Traveling waves of a delayed epidemic model with spatial diffusion

Wu Pei[1], **Qiaoshun Yang**[✉ 1] and **Zhiting Xu**[2]

[1]Department of Mathematics and Computer Science, Normal College of Jishou University
Jishou, Hunan, 416000, P. R. China
[2]School of Mathematical Sciences, South China Normal University, Guangzhou, 510631, P. R. China

Abstract. In this paper, we study the existence and non-existence of traveling waves for a delayed epidemic model with spatial diffusion. That is, by using Schauder's fixed-point theorem and the construction of Lyapunov functional, we prove that when the basic reproduction number $R_0 > 1$, there exists a critical number $c^* > 0$ such that for all $c > c^*$, the model admits a non-trivial and positive traveling wave solution with wave speed c. And for $c < c^*$, by the theory of asymptotic spreading, we further show that the model admits no non-trivial and non-negative traveling wave solution. And also, some numerical simulations are performed to illustrate our analytic results.

Keywords: traveling wave solutions, delayed epidemic model, Schauder fixed point theorem, Lyapunov functional.

2010 Mathematics Subject Classification: 35K57, 35C07, 92D30.

1   Introduction

In [3], the authors derived the following delayed epidemic model with the Beddington–DeAngelis incidence rate

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = A - \mu S(t) - \dfrac{\beta S(t)I(t)}{1 + \alpha_1 S(t) + \alpha_2 I(t)}, \\[2mm] \dfrac{\mathrm{d}I}{\mathrm{d}t} = \dfrac{\beta e^{-\mu\tau} S(t-\tau)I(t-\tau)}{1 + \alpha_1 S(t-\tau) + \alpha_2 I(t-\tau)} - (\mu + \alpha + \gamma)I(t), \end{cases} \tag{1.1}$$

where $S(t)$, $I(t)$ represent the number of susceptible individuals and infective individuals at time t, respectively. A is the recruitment rate of the population, μ is the natural death of the population, α is the death rate due to disease, β is the transmission rate, α_1 and α_2 are the parameters that measure the inhibitory effect, γ is the recovery rate of the infectious individuals, and τ is the incubation period.

[✉]Corresponding author. Email: yqs244@163.com

By constructing the suitable Lyapunov functional, the authors [3] determined the global asymptotic stability of model (1.1). Clearly, model (1.1) is one of ODE type, which could only reflect the epidemiological and demographic process as the time changes. We note that the spatial content of the environment has been ignored in model (1.1). To closely match the reality, considering a diffusive epidemic model of PDE type is natural and reasonable, therefore, it gives us the motivation to investigate the PDE type of model (1.1). Here, we propose the following delayed disease model with spatial diffusion

$$
\begin{cases}
\dfrac{\partial S}{\partial t} = d_1 \Delta S(t,x) + A - \mu S(t,x) - \dfrac{\beta S(t,x) I(t,x)}{1 + \alpha_1 S(t,x) + \alpha_2 I(t,x)}, \\[2mm]
\dfrac{\partial I}{\partial t} = d_2 \Delta I(t,x) + \dfrac{\beta e^{-\mu\tau} S(t-\tau,x) I(t-\tau,x)}{1 + \alpha_1 S(t-\tau,x) + \alpha_2 I(t-\tau,x)} - (\mu + \alpha + \gamma) I(t,x),
\end{cases}
\tag{1.2}
$$

in which $S(t,x)$ and $I(t,x)$ denote the number of susceptible individuals and infective individuals at time t and position $x \in \mathbb{R}^n$, respectively. $d_1, d_2 > 0$ are the diffusion rates, Δ is the Laplacian operator. The parameters $A, \mu, \beta, \alpha_1, \alpha_2, \gamma, \tau$ are positive constants as in model (1.1).

In the biological context, to better understand the geographic spread of infectious diseases, epidemic waves play a key role in studying the spatial spread of infectious diseases. Biologically speaking, the existence of an epidemic wave implies that the disease can invade successfully and an epidemics arises. The traveling wave describes the epidemic wave moving out from an initial disease-free equilibrium to the endemic equilibrium with a constant speed. The wave speed c may explain the spatial spread speed of the disease, which may measure how fast the disease invades geographically. Recently, many authors have studied the existence of traveling wave solutions of various epidemic models, see, for example, [1,2,4,5,7,9,10,13,15–19,21–25] and references therein.

