Gamma(\alpha)} \Big(\int_0^T (T-s)^\beta ds \Big)^{1-\alpha_1} + \frac{M_\rho}{\Gamma(\alpha)} \Big(\int_0^t (t-s)^\beta ds \Big)^{1-\alpha_1} \\
&\leq \frac{\phi(0)}{A-1} + \frac{M_\rho}{(A-1)\Gamma(\alpha)} \frac{1}{(1+\beta)^{1-\alpha_1}} T^{(1+\beta)(1-\alpha_1)} + \frac{M_\rho}{\Gamma(\alpha)} \frac{1}{(1+\beta)^{1-\alpha_1}} T^{(1+\beta)(1-\alpha_1)} \\
&\leq \frac{\phi(0)}{A-1} + \frac{AM_\rho}{(A-1)\Gamma(\alpha)(1+\beta)^{1-\alpha_1}} T^{(1+\beta)(1-\alpha_1)}.
\end{aligned}$$

Also, for every $t \in J$, we have

$$\begin{aligned}
|x(t)| &\leq \lambda |x(t)| \\
&\leq \frac{\phi(0)}{A(A-1)} + \frac{\|\phi\|}{A} + \frac{M_\rho}{A(A-1)\Gamma(\alpha)} \Big(\int_0^T (T-s)^\beta ds \Big)^{1-\alpha_1} \\
&\quad + \frac{M_\rho}{A\Gamma(\alpha)} \Big(\int_0^{T+t} (T+t-s)^\beta ds \Big)^{1-\alpha_1} \\
&\leq \frac{\phi(0)}{A(A-1)} + \frac{\|\phi\|}{A} + \frac{M_\rho}{A(A-1)\Gamma(\alpha)} \frac{1}{(1+\beta)^{1-\alpha_1}} T^{(1+\beta)(1-\alpha_1)} \\
&\quad + \frac{M_\rho}{A\Gamma(\alpha)(1+\beta)^{1-\alpha_1}} T^{(1+\beta)(1-\alpha_1)} \\
&\leq \frac{\phi(0)}{A(A-1)} + \frac{\|\phi\|}{A} + \frac{M_\rho}{(A-1)\Gamma(\alpha)(1+\beta)^{1-\alpha_1}} T^{(1+\beta)(1-\alpha_1)}.
\end{aligned}$$

Consequently, for every $t \in J \cup I$, it holds

$$\|x\|_{J\cup I} < \rho,$$

which contradicts with $x \in \partial P_\rho$.

So applying Lemma 2.2, we can obtain that L has at least a fixed point, what means that the boundary value problem $(1)-(2)$ has at least one positive solution x, such that

$$\|x\|_{J\cup I} < \rho.$$

Then, taking into account the formula of L and the fact $A > 1$, we easily conclude that

$$x(t) \geq \frac{\phi(0)}{A-1}, \quad t \in I,$$

which implies that

$$\|x\|_{J\cup I} \geq \frac{\phi(0)}{A-1}.$$

Observe that $\frac{\phi(0)}{A-1} < \rho$. Therefore, we finally have

$$\frac{\phi(0)}{A-1} \leq \|x\|_{J\cup I} < \rho.$$

The proof is complete.

In order to gain our second result, we need the following assumption:

(H_6) There exists an interval $E \subseteq I$, and functions $u : E \to [0, r]$, continuous $v : E \to [0, +\infty)$ with $\sup\{v(t) : t \in E\} > 0$ and nonincreasing $w : [0, +\infty) \to [0, +\infty)$ such that

$$f(t, y) \geq v(t)w(y(-u(t))), \qquad (t, y) \in E \times C^+(J).$$

Let

$$\mu := \frac{1}{A(A-1)} \int_E v(s)ds, \ \Lambda := \frac{\phi(0)}{A(A-1)} + \frac{\phi(-r)}{A}.$$

