Solvability of a Volterra–Stieltjes integral equation in the class of functions having limits at infinity

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Abstract. The paper is devoted to the study of the solvability of a nonlinear Volterra–Stieltjes integral equation in the class of real functions defined, bounded and continuous on the real half-axis $\mathbb{R}_+$ and having finite limits at infinity. The considered class of integral equations contains, as special cases, a few types of nonlinear integral equations. In particular, that class contains the Volterra–Hammerstein integral equation and the Volterra–Wiener–Hopf integral equation, among others. The basic tools applied in our study is the classical Schauder fixed point principle and a suitable criterion for relative compactness in the Banach space of real functions defined, bounded and continuous on $\mathbb{R}_+$. Moreover, we will utilize some facts and results from the theory of functions of bounded variation.

Keywords: space of continuous and bounded functions, variation of function, function of bounded variation, Riemann–Stieltjes integral, criterion of relative compactness, integral equation, Schauder fixed point principle.

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1 Introduction

Integral equations appear in several branches of mathematics. They can be encountered especially in nonlinear analysis and its numerous applications in mathematical physics, engineering, mechanics, economics, biology, the theory of radiative transfer, vehicular traffic theory, queuing theory, etc. (see [9,10,12,15,21]). Obviously, the theory of integral equations is highly developed and forms a very important and applicable branch of nonlinear analysis. The survey of various types of integral equations and their applications can be found in [10,12,13,19–21], for example.

The goal of the paper is to discuss the solvability of a certain class of nonlinear integral equations of Volterra–Stieltjes type. The interest in the study of such integral equations was initiated mainly by the papers [3,7,8,17,22]. Indeed, it turns out that a lot of integral equations considered separately can be treated as special cases of the integral equations of Volterra–Stieltjes, Hammerstein–Stieltjes, and Urysohn–Stieltjes type. Moreover, the study of those

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types of integral equations is much simpler and allows to obtain deeper results than those found in the aforementioned papers. It is worthwhile mentioning that the review of results concerning integral equations of Volterra–Stieltjes type in contained in [2].

In this paper we are going to investigate the existence of solutions of a Volterra–Stieltjes integral equation having rather general form and including some important special cases of nonlinear integral equations. For example, the Volterra–Hammerstein integral equation and the nonlinear Volterra–Wiener–Hopf integral equation appear to be special cases of the integral equation in question.

In contrast to results obtained in other papers we will not assume that the integrands in the Volterra–Stieltjes integral equations investigated here satisfy Lipschitz (Hölder) conditions. Such an assumption was imposed in earlier mentioned papers and that fact caused that the results obtained in those papers were not sufficiently general and satisfactory. The details concerning the approach utilized in this paper will be presented later on.

We will look for solutions of the mentioned Volterra–Stieltjes integral equation in the Banach space $BC(\mathbb{R}_+)$ consisting of real functions defined, bounded and continuous on the interval $\mathbb{R}_+ = [0, \infty)$. We will be interested in finding such solutions in that Banach space which tend to finite limits at infinity.

In our considerations we will utilize the classical Schauder fixed point principle (cf. [12]) in conjunction with a certain criterion for relative compactness in the space $BC(\mathbb{R}_+)$. That criterion is associated with the required property of solutions mentioned above. Apart from this we will also apply some results from the theory of functions of bounded variation.

The results of the paper extend and generalize those obtained in [2,3,8,22] and in a lot of other papers as well. Moreover, we correct also some result obtained in [3].

2   Notation, definitions and auxiliary facts

This section is dedicated to recall auxiliary facts and results which will be utilized in the paper. At first we establish some notation.

We will use the symbols $\mathbb{R}$ and $\mathbb{R}_+$ to denote the sets of real and nonnegative real numbers, respectively. Our considerations will take place in the Banach space $BC(\mathbb{R}_+)$ consisting of all real functions defined, continuous and bounded on $\mathbb{R}_+$. The space $BC(\mathbb{R}_+)$ is equipped with the standard supremum norm

$$\|x\| = \sup \left\{ |x(t)| : t \in \mathbb{R}_+ \right\}.$$

It is worthwhile mentioning that the famous Arzelà–Ascoli criterion for relative compactness does not work in the space $BC(\mathbb{R}_+)$. Even more, in this space we do not know a criterion (i.e. necessary and sufficient condition) for relative compactness. However, we know a few convenient sufficient conditions for relative compactness [6]. For our further purposes we recall such a condition.

Theorem 2.1. *Let X be a nonempty and bounded subset of the space $BC(\mathbb{R}_+)$. Assume that functions belonging to X are locally equicontinuous on $\mathbb{R}_+$, i.e., for each $T > 0$ the functions from X are equicontinuous on the interval $[0, T]$. Moreover, assume that the following condition is satisfied: for any $\varepsilon > 0$ there exists a number $T > 0$ such that for every function $x \in X$ and for all $t, s \in [T, \infty)$ the inequality $|x(t) - x(s)| \leq \varepsilon$ is satisfied. Then the set X is relatively compact in the space $BC(\mathbb{R}_+)$.*

Remark 2.2. Let us observe that in the case when functions from the set X satisfy conditions indicated in Theorem 2.1 then those functions tend to finite limits at infinity uniformly with respect to the set X (cf. [5]).

In our further investigations we will frequently use the concept of the modulus of continuity. To define this concept take a function $x \in BC(\mathbb{R}_+)$ and fix arbitrarily $T > 0$. For $\varepsilon > 0$ define the following quantity:

$$\omega^T(x, \varepsilon) = \sup \left\{ |x(t) - x(s)| : t, s \in [0, T], \ |t - s| \leq \varepsilon \right\}.$$

This quantity is called the modulus of continuity of the function x on the interval $[0, T]$. Observe that $\omega^T(x, \varepsilon) \to 0$ as $\varepsilon \to 0$ which is a simple consequence of the uniform continuity of x on the interval $[0, T]$.