In this paper, we will study the existence and non-existence of traveling waves for model (1.2). We employ Schauder's fixed point theorem combining with the upper-lower solutions to establish the existence theorem (Theorem 3.2). Namely, we will show that when the basic reproduction number $R_0 > 1$, there exists $c^* > 0$ such that (1.2) has a positive traveling wave solution if $c > c^*$. Further, we shall construct the appropriate Lyapunov functional to show that the traveling wave converges to the endemic steady state $E^* = (S^*, I^*)$ as $t \to +\infty$. Moreover, by the theory of asymptotic spreading, we conclude the non-existence of traveling wave solutions for model (1.2) when $R_0 > 1$ and $c \in (0, c^*)$ (Theorem 3.3). Some numerical simulations are carried out to validate the theoretical results.

This paper is organized as follows. In Section 2, we give some preliminaries, that is, we establish the well-posedness for model (1.2), construct a pair of upper-lower solutions, and verify the conditions of the Schauder fixed point theorem. In Section 3, we give and show the existence and non-existence of traveling waves of model (1.2). Some numerical simulations are given in Section 4.

2 Preliminaries

2.1 The well-posedness

For simplicity, let

$$
\bar{S} = \frac{\mu}{A} S, \quad \bar{I} = \frac{\mu}{A} I, \quad \bar{\alpha}_1 = \frac{\alpha_1}{\mu} A, \quad \bar{\alpha}_2 = \frac{\alpha_2}{\mu} A, \quad \bar{\beta} = \frac{\beta}{\mu} A, \quad r = \mu + \alpha + \gamma,
$$

and dropping the bars on S, I, α_1, α_2, β, we obtain the following model

$$\begin{cases} \dfrac{\partial S}{\partial t} = d_1 \Delta S(t,x) + \mu(1 - S(t,x)) - \beta f(S,I)(t,x), \\ \dfrac{\partial I}{\partial t} = d_2 \Delta I(t,x) + \beta e^{-\mu\tau} f(S,I)(t-\tau,x) - rI(t,x), \end{cases} \tag{2.1}$$

where

$$f(S,I)(t,x) = \frac{S(t,x)I(t,x)}{1 + \alpha_1 S(t,x) + \alpha_2 I(t,x)},$$

under the initial conditions

$$S(t,x) = \varphi_1(t,x) \geq 0, \qquad I(t,x) = \varphi_2(t,x) \geq 0 \tag{2.2}$$

for all $(t,x) \in [-\tau,0] \times \mathbb{R}^n$, where $\varphi_i(t,x)$ $(i = 1,2)$ are nonnegative and continuous in $[-\tau,+\infty) \times \mathbb{R}$, but not identically zero.

As in [3], we define the basic reproduction number R_0 as

$$R_0 = \frac{\beta e^{-\mu\tau}}{r(1 + \alpha_1)}.$$

By a direct computation, we get the following conclusion.

Lemma 2.1.

(1) *System* (2.1) *always has a disease-free equilibrium* $E_0 = (1,0)$.

(2) *If* $R_0 > 1$, *then system* (2.1) *has a unique endemic equilibrium* $E^* = (S^*, I^*)$, *where*

$$S^* = \frac{r + \mu\alpha_2 e^{-\mu\tau}}{r[\alpha_1(R_0 - 1) + R_0] + \mu\alpha_2 e^{-\mu\tau}}, \qquad I^* = \frac{\mu e^{-\mu\tau}(\alpha_1 + 1)(R_0 - 1)}{r[\alpha_1(R_0 - 1) + R_0] + \mu\alpha_2 e^{-\mu\tau}}.$$

Next, we consider the positive invariance and uniform boundedness of solutions for the initial value problem of system (2.1)–(2.2).