Theorem 3.2. Suppose that $(H_1) - (H_6)$ hold, also suppose that there exists $\tau > 0$, such that

$$\tau \leq \frac{T^{\alpha-1}}{\Gamma(\alpha)} \mu w(\tau). \tag{8}$$

Then the boundary value problem $(1) - (2)$ has at least one positive solution x, such that

$$d \leq \|x\|_{J\cup I} \leq \max\{\tau, \rho\},$$

where

$$d = \begin{cases} \rho, & \text{if } \tau > \rho, \\ \max\{\tau, \Lambda\}, & \text{if } \tau < \rho, \end{cases}$$

and ρ is the constant involved in (H_5) and $\tau \neq \rho$.

Proof. From Lemma 3.2, $F : \overline{P}_{\max\{\tau,\rho\}} \to P$ is a completely continuous map.

As we did in Theorem 3.1, we can prove that $Fx \neq \lambda x$ for every $\lambda \geq 1$ and $x \in \partial P_\rho$.

Now we will prove that $\|Fx\|_{J\cup I} \geq \|x\|_{J\cup I}$ for every $x \in \partial P_\tau$ and $t \in J \cup I$. By (H_4) and (H_6), we have

$$\begin{aligned} Fx(-r) &= \frac{\phi(0)}{A(A-1)} + \frac{1}{A(A-1)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, x_s)ds \\ &\quad + \frac{1}{A\Gamma(\alpha)} \int_0^{T-r} (T-r-s)^{\alpha-1} f(s, x_s)ds + \frac{\phi(-r)}{A} \\ &\geq \frac{1}{A(A-1)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, x_s)ds \end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{A(A-1)\Gamma(\alpha)} \int_E (T-s)^{\alpha-1} v(s) w(x(s-u(s)))ds \\
&\geq \frac{1}{A(A-1)\Gamma(\alpha)} T^{\alpha-1} \int_E v(s) w(x(s-u(s)))ds \\
&\geq \frac{T^{\alpha-1}}{\Gamma(\alpha)} \mu w(\tau) \\
&\geq \tau.
\end{aligned}$$

Therefore, for every $x \in \partial P_\tau$, we have $\|Fx\|_{J\cup I} \geq \|x\|_{J\cup I} = \tau$.

Applying Lemma 2.3, we get that L has a fixed point, which means that the boundary value problem $(1)-(2)$ has at least one positive solution x, such that

$$\min\{\tau, \rho\} \leq \|x\|_{J\cup I} \leq \max\{\tau, \rho\}.$$

But x is a positive solution of the boundary value problem$(1)-(2)$, this means that $x = Fx$ and it is easy to see that $x(-r) = Fx(-r) \geq \Lambda$, which implies that

$$\|x\|_{J\cup I} \geq \Lambda.$$

Moreover, it is clear that $\Lambda \leq \rho$. Hence

$$d \leq \|x\|_{J\cup I} \leq \max\{\tau, \rho\}.$$

The proof is complete.

Now, we give the following assumption $(H_6)'$, which is similar to assumption (H_6), when the function w is nondecreasing.

$(H_6)'$ There exists an interval $E \subseteq I$, and functions $u : E \to [0, r]$, continuous $v : E \to [0, +\infty)$ with $\sup\{v(t) : t \in E\} > 0$ and nondecreasing $w : [0, +\infty) \to [0, +\infty)$ such that

$$f(t, y) \geq v(t) w(y(-u(t))), \qquad (t, y) \in E \times C^+(J).$$

Theorem 3.3. Suppose that $(H_1)-(H_5), (H_6)'$ hold, and there exists $\tau > 0$ such that

$$\tau \leq \frac{T^{\alpha-1}}{\Gamma(\alpha)} \mu w(0). \tag{9}$$

Then the boundary value problem $(1)-(2)$ has at least one positive solution x, such that

$$d \leq \|x\|_{J\cup I} \leq \max\{\tau, \rho\},$$

where d is defined in Theorem 3.2 , ρ is the constant involved in (H_5) and $\tau \neq \rho$.