In what follows we discuss a few auxiliary facts concerning functions of bounded variation [1]. To this end assume that x is a real function defined on a fixed interval $[a, b]$. By the symbol $\bigvee_a^b x$ we will denote the variation of the function x on the interval $[a, b]$. In the case when $\bigvee_a^b x$ is finite we say that x is of bounded variation on $[a, b]$. In the case of a function $u(t, s) = u : [a, b] \times [c, d] \to \mathbb{R}$ we can consider the variation $\bigvee_{t=p}^q u(t, s)$ of the function $t \mapsto u(t, s)$ (i.e., the variation of the function $u(t, s)$ with respect to the variable t) on the interval $[p, q] \subset [a, b]$. In the similar way we define the quantity $\bigvee_{s=p}^q u(t, s)$.

We will not discuss the properties of the variation of functions of bounded variation. We refer to [1] for the mentioned properties.

Furthermore, assume that x and φ are two real functions defined on the interval $[a, b]$. Then, under some extra conditions (cf. [1]) we can define the Stieltjes integral (more precisely, the Riemann–Stieltjes integral) of the function x with respect to the function φ on the interval $[a, b]$ which is denoted by the symbol

$$\int_a^b x(t) d\varphi(t).$$

In such a case we say that x is Stieltjes integrable on the interval $[a, b]$ with respect to φ.

In the literature we may encounter a lot of conditions guaranteeing the Stieltjes integrability [1, 16, 18]. One of the most frequently exploited condition requires that x is continuous and φ is of bounded variation on $[a, b]$.

Next, we recall a few properties of the Stieltjes integral which will be used in our considerations (cf. [1]).

Lemma 2.3. *Assume that x is Stieltjes integrable on the interval $[a, b]$ with respect to a function φ of bounded variation. Then*

$$\left| \int_a^b x(t) d\varphi(t) \right| \leq \int_a^b |x(t)| \ d \left(\bigvee_a^t \varphi \right).$$

Lemma 2.4. *Let x_1, x_2 be Stieltjes integrable functions on the interval $[a, b]$ with respect to a nondecreasing function φ such that $x_1(t) \leq x_2(t)$ for $t \in [a, b]$. Then the following inequality is satisfied:*

$$\int_a^b x_1(t) d\varphi(t) \leq \int_a^b x_2(t) d\varphi(t).$$

In what follows we will use the Stieltjes integrals of the form

$$\int_a^b x(s) d_s g(t, s),$$

where $g : [a,b] \times [a,b] \to \mathbb{R}$ and the symbol d_s indicates the integration (in the Riemann–Stieltjes sense) with respect to the variable s.

Obviously, we can also consider the Stieltjes integral with integrand functions depending on two variables, for example

$$\int_a^b y(t,s) d_s g(t,s)$$

and so on.

3 Main result

At the beginning we recall a few facts concerning the nonlinear Volterra integral equation of the form

$$x(t) = a(t) + \int_a^t v(t,s,x(s)) ds, \tag{3.1}$$

where $t \in [a,b]$ and v is a given function defined on the set $\overline{\Delta} \times \mathbb{R}$ with real values. Here the symbol $\overline{\Delta}$ denotes the triangle

$$\overline{\Delta} = \{(t,s) : 0 \le s \le t \le b\}.$$

Obviously, instead of the bounded interval $[a,b]$ we may consider the nonlinear Volterra integral equation (3.1) on an unbounded interval $[a,\infty)$ i.e., $t \in [a,\infty)$ in Eq. (3.1). Further, for simplicity, we will use the interval $\mathbb{R}_+ = [0,\infty)$ instead of $[a,\infty)$.

On the other hand taking into account the classical linear Volterra integral equation having the form [21]

$$x(t) = a(t) + \int_a^t k(t,s)x(s) ds, \tag{3.2}$$

we will next investigate in the sequel the nonlinear Volterra integral equation having the form

$$x(t) = a(t) + \int_0^t k(t,s) f(s,x(s)) ds \tag{3.3}$$

for $t \in \mathbb{R}_+$. Obviously Eq. (3.2) is a special case of Eq. (3.3).

Let us observe that the nonlinear integral Volterra equation (3.3) can be treated as a counterpart of the so-called Hammerstein integral equation having the form [12,21]

$$x(t) = a(t) + \int_a^b k(t,s) f(s,x(s)) ds.$$

To make our considerations sufficiently general, we will investigate in the sequel to this paper the nonlinear Volterra integral equation having the form

$$x(t) = a(t) + \int_0^t k(t,s) f(t,s,x(s)) ds \tag{3.4}$$

for $t \in \mathbb{R}_+$. Assumptions concerning the kernel $k = k(t,s)$ for $(t,s) \in \Delta = \{(t,s) : 0 \le s \le t < \infty\}$ and the function $f(t,s,x) = f : \Delta \times \mathbb{R} \to \mathbb{R}$ will be formulated later.

Integral equations of form (3.4) were investigated in several papers and monographs. The general approach to Eq. (3.4) in classical function spaces comprising of functions continuous on $\mathbb{R}_+$ and satisfying some additional assumptions was presented in the papers [4, 11, 14], among others.

In those papers Eqs. (3.1) and (3.3) have been considered in the Banach space consisting of real functions defined, continuous on $\mathbb{R}_+$ and tempered by a suitable tempering function. But such an approach requires the assumption that the nonlinear part of Eqs. (3.3) or (3.4), i.e., the function $f = f(t,s,x)$ be sublinear. This means that there exist functions $L_1 = L_1(t)$, $L_2 = L_2(t)$ defined and continuous on $\mathbb{R}_+$ and such that

$$\left|f(t,s,x)\right| \leq L_1(s) + L_2(s)|x|$$

for $(t,s) \in \Delta$ and $x \in \mathbb{R}$.

Obviously, let us notice that the imposed assumption on sublinearity of the function f is rather restrictive in some situations and it does not allow us to obtain sufficiently general results concerning Eqs. (3.1), (3.3) and (3.4).

By the above indicated reasons we use a different approach in this paper and that is to replace the nonlinear Volterra integral equation (3.4) by the integral equation having the form

$$x(t) = a(t) + \int_0^t f\big(t,s,x(s)\big)d_sK(t,s), \tag{3.5}$$

where the integral appearing in the above equation is understood in the Stieltjes sense. Further we formulate suitable assumptions concerning the function $K = K(t,s)$ in Eq. (3.5) and we show that Eq. (3.4) can be treated as a special case of Eq. (3.5).