Let $\mathbb{X} := BUC(\mathbb{R}^n, \mathbb{R}^2)$ be the set of all bounded and uniformly continuous functions from $\mathbb{R}^n$ to $\mathbb{R}^2$, and $\mathbb{X}_+ := BUC(\mathbb{R}^n, \mathbb{R}^2_+)$. Then $\mathbb{X}_+$ is a closed cone of $\mathbb{X}$ and induces a partial ordering on $\mathbb{X}$. With the usual supremum norm, it follows that $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ is a Banach space. Clearly, any vector in $\mathbb{R}^2$ can be regarded as an element in $\mathbb{X}$. For $u = (u_1, u_2), v = (v_1, v_2) \in \mathbb{X}$, we write $u \geq v(u \leq v)$ provided $u_i(x) \geq v_i(x)$ $(u_i(x) \leq v_i(x)), i = 1, 2, x \in \overline{\Omega}$. For $\tau \geq 0$, we define $\mathbb{C} = C([-\tau,0], \mathbb{X})$ with the supremum norm and $\mathbb{C}_+ = C([-\tau,0], \mathbb{X}_+)$. Then $(\mathbb{C}, \mathbb{C}_+)$ is an ordered Banach space. As usual, we identify an element $\varphi \in \mathbb{C}$ as a function from $[-\tau,0] \times \mathbb{X}_+$ into $\mathbb{R}^2$ by $\varphi(x,s) = \varphi(s)(x)$. For any given function $u : [-\tau,\sigma) \to \mathbb{X}$ for $\sigma > 0$, we define $u_t \in \mathbb{C}$ by $u_t(\theta) = u(t + \theta), \theta \in [-\tau,0]$. Let $\mathbf{D} = (d_1, d_2)^T$. By [6, Theorem 1.5], it follows that $\mathbb{X}$-realization $\mathbf{D}\Delta X$ of $\mathbf{D}\Delta$ generates an analytic semigroup $\mathscr{T}(t)$ on $\mathbb{X}$.

For any $\varphi = (\varphi_1, \varphi_2) \in \mathbb{C}_+$ and $x \in \mathbb{R}^n$, define $F = (f_1, f_2) : \mathbb{C}_+ \to \mathbb{X}$ by

$$f_1(\varphi)(x) = \mu(1 - \varphi_1(0,x)) - \beta f(\varphi_1, \varphi_2)(0,x),$$
$$f_2(\varphi)(x) = \beta e^{-\mu\tau} f(\varphi_1, \varphi_2)(-\tau, x) - r\varphi_2(0,x).$$

Then F is Lipschitz continuous in any bounded subset of $\mathbb{C}_+$. Rewriting (2.1) and (2.2) as the following abstract functional differential equation

$$\begin{cases} \dfrac{du}{dt} = \mathbf{A}u + F(u_t), t \geq 0, \ u_t \in \mathbb{C}, \\ u_0 = \varphi \in \mathbb{C}, \end{cases}$$

where $u = (S, I)$, $\mathbf{A}u := (d_1 \Delta u_1, d_2 \Delta u_2)^T$, $\varphi = (\varphi_1, \varphi_2)$. Define

$$[\mathbf{0}, \mathbf{M}]_{\mathbb{C}} = \left\{ \varphi \in \mathbb{C} : 0 \leq \varphi(\theta, x) \leq \mathbf{M}, \quad \forall x \in \mathbb{R}^n, \theta \in [-\tau, 0] \right\}$$

with $\mathbf{0} := (0, 0)$, and $\mathbf{M} := \left(1, \frac{1}{\alpha_2} \left(\frac{\beta}{r} e^{-\mu\tau} - \alpha_1 - 1 \right) \right)$ for $R_0 > 1$.

Theorem 2.2. *For any given initial function $\varphi = (\varphi_1, \varphi_2) \in [\mathbf{0}, \mathbf{M}]_{\mathbb{C}}$, there exists a unique nonnegative solution $u(t, x; \varphi)$ of (2.1)–(2.2) on $[0, \infty)$ and $u_t \in [\mathbf{0}, \mathbf{M}]_{\mathbb{C}}$ for $t \geq 0$.*

Proof. For any given $\varphi = (\varphi_1, \varphi_2) \in [\mathbf{0}, \mathbf{M}]_{\mathbb{C}}$ and $\kappa \geq 0$, we have

$$\varphi(0, x) + \kappa F(\varphi)(x) = \begin{pmatrix} \varphi_1(0, x) + \kappa(\mu(1 - \varphi_1(0, x) - \beta f(\varphi_1, \varphi_2)(0, x) \\ \varphi_2(0, x) + \kappa(\beta e^{-\mu\tau} f(\varphi_1, \varphi_2)(-\tau, x) - r\varphi_2(0, x)) \end{pmatrix}$$

$$\geq \begin{pmatrix} \varphi_1(0, x) \left(1 - \kappa \left(\mu + \frac{\beta}{\alpha_2} \right) \right) \\ (1 - \kappa r)\varphi_2(0, x) \end{pmatrix}.$$