Proof. $F : \overline{P}_{\max\{\tau,\rho\}} \to P$ is a completely continuous map.

As we did in Theorem 3.1, we can prove that $Fx \neq \lambda x$ for every $\lambda \geq 1$ and $x \in \partial P_\rho$.

Now we will prove that $\|Fx\|_{J\cup I} \geq \|x\|_{J\cup I}$ for every $x \in \partial P_\tau$.

As in Theorem 3.2, using (H_4) and $(H_6)'$, we obtain

$$\begin{aligned}
Fx(-r) &\geq \frac{1}{A(A-1)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, x_s) ds \\
&\geq \frac{1}{A(A-1)\Gamma(\alpha)} T^{\alpha-1} \int_E v(s) w(x(s-u(s))) ds \\
&\geq \frac{T^{\alpha-1}}{\Gamma(\alpha)} \mu w(0) \\
&\geq \tau.
\end{aligned}$$

Therefore, for every $x \in \partial P_\tau$, we have $\|Fx\|_{J\cup I} \geq \|x\|_{J\cup I} = \tau$.

Applying Lemma 2.3, we get that L has at least a fixed point, which means that the boundary value problem $(1)-(2)$ has at least one positive solution x, such that

$$\min\{\tau, \rho\} \leq \|x\|_{J\cup I} \leq \max\{\tau, \rho\}.$$

Then

$$\|x\|_{J\cup I} \geq \Lambda, \quad \Lambda \leq \rho.$$

So $d \leq \|x\|_{J\cup I} \leq \max\{\tau, \rho\}$. The proof is complete.

Theorem 3.4. Suppose that $(H_1)-(H_6)$ (respectively $(H_1)-(H_5), (H_6)'$) hold, additionally, there exist $\tau > 0$ such that (9) (respectively (10)) holds. Then if $\rho < \tau$, the boundary value problem $(1)-(2)$ has at least two positive solutions x_1, x_2, such that

$$\frac{\phi(0)}{A-1} \leq \|x_1\|_{J\cup I} < \rho < \|x_2\|_{J\cup I} < \tau.$$

Example 3.1. Consider the boundary value problem

$$D^{\frac{2}{3}} x(t) = (\sin t)\sqrt{x_t}, \quad t \in I := [0,1], \tag{10}$$

$$5x_0 - x_1 = \frac{1}{2}, \tag{11}$$

where $f(t, x_t) = (\sin t)\sqrt{x_t}$, for $t \in I$, $\alpha = \frac{2}{3}$, $A = 5$ and $\phi(t) = \frac{1}{2}$. For any $x \in C[-\frac{1}{2}, 1]$, The x_t defined by $x_t(\theta) = x(t+\theta), -\frac{1}{2} \leq \theta \leq 0$.

For any given $l > 0, x \in P_l$, choose $m_l(t) = \sqrt{l}t$ and $\alpha_1 = \frac{1}{2}$ such that $m_l(t) \in L^{\frac{1}{\alpha_1}} I$. So the assumption (H_2) holds and $M_l = \|m_l(t)\|_{L^2 I} = \sqrt{\frac{l}{3}}$.

Now observe that assumptions $(H_1), (H_3) - (H_4)$ hold. For $\rho = 1.27$,

$$\frac{\phi(0)}{A-1} + \frac{\|\phi\|}{A} + \frac{AM_\rho}{(A-1)\Gamma(\alpha)(1+\beta)^{1-\alpha_1}} T^{(1+\beta)(1-\alpha_1)} = \frac{9}{40} + \frac{5\sqrt{\rho}}{4\Gamma(\frac{2}{3})} < \rho.$$

Hence, the condition (H_5) holds. Therefore, by applying Theorem 3.1, we can get that the boundary value problem (11)-(12) has at least one positive solution x satisfying $0.125 \leq \|x\|_{J\cup I} \leq 1.27$.

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