Notice that keeping in mind the form of Eq. (3.5) we can call it the Volterra–Stieltjes integral equation.

It is worthwhile mentioning that the Volterra–Stieltjes integral equation (3.5) was studied in the paper [22]. In that paper the author assumed, among others, that the function $f = f(t,s,x)$ appearing under the integral in (3.5) satisfies the condition

$$\left|f(t,s,x) - f(t,s,y)\right| \leq n(t,s)\phi\big(|x-y|\big),$$

where $n = n(t,s)$ is a continuous function on the triangle Δ and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing, $\phi(0) = 0$, and ϕ is continuous at zero.

Observe that the above assumption slightly generalizes the notion of Lipschitz (Hölder) continuity and is very restrictive. In the present paper we dispense with this assumption and, in general, impose assumptions other than those found in [22]. In this regard, our results essentially generalize the ones in [22].

Now, as we announced above, we investigate the solvability of the Volterra–Stieltjes integral equation (3.5). Our investigations will be located in the Banach space $BC(\mathbb{R}_+)$. We will study Eq. (3.5) assuming that the following conditions are satisfied.

(i) The function $a = a(t)$ is a member of the space $BC(\mathbb{R}_+)$ and $\lim_{t\to\infty} a(t)$ exists and is finite.

(ii) $f : \Delta \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there exists a nondecreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\left|f(t,s,x)\right| \leq \phi\big(|x|\big)$$

for all $(t,s) \in \Delta$ and $x \in \mathbb{R}$. Moreover, we assume that the function f is uniformly continuous on each set of the form $\Delta \times [-R, R]$, for arbitrary $R > 0$.

(iii) $K(t,s) = K : \Delta \to \mathbb{R}$ is a continuous function on the triangle Δ.

(iv) For arbitrarily fixed $t \in \mathbb{R}_+$ the function $s \mapsto K(t,s)$ is of bounded variation on the interval $[0,t]$.

(v) For any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $t_1, t_2 \in \mathbb{R}_+$, $t_1 < t_2$, $t_2 - t_1 \leq \delta$, the following inequality holds

$$\bigvee_{s=0}^{t_1} \left[K(t_2, s) - K(t_1, s) \right] \leq \varepsilon.$$

(vi) $K(t,0) = 0$ for each $t \geq 0$.

(vii) The function $t \mapsto \bigvee_{s=0}^{t} K(t,s)$ is bounded on $\mathbb{R}_+$.

(viii) The following limits hold:

$\lim_{T \to \infty} \left\{ \sup \left[\bigvee_{\tau=s}^{t} K(t,\tau) : T \leq s < t \right] \right\} = 0,$

$\lim_{T \to \infty} \left\{ \sup \left[\bigvee_{\tau=0}^{s} \left[K(t,\tau) - K(s,\tau) \right] : T \leq s < t \right] \right\} = 0,$

$\lim_{T \to \infty} \left\{ \sup \left[\left| f(t,\tau,y) - f(s,\tau,y) \right| : t, s \geqslant T, \tau \in \mathbb{R}_+, \tau \leqslant s, \tau \leqslant t, y \in [-R, R] \right] \right\} = 0,$

for each fixed $R > 0$.

In order to formulate our last assumption let us denote by $\overline{K}$ the following constant:

$$\overline{K} = \sup \left\{ \bigvee_{s=0}^{t} K(t,s) : t \in \mathbb{R}_+ \right\}.$$

Observe that $\overline{K} < \infty$, a consequence of assumption (vii). Because of this, we can state our last assumption as follows.

(ix) There exists a positive number r_0 satisfying the inequality

$$\|a\| + \overline{K} \phi(r) \leq r,$$

where ϕ is the nondecreasing function defined in (ii).

Now we are in a position to present the main result of the paper.

Theorem 3.1. *Under the assumptions (i)–(ix), the integral equation* (3.5) *has at least one solution* $x = x(t)$ *in the space* $BC(\mathbb{R}_+)$ *such that* $\|x\| \leq r_0$ *for some* $r_0 > 0$ *and for which* $\lim_{t \to \infty} x(t)$ *exists and is finite.*

In the proof of the above theorem we will use a few facts contained in the following lemmas, which can be found in [1,3].

Lemma 3.2. *Under assumptions (iii) and (iv), the function*

$$p \mapsto \bigvee_{s=0}^{p} K(t,s)$$

is continuous on the interval $[0,t]$ *for any fixed* $t \in \mathbb{R}_+$.

Lemma 3.3. *Let assumptions (iii)–(v) be satisfied. Then, for arbitrary fixed numbers $t_2 > 0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that if $t_1 < t_2$ and $t_2 - t_1 \leq \delta$, then*

$$\bigvee_{s=t_1}^{t_2} K(t_2, s) \leq \varepsilon.$$

Proof of Theorem 3.1. Let us consider the operator F defined on the space $BC(\mathbb{R}_+)$ by the formula

$$(Fx)(t) = a(t) + \int_0^t f(t, \tau, x(\tau)) d_\tau K(t, \tau),$$

where $t \in \mathbb{R}_+$. Notice that the function Fx is well-defined on the interval $\mathbb{R}_+$. We are going to use Schauder's theorem to prove that the operator F has a fixed point.