Hence, for $0 \leq \kappa < \min\left\{ \frac{1}{r}, \frac{\alpha_2}{\alpha_2 \mu + \beta} \right\}$, it follows

$$\varphi(0, x) + \kappa F(\varphi)(x) \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

On the other hand, for any sufficiently small $\kappa > 0$ and any fixed $u_2 > 0$, the functions $u_1 + \kappa(\mu(1 - u_1) - \beta f(u_1, u_2)$ is increasing for $u_1 > 0$; and $(1 - \kappa r)u_2 + \kappa\beta e^{-\mu\tau} f(u_1, u_2)$ is increasing for u_1. Then, for $0 < \kappa < \frac{1}{r}$,

$$\varphi(0, x) + \kappa F(\varphi)(x) \leq \begin{pmatrix} 1 \\ (1 - \kappa r)\varphi_2(0, x) + \kappa\beta e^{-\mu\tau} f(1, \varphi_2)(-\tau, x) \end{pmatrix}$$

$$\leq \begin{pmatrix} 1 \\ \frac{1}{\alpha_2} \left(\frac{\beta}{r} e^{-\mu\tau} - \alpha_1 - 1 \right) \end{pmatrix}.$$

Hence, $\varphi(0, x) + \kappa F(\varphi)(x) \in [\mathbf{0}, \mathbf{M}]_{\mathbb{C}}$. This implies

$$\lim_{\kappa \to 0+0} \frac{1}{\kappa} \operatorname{dist}(\varphi(0, x) + \kappa F(\varphi)(x), [\mathbf{0}, \mathbf{M}]_{\mathbb{C}}) = 0, \qquad \forall \varphi \in [\mathbf{0}, \mathbf{M}]_{\mathbb{C}}.$$

Let $\mathbb{K} = [\mathbf{0}, \mathbf{M}]_{\mathbb{C}}$, $S(t, x) = \mathscr{T}(t - s)$ and $B(t, \varphi) = F(\varphi)$. It follows from [11, Corollary 4] that (2.1)–(2.2) admit a unique mild solution $u(t, \varphi)$ with $u(t, \varphi) \in [\mathbf{0}, \mathbf{M}]_{\mathbb{C}}$ for any $t \in [0, \infty)$. Furthermore, since the semigroup $\mathscr{T}(t)$ is analytic, the mild solution $u(t, \varphi)$ of (2.1)–(2.2) is classic for $t > \tau$ (see [20, Corollary 2.2.5]). $\qquad \square$

2.2 The wave equations and the upper and lower solutions

In this paper, we mainly deal with the existence of traveling waves of system (2.1) connecting the disease-free equilibrium $E_0(1, 0)$ and the endemic equilibrium $E^*(S^*, I^*)$. Without loss generality, we consider $n = 1$. A traveling wave solution of (2.1) is a special type of solution of system (2.1) with the form $(S(t, x), I(t, x)) = (S(x + ct), I(x + ct))$, here $c > 0$ is the wave speed, and letting $x + ct$ by t, which satisfies the following wave equation

$$\begin{cases} cS'(t) = d_1 S''(t) + \mu(1 - S(t)) - \beta f(S, I)(t), \\ cI'(t) = d_2 I''(t) + \beta e^{-\mu\tau} f(S, I)(t - c\tau) - r I(t), \end{cases} \tag{2.3}$$

and the boundary conditions

$$S(-\infty) = 1, \quad S(+\infty) = S^*, \quad I(-\infty) = 0, \quad I(+\infty) = I^*. \tag{2.4}$$

Linearizing of the second equation of (2.3) at $E_0(1,0)$, we get the characteristic equation

$$\Delta(\lambda, c) = d_2\lambda^2 - c\lambda + \frac{\beta}{1 + \alpha_1}e^{-(\lambda c + \mu)\tau} - r = 0.$$

It is easy to show the following lemma, see, [13, Lemma 4.4] or [21, Lemma 3.1].