First we show that Fx is continuous on the interval $\mathbb{R}_+$. To this end, fix arbitrary numbers $T > 0$ and $\varepsilon > 0$. Further, take $t, s \in [0, T]$ such that $|t - s| \leq \varepsilon$. Without loss of generality we may assume that $s < t$. Then, taking into account our assumptions and Lemmas 2.3 and 2.4, we obtain:

$$
\begin{aligned}
&\left|(Fx)(t) - (Fx)(s)\right| \\
&\leq \left|a(t) - a(s)\right| + \left|\int_0^t f(t, \tau, x(\tau)) d_\tau K(t, \tau) - \int_0^s f(s, \tau, x(\tau)) d_\tau K(s, \tau)\right| \\
&\leq \omega^T(a, \varepsilon) + \left|\int_0^t f(t, \tau, x(\tau)) d_\tau K(t, \tau) - \int_0^s f(t, \tau, x(\tau)) d_\tau K(t, \tau)\right| \\
&\quad + \left|\int_0^s f(t, \tau, x(\tau)) d_\tau K(t, \tau) - \int_0^s f(t, \tau, x(\tau)) d_\tau K(s, \tau)\right| \\
&\quad + \left|\int_0^s f(t, \tau, x(\tau)) d_\tau K(s, \tau) - \int_0^s f(s, \tau, x(\tau)) d_\tau K(s, \tau)\right| \\
&\leq \omega^T(a, \varepsilon) + \left|\int_s^t f(t, \tau, x(\tau)) d_\tau K(t, \tau)\right| + \left|\int_0^s f(t, \tau, x(\tau)) d_\tau [K(t, \tau) - K(s, \tau)]\right| \\
&\quad + \left|\int_0^s \left[f(t, \tau, x(\tau)) - f(s, \tau, x(\tau))\right] d_\tau K(s, \tau)\right| \\
&\leq \omega^T(a, \varepsilon) + \phi(\|x\|) \int_s^t d_\tau \left(\bigvee_{p=s}^\tau K(t, p)\right) \\
&\quad + \phi(\|x\|) \int_0^s d_\tau \left(\bigvee_{p=0}^\tau [K(t, p) - K(s, p)]\right) + \omega_{\|x\|}^1(f, \varepsilon) \int_0^s d_\tau \left(\bigvee_{p=0}^\tau K(s, p)\right) \\
&\leq \omega^T(a, \varepsilon) + \phi(\|x\|) \bigvee_{\tau=s}^t K(t, \tau) + \phi(\|x\|) \bigvee_{\tau=0}^s [K(t, \tau) - K(s, \tau)] \\
&\quad + \omega_{\|x\|}^1(f, \varepsilon) \bigvee_{\tau=0}^s K(s, \tau), \tag{3.6}
\end{aligned}
$$

where we denoted

$$\omega_\beta^1(f, \varepsilon) = \sup\left\{\left|f(t, \tau, y) - f(s, \tau, y)\right| : t, s, \tau \in [0, T], \ y \in [-\beta, \beta], \ |t - s| \leq \varepsilon\right\}$$

for arbitrary $\beta > 0$.

Observe that in view of assumption (ii) we infer that $\omega_{\|x\|}^1(f, \varepsilon) \to 0$ as $\varepsilon \to 0$. Linking these facts with assumptions (i), (v), (vii) and Lemma 3.3, on the basis of estimate (3.6) we conclude

that the function Fx is continuous on the interval $[0, T]$. Since T was chosen arbitrarily this implies that Fx is continuous on the interval $\mathbb{R}_+$.

Now, we prove that the function Fx is bounded on $\mathbb{R}_+$, where $x \in BC(\mathbb{R}_+)$ is arbitrarily fixed. For the proof take $t \in \mathbb{R}_+$. Then, keeping in mind Lemmas 2.3 and 2.4, we deduce the following estimates:

$$
\begin{aligned}
|(Fx)(t)| &\leq |a(t)| + \left| \int_0^t f(t, \tau, x(\tau)) d_\tau K(t, \tau) \right| \\
&\leq \|a\| + \int_0^t |f(t, \tau, x(\tau))| d_\tau \left(\bigvee_{p=0}^\tau K(t, p) \right) \\
&\leq \|a\| + \phi(\|x\|) \int_0^t d_\tau \left(\bigvee_{p=0}^\tau K(t, p) \right) \\
&\leq \|a\| + \phi(\|x\|) \bigvee_{\tau=0}^t K(t, \tau) \leq \|a\| + \phi(\|x\|) \overline{K},
\end{aligned}
$$

where the constant $\overline{K}$ was introduced earlier.

The above inequality implies the following one:

$$\|Fx\| \leq \|a\| + \overline{K} \phi(\|x\|). \tag{3.7}$$

Hence we conclude that the function Fx is bounded on $\mathbb{R}_+$. Combining this fact with the continuity of the function Fx we infer that $Fx \in BC(\mathbb{R}_+)$ i.e., the operator F transforms the space $BC(\mathbb{R}_+)$ into itself. Additionally, keeping in mind that the function ϕ is nondecreasing on $\mathbb{R}_+$ (cf. assumption (ii)), from estimate (3.7) and assumption (ix) we deduce that there exists a positive number r_0 such that the operator F maps the ball $B_{r_0} = \{x \in BC(\mathbb{R}_+) : \|x\| \leq r_0\}$ into itself.

In what follows we show that the operator F is continuous on the ball B_{r_0}. To this end fix $\varepsilon > 0$. Next, take arbitrary functions $x, y \in B_{r_0}$ such that $\|x - y\| \leq \varepsilon$. Then, keeping in mind the imposed assumptions, for arbitrarily fixed $t \in \mathbb{R}_+$ we get:

$$
\begin{aligned}
|(Fx)(t) - (Fy)(t)| &\leq \left| \int_0^t f(t, \tau, x(\tau)) d_\tau K(t, \tau) - \int_0^t f(t, \tau, y(\tau)) d_\tau K(t, \tau) \right| \\
&\leq \int_0^t |f(t, \tau, x(\tau)) - f(t, \tau, y(\tau))| d_\tau \left(\bigvee_{p=0}^\tau K(t, p) \right) \\
&\leq \int_0^t \omega_{r_0}^3(f, \varepsilon) d_\tau \left(\bigvee_{p=0}^\tau K(t, p) \right) \\
&\leq \omega_{r_0}^3(f, \varepsilon) \bigvee_{\tau=0}^t K(t, \tau) \leq \overline{K} \omega_{r_0}^3(f, \varepsilon), \tag{3.8}
\end{aligned}
$$

where we denoted

$$\omega_{r_0}^3(f, \varepsilon) = \sup \left\{ |f(t, \tau, x) - f(t, \tau, y)| : t, \tau \in \mathbb{R}_+, \ x, y \in [-r_0, r_0], \ |x - y| \leq \varepsilon \right\}.$$

In view of the second part of assumption (ii) it is clear that $\omega_{r_0}^3(f, \varepsilon) \to 0$ as $\varepsilon \to 0$. This fact in conjunction with estimate (3.8) allows us to infer the desired conclusion concerning the continuity of the operator F on the ball B_{r_0}.

The next step in our proof depends on showing that the image of the ball B_{r_0} under the operator F i.e., the set $F(B_{r_0})$, is relatively compact in the space $BC(\mathbb{R}_+)$.

In the proof of this claim we will utilize the criterion of relative compactness contained in Theorem 2.1.

At first, we introduce two auxiliary functions $M(\varepsilon)$ and $N(\varepsilon)$ defined as follows:

$$M(\varepsilon) = \sup \left\{ \bigvee_{s=0}^{t_1} \left[K(t_2, s) - K(t_1, s) \right] : t_1, t_2 \in \mathbb{R}_+, \ t_1 < t_2, \ t_2 - t_1 \leq \varepsilon \right\},$$

$$N(\varepsilon) = \sup \left\{ \bigvee_{s=t_1}^{t_2} K(t_2, s) : t_1, t_2 \in \mathbb{R}_+, \ t_1 < t_2, \ t_2 - t_1 \leq \varepsilon \right\}.$$

Notice that in view of assumption (v) and Lemma 3.3 we have that $M(\varepsilon) \to 0$ and $N(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Further, fix arbitrarily $\varepsilon > 0$ and $T > 0$ and take a function $x \in B_{r_0}$. Choose $t, s \in [0, T]$ such that $|t - s| \leq \varepsilon$. Without loss of generality we may assume that $s < t$. Then, in virtue of estimate (3.6) we obtain

$$\left| (Fx)(t) - (Fx)(s) \right| \leq \omega^T(a, \varepsilon) + \phi(r_0) \left[M(\varepsilon) + N(\varepsilon) \right] + \overline{K} \omega_{r_0}^1(f, \varepsilon),$$

where the constant $\overline{K}$ and the modulus of continuity $\omega_{r_0}^1(f, \varepsilon)$ were defined earlier. Obviously, in view of the properties of the functions $M(\varepsilon)$, $N(\varepsilon)$ and $\omega_{r_0}^1(f, \varepsilon)$ (cf. assumption (ii)), the above estimate implies that functions from the set $F(B_{r_0})$ are equicontinuous on the interval $[0, T]$.

Now, utilizing assumptions (i) and (viii), we can find a number $T > 0$ such that for arbitrary $t, s \in [T, \infty)$ such that $s \leq t$, we have

$$\left| a(t) - a(s) \right| \leq \frac{\varepsilon}{4},$$

$$\bigvee_{\tau=s}^{t} K(t, \tau) \leq \frac{\varepsilon}{4\phi(r_0)}, \tag{3.9}$$

$$\bigvee_{\tau=s}^{t} \left[K(t, \tau) - K(s, \tau) \right] \leq \frac{\varepsilon}{4\phi(r_0)}$$

and

$$\left| f(t, u, y) - f(s, u, y) \right| \leq \frac{\varepsilon}{4\overline{K}}, \tag{3.10}$$

for $y \in [-r_0, r_0]$ and for arbitrary $u \in \mathbb{R}_+$, $u \leq t$.

Further, arguing similarly as we have done in order to obtain estimate (3.6), for arbitrarily fixed t, s such that $T \leq s < t$, for arbitrary $u \in \mathbb{R}_+$ such that $u \leq s$ and for $x \in B_{r_0}$, in view of (3.9) and (3.10) we obtain

$$\left| (Fx)(t) - (Fx)(s) \right| \leq \left| a(t) - a(s) \right| + \phi(r_0) \left(\bigvee_{\tau=s}^{t} K(t, \tau) + \bigvee_{\tau=0}^{s} \left[K(t, \tau) - K(s, \tau) \right] \right)$$

$$+ \int_0^s \frac{\varepsilon}{4\overline{K}} d_\tau \left(\bigvee_{p=0}^{\tau} K(s, p) \right) \leq \varepsilon.$$

Joining the above established properties of the set $F(B_{r_0})$ and keeping in mind Theorem 2.1 we conclude that the set $F(B_{r_0})$ is relatively compact in the space $BC(\mathbb{R}_+)$. Next, taking into account the continuity of the operator F on the set B_{r_0} and applying the classical Schauder fixed point principle we infer that there exists at least one fixed point x of the operator F

belonging to the ball B_{r_0}. Obviously, the function $x = x(t)$ is a solution of the Volterra–Stieltjes integral equation (3.5). Moreover, the function x belongs to the set $F(B_{r_0})$. Since, as we showed above, the set $F(B_{r_0})$ is relatively compact in the sense of Theorem 2.1 this implies that the function $x = x(t)$ has a finite limit at infinity. The proof is complete.   □

4   Remarks, further results and examples

Let us pay attention to the fact that the existence result contained in Theorem 3.1 generalizes results of the similar type contained in the papers [3,22].

Recall that in [3] we considered the Volterra–Stieltjes integral equation having the form

$$x(t) = a(t) + \int_0^t f(s, x(s)) d_s K(t, s). \tag{4.1}$$

Obviously, Eq. (4.1) is a particular case of Eq. (3.5). In this regard our existence result concerning Eq. (3.5) generalizes that form [3].

Unfortunately, the result obtained in [3] is not correct. Indeed, in the main existence result of that paper we had overlooked assumption (viii). Because of this, the reasonings in the proof of Theorem 5 in [3], which are located at the end of the proof of the mentioned theorem, are not correct.

It is worthwhile mentioning that the third equality from assumption (viii), in the case of Eq. (4.1) is superfluous since the function $f = f(s, x)$ appearing in that equation does not depend on the variable t as in the case of the function f in Eq. (3.5).

Thus, if we consider Eq. (4.1), then we should impose the same assumptions as those in Theorem 3.1 in the present paper but we should delete the third equality from assumption (viii).

Moreover, let us pay attention to the fact that in [3, 22] instead of assumptions (ii) the following requirements concerning Eq. (4.1) were imposed:

(ii′) $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ which is nondecreasing, $\psi(0) = 0$, $\lim_{t \to 0} \psi(t) = 0$ and such that

$$\left| f(s, x) - f(s, y) \right| \le \psi(|x - y|)$$

for all $s \in \mathbb{R}_+$ and $x, y \in \mathbb{R}$.

(ii″) The function $t \mapsto f(t, 0)$ is a member of $BC(\mathbb{R}_+)$.

It is easily seen that assumptions (ii′) and (ii″) imply assumption (ii). In fact, putting $y = 0$ in (ii′) we get

$$\left| f(s, x) \right| \le \left| f(s, 0) \right| + \psi(|x|).$$

In view of assumptions (ii″) we infer that there exists a constant H such that

$$\left| f(s, x) \right| \le H + \psi(|x|).$$

Thus, if we put $\phi(r) = H + \psi(r)$, we conclude that assumption (ii) is satisfied. It is easy to check that assumption (ii) implies assumption (ii″) but dose not implies assumption (ii′).

Further, we pay our attention to an important consequence of the first two equalities in assumption (viii). Indeed, we have the following theorem.

Theorem 4.1. *Assume that the function $K : \Delta \to \mathbb{R}$ satisfies assumptions (iv), (vii) and the first two equalities from assumption (viii). Then*

$$\lim_{t \to \infty} \bigvee_{\tau=0}^{t} K(t, \tau)$$

exists and is finite.

Proof. Let us fix a number $\varepsilon > 0$. In view of the first two limits in assumption (viii), we can find $T > 0$ such that for arbitrary t, s with $T \le s < t$ the following inequalities hold:

$$\bigvee_{\tau=s}^{t} K(t, \tau) \le \frac{\varepsilon}{2},$$

$$\bigvee_{\tau=0}^{s} \left[K(t, \tau) - K(s, \tau) \right] \le \frac{\varepsilon}{2}.$$

(4.2)

Further, taking t, s such that $T \le s < t$, in view of the properties of the variation of a function [1] and inequalities (4.2), we get

$$\left| \bigvee_{\tau=0}^{t} K(t, \tau) - \bigvee_{\tau=0}^{s} K(s, \tau) \right|$$

$$= \left| \bigvee_{\tau=0}^{s} K(t, \tau) + \bigvee_{\tau=s}^{t} K(t, \tau) - \bigvee_{\tau=0}^{s} K(s, \tau) \right|$$

$$\le \left| \bigvee_{\tau=s}^{t} K(t, \tau) \right| + \left| \bigvee_{\tau=0}^{s} K(t, \tau) - \bigvee_{\tau=0}^{s} K(s, \tau) \right|$$

$$\le \bigvee_{\tau=s}^{t} K(t, \tau) + \bigvee_{\tau=0}^{s} \left[K(t, \tau) - K(s, \tau) \right] \le \varepsilon.$$

This shows that the function $t \mapsto \bigvee_{\tau=0}^{t} K(t, \tau)$ satisfies the Cauchy condition (at infinity) on the interval $\mathbb{R}_+$ and completes the proof. $\square$

It is rather difficult to check if the converse assertion to that contained in Theorem 4.1 is true.

In what follows we are going to formulate a condition that will be convenient in applications and which will ensure that the function $K = K(t, s)$ satisfies assumption (v) (cf. [3]). To this end assume, as before, that $K(t, s) = K : \Delta \to \mathbb{R}$, where $\Delta = \left\{ (t, s) : 0 \le s \le t \right\}$. Then the announced condition can be formulated as follows:

(v′) For arbitrary $t_1, t_2 \in \mathbb{R}_+$ with $t_1 < t_2$ the function $s \mapsto K(t_2, s) - K(t_1, s)$ is nondecreasing (nonincreasing) on the interval $[0, t_1]$.

It can be shown that if the function $K(t, s)$ satisfies assumptions (v′) and (vi) then for arbitrarily fixed $s \in \mathbb{R}_+$ the function $t \mapsto K(t, s)$ is nondecreasing (nonincreasing) on the interval $[s, \infty]$ (cf. [3]). Moreover, under assumptions (iii), (v′) and (vi) the function K satisfies assumptions (v).

Remark 4.2. Let us mention that under assumptions (iii), (v′), and (vi) the second equality in assumption (viii) can be replaced by the following requirement:

(x_1) $\lim_{T\to\infty} \left\{ \sup\left[K(t,s) - K(s,s) : T \le s < t \right] \right\} = 0$

in the case when we assume in (v′) that the function $s \mapsto K(t_2,s) - K(t_1,s)$ is nondecreasing. Moreover in the case when we assume that the mentioned function is nonincreasing then the second equality in (viii) can be formulated in the form

(x_2) $\lim_{T\to\infty} \left\{ \sup\left[K(s,s) - K(t,s) : T \le s < t \right] \right\} = 0.$

Now, we will consider a few special cases of the Volterra–Stieltjes integral equation (3.5).

Firstly, let us take into account the nonlinear Volterra–Hammerstein integral equation having the form

$$x(t) = a(t) + \int_0^t k(t,s) f\bigl(t,s,x(s)\bigr) ds \tag{4.3}$$

for $t \in \mathbb{R}_+$. Assuming that the function $k(t,s) = k : \Delta \to \mathbb{R}$ is such that the function $s \mapsto k(t,s)$ is integrable on the interval $[0,t]$ for any fixed $t \in \mathbb{R}_+$, we can treat Eq. (4.3) as a special case of Eq. (3.5) if we put

$$K(t,s) = \int_0^s k(t,z) dz \tag{4.4}$$

for $(t,s) \in \Delta$. In particular, the well-known Volterra–Wiener–Hopf integral equation which has the form [3]

$$x(t) = a(t) + \int_0^t k(t-s) f\bigl(s,x(s)\bigr) ds$$

for $t \in \mathbb{R}_+$, can be regarded as a special case of Eq. (3.5) (even of Eq. (4.1)) if we put

$$K(t,s) = \int_0^s k(t-z) dz.$$

In what follows we focus on Eq. (4.3) and we formulate an existence theorem concerning this equation adapting assumptions of Theorem 3.1 appropriately. Obviously, to this end we have to replace assumptions involving the function $K = K(t,s)$, i.e., assumptions (iii)–(viii). Thus, in the light of (4.4) it is sufficient to require that:

(iii$_1$) The function $k : \Delta \to \mathbb{R}$ is continuous on the triangle Δ.

Moreover, due to the properties of the variation of a function [1] we conclude that assumption (iv) is superfluous in our situation.

Now, let us reformulate assumption (v). Keeping in mind the above mentioned property of the variation of a function we can state the following version of (v):

(v$_1$) For any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $t_1, t_2 \in \mathbb{R}_+$, $t_1 < t_2$, $t_2 - t_1 \le \delta$, the following inequality holds

$$\int_0^{t_1} \bigl| k(t_2,s) - k(t_1,s) \bigr| ds \le \varepsilon.$$

Indeed, for arbitrarily fixed $t_1, t_2 \in \mathbb{R}_+$, in view of the mentioned property of the variation of a function (cf. [1, Proposition 3.22]) and (4.4), we have

$$\bigvee_{s=0}^{t_1} \bigl[K(t_2,s) - K(t_1,s) \bigr] = \bigvee_{s=0}^{t_1} \int_0^s \bigl[k(t_2,z) - k(t_1,z) \bigr] dz$$
$$= \int_0^{t_1} \bigl| k(t_2,s) - k(t_1,s) \bigr| ds,$$

which justifies (v_1).

Further notice, that in virtue of (4.4), for arbitrarily fixed $t \in \mathbb{R}_+$ we have

$$K(t,0) = \int_0^0 k(t,z)dz = 0.$$

Thus, assumption (vi) is automatically satisfied.

Similarly as above we can reformulate assumption (vii) which takes the form:

(vii$_1$) The function $t \mapsto \int_0^t |k(t,s)| ds$ is bounded on $\mathbb{R}_+$.

Finally, we present new versions of the first two equalities from assumption (viii). Indeed, these versions have the following form:

(viii$_1$) $\lim_{T \to \infty} \left\{ \sup \left[\int_s^t |k(t,z)| dz : T \leq s < t \right] \right\} = 0.$

(viii$_1'$) $\lim_{T \to \infty} \left\{ \sup \left[\int_0^s |k(t,z) - k(s,z)| dz : T \leq s < t \right] \right\} = 0.$

Obviously , the last equality from assumption (viii) does not change and we present it in the following form:

(viii$_1''$) $\lim_{T \to \infty} \left\{ \sup \left[|f(t,\tau,y) - f(s,\tau,y)| : t,s \geqslant T, \ \tau \leqslant s, \ \tau \leqslant t, \ y \in [-R,R] \right] \right\} = 0$ for each $R > 0$.

Now we are prepared to formulate the existence theorem concerning Eq. (4.3).

Theorem 4.3. *Under assumptions (i), (ii), (iii$_1$), (v_1), (vii$_1$), (viii$_1$), (viii$_1'$), (viii$_1''$) and (ix), Eq. (4.3) has at least one solution $x = x(t)$ in the space $BC(\mathbb{R}_+)$ such that $\|x\| \leq r_0$ and the finite limit $\lim_{t \to \infty} x(t)$ does exist.*

Next we give an example illustrating our results.

Example 4.4. Consider the following Volterra–Hammerstein integral equation

$$x(t) = \frac{t^2+1}{10t^2+11} + \int_0^t \left(\frac{1}{(t^2+4)(s^2+1)} + se^{-2t} \right) \left(t^2 e^{-s-t} + \frac{s}{s^2+1} x^2(s) \right) ds, \quad (4.5)$$

where $t \in \mathbb{R}_+$.

Observe that the above integral equation is a particular case of Eq. (4.3) if we put:

$$a(t) = \frac{t^2+1}{10t^2+11}, \quad (4.6)$$

$$k(t,s) = \frac{1}{(t^2+4)(s^2+1)} + se^{-2t}, \quad (4.7)$$

$$f(t,s,x) = t^2 e^{-s-t} + \frac{s}{s^2+1} x^2. \quad (4.8)$$

We show that the functions involved in Eq. (4.5) satisfy assumptions of Theorem 4.3.

At the beginning notice that the function $a = a(t)$ defined by (4.6) satisfies assumption (i). Moreover, $\|a\| = \frac{1}{10}$.

Obviously, the function $f = f(t,s,x)$ is continuous on the set $\Delta \times \mathbb{R}$. Further, fixing arbitrarily $(t,s) \in \Delta$ and $x \in \mathbb{R}$, we obtain the following estimate

$$|f(t,s,x)| \leq t^2 e^{-t} + \frac{s}{s^2+1} x^2 \leq t^2 e^{-t} + \frac{1}{2}|x|^2 \leq \frac{4}{e} + \frac{1}{2}|x|^2.$$

Thus we see that the function f satisfies the inequality from assumption (ii) with $\phi(r) = \frac{4}{e} + \frac{1}{2}r^2$. Moreover, it is easy to check that the function f is uniformly continuous on every set of the form $\Delta \times [-R, R]$, for $R > 0$. Hence we conclude that the function f defined (4.8) satisfies assumption (ii).

The fact that the function $k = k(t, s)$ given by (4.7) is continuous on Δ is obvious. Thus assumption (iii$_1$) is satisfied.

To check assumption (v$_1$) let us fix arbitrarily $\varepsilon > 0$ and take $t_1, t_2 \in \mathbb{R}_+$ such that $t_1 < t_2$ and $t_2 - t_1 \leq \varepsilon$. Then we have:

$$
\begin{aligned}
\int_0^{t_1} |k(t_2, s) - k(t_1, s)| ds &= \int_0^{t_1} \left| \frac{1}{(t_2^2 + 4)(s^2 + 1)} + se^{-2t_2} - \frac{1}{(t_1^2 + 4)(s^2 + 1)} - se^{-2t_1} \right| ds \\
&\leq \int_0^{t_1} \left(\frac{1}{t_1^2 + 4} - \frac{1}{t_2^2 + 4} \right) \frac{1}{s^2 + 1} ds + \int_0^{t_1} \left(e^{-2t_1} - e^{-2t_2} \right) s \, ds \\
&= \left(\frac{1}{t_1^2 + 4} - \frac{1}{t_2^2 + 4} \right) \arctan t_1 + \left(e^{-2t_1} - e^{-2t_2} \right) \frac{t_1^2}{2} \\
&\leq \frac{\pi}{2} \left(\frac{1}{t_1^2 + 4} - \frac{1}{t_2^2 + 4} \right) + \frac{t_1^2}{2} e^{-2t_1} \left[1 - e^{-2(t_2 - t_1)} \right] \\
&\leq \frac{\pi}{2} \omega(g, \varepsilon) + \frac{1}{2e^2} \left(1 - e^{-2\varepsilon} \right),
\end{aligned}
\tag{4.9}
$$

where $\omega(g, \varepsilon)$ stands for the modulus of continuity of the function $g(t) = \frac{1}{t^2 + 4}$ on the interval $\mathbb{R}_+$. Taking into account that g is uniformly continuous on $\mathbb{R}_+$, from (4.9) we deduce that assumption (v$_1$) is satisfied.

Now we verify assumption (vii$_1$). To this end, let $t \in \mathbb{R}_+$ be arbitrary. Then we have:

$$
\begin{aligned}
\int_0^t |k(t, s)| ds &= \int_0^t \left[\frac{1}{(t^2 + 4)(s^2 + 1)} + se^{-2t} \right] ds \\
&= \frac{1}{t^2 + 4} \arctan t + \frac{t^2}{2} e^{-2t} \leq \frac{\pi}{8} + \frac{1}{2e^2}.
\end{aligned}
$$

This shows that assumption (vii$_1$) is satisfied.

To check assumption (viii$_1$) fix $T \geqslant 0$ and take t, s such that $T \leq s < t$. Then we have:

$$
\begin{aligned}
\int_s^t |k(t, z)| dz &= \int_s^t \left[\frac{1}{(t^2 + 4)(z^2 + 1)} + ze^{-2t} \right] dz \\
&= \frac{1}{t^2 + 4} \left(\arctan t - \arctan s \right) + \frac{1}{2} (t^2 - s^2) e^{-2t} \\
&\leq \frac{\arctan t}{t^2 + 4} + \frac{1}{2} t^2 e^{-2t}.
\end{aligned}
$$

Hence we deduce easily that assumption (viii$_1$) is fulfilled.

In a similar way we obtain:

$$
\begin{aligned}
\int_0^s |k(t, z) - k(s, z)| dz &= \int_0^s \left| \frac{1}{(t^2 + 4)(z^2 + 1)} + ze^{-2t} - \frac{1}{(s^2 + 4)(z^2 + 1)} - ze^{-2s} \right| dz \\
&= \left(\frac{1}{s^2 + 4} - \frac{1}{t^2 + 4} \right) \arctan s + \left(e^{-2s} - e^{-2t} \right) \frac{s^2}{2} \\
&\leq \frac{\pi}{2} \frac{1}{s^2 + 4} + \frac{s^2}{2} e^{-2s}.
\end{aligned}
$$

From the above estimate it is easy to deduce that assumption (viii$'_1$) is satisfied.

Further, for arbitrary fixed $R > 0$, $T > 0$ and $t, s, \tau \in \mathbb{R}_+$ such that $t, s \geqslant T$, $\tau \leq s$, $\tau \leq t$ and for $y \in [-R, R]$, we get

$$\begin{aligned} |f(t, \tau, y) - f(s, \tau, y)| &\leq |t^2 e^{-\tau - t} - s^2 e^{-\tau - s}| \\ &\leq |t^2 e^{-t} - s^2 e^{-s}|. \end{aligned}$$

Taking into account the fact that the function $h(t) = t^2 e^{-t}$ is continuous on $\mathbb{R}_+$ and $\lim_{t \to \infty} h(t) = 0$, from the above estimate we easily conclude that assumption (viii$''_1$) is satisfied.

Finally, let us take into account assumption (ix). Keeping in mind (4.4) we obtain:

$$\begin{aligned} \bigvee_{s=0}^{t} K(t, s) = \bigvee_{s=0}^{t} \int_0^s k(t, z) dz &= \int_0^t |k(t, s)| ds \\ &= \int_0^t \left(\frac{1}{(t^2 + 4)(s^2 + 1)} + s e^{-2t} \right) ds = \frac{1}{t^2 + 4} \arctan t + \frac{t^2}{2} e^{-2t}. \end{aligned}$$

Hence, we have

$$\overline{K} = \sup \left\{ \bigvee_{s=0}^{t} K(t, s) : t \in \mathbb{R}_+ \right\} \leq \frac{\pi}{8} + \frac{1}{2e^2} = 0.460366763 \ldots$$

Thus, inequality from assumption (ix) has now the form

$$\frac{1}{10} + \overline{K} \left(\frac{4}{e} + \frac{1}{2} r^2 \right) \leq r.$$

Equivalently, we get

$$\frac{\overline{K}}{2} r^2 - r + \frac{4\overline{K}}{e} + \frac{1}{10} \leq 0. \tag{4.10}$$

The discriminant of the quadratic polynomial from (4.10) can be evaluated as below:

$$1 - \frac{8\overline{K}^2}{e} - \frac{\overline{K}}{5} = 0.284186696 \ldots$$

Hence we can obtain the values of its roots:

$$r_1 = 1.014209535 \ldots,$$

and

$$r_2 = 3.330152744 \ldots$$

It is easily seen that for any $r_0 \in (r_1, r_2)$ Eq. (4.5) has at least one solution $x = x(t)$ in the space $BC(\mathbb{R}_+)$ such that $\|x\| \leq r_0$ and $\lim_{t \to \infty} x(t)$ exists and is finite.

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