Lemma 2.3. *Assume that $R_0 > 1$. Then there exist two positive constants $\lambda^* > 0$ and $c^* > 0$ such that*

$$\Delta(\lambda^*, c^*) = 0, \qquad \frac{\partial \Delta}{\partial \lambda}(\lambda, c)|_{(\lambda^*, c^*)} = 0.$$

Furthermore,

(1) *If $0 < c < c^*$, then $\Delta(\lambda, c) > 0$ for all $\lambda \in [0, \infty)$.*

(2) *If $c > c^*$, then the equation $\Delta(\lambda, c) = 0$ has two positive roots $\lambda_1(c)$ and $\lambda_2(c)$ with $0 < \lambda_1(c) < \lambda^* < \lambda_2(c)$ such that*

$$\Delta(\lambda, c) \begin{cases} > 0, & \forall \, \lambda \in (0, \lambda_1(c)) \cup (\lambda_2(c), +\infty), \\ < 0, & \forall \, \lambda \in (\lambda_1(c), \lambda_2(c)). \end{cases}$$

In this subsection, we assume that $R_0 > 1$. In addition, we fix a positive constant $c > c^*$ and always denote $\lambda_i(c) = \lambda_i$, $i = 1, 2$.

Now, we define four continuous functions as following

$$\overline{S}(t) = 1, \qquad \underline{S}(t) = \max\left\{1 - \frac{1}{\sigma}e^{\sigma t}, \frac{\mu\alpha_2}{\mu\alpha_2 + \beta}\right\},$$

and

$$\overline{I}(t) = \min\left\{e^{\lambda_1 t}, \frac{1}{\alpha_2}\left(\frac{\beta}{r}e^{-\mu\tau} - \alpha_1 - 1\right)\right\}, \qquad \underline{I}(t) = \max\{e^{\lambda_1 t}(1 - Me^{\varepsilon t}), 0\}$$

for $t \in \mathbb{R}$, where σ, M, ε are three positive constants to be determined in the following lemmas.

Lemma 2.4. *The functions $\overline{S}(t)$ and $\overline{I}(t)$ satisfy the inequality*

$$d_2\,\overline{I}''(t) - c\,\overline{I}'(t) + \beta e^{-\mu\tau}f(\overline{S}, \overline{I})(t - c\tau) - r\overline{I}(t) \leq 0, \tag{2.5}$$

for all $t \neq t_1 := \frac{1}{\lambda_1}\ln\frac{1}{\alpha_2}\left(\frac{\beta}{r}e^{-\mu\tau} - \alpha_1 - 1\right)$.

Proof. If $t < t_1$, then $\overline{I}(t) = e^{\lambda_1 t}$. Note that $\overline{S}(t) = 1$ and $\overline{I}(t) \leq e^{\lambda_1 t}$ for all $t \in \mathbb{R}$, then

$$\begin{aligned}
d_2\overline{I}''(t) &- c\overline{I}'(t) + \beta e^{-\mu\tau}f(\overline{S}, \overline{I})(t - c\tau) - r\,\overline{I}(t) \\
&\leq d_2\overline{I}''(t) - c\overline{I}'(t) + \frac{\beta}{1 + \alpha_1}e^{-\mu\tau}\overline{I}(t - c\tau) - r\,\overline{I}(t) \\
&\leq e^{\lambda_1 t}\Delta(\lambda_1, c) \\
&= 0.
\end{aligned}$$

If $t > t_1$, then $\bar{I}(t) = \frac{1}{\alpha_2}\left(\frac{\beta}{r}e^{-\mu\tau} - \alpha_1 - 1\right)$. In view of the fact that $\bar{S}(t) = 1$ and $\bar{I}(t) \leq \frac{1}{\alpha_2}\left(\frac{\beta}{r}e^{-\mu\tau} - \alpha_1 - 1\right)$ for all $t \in \mathbb{R}$, then

$$d_2\bar{I}''(t) - c\bar{I}'(t) + \beta e^{-\mu\tau}f(\bar{S}, \bar{I})(t - c\tau) - r\bar{I}(t)$$

$$\leq d_2\bar{I}''(t) - c\bar{I}'(t) + \beta e^{-\mu\tau}\frac{\frac{1}{\alpha_2}\left(\frac{\beta}{r}e^{-\mu\tau} - \alpha_1 - 1\right)}{1 + \alpha_1 + \frac{\alpha_2}{\alpha_2}\left(\frac{\beta}{r}e^{-\mu\tau} - \alpha_1 - 1\right)} - r\bar{I}(t)$$

$$= \frac{r}{\alpha_2}\left(\frac{\beta}{r}e^{-\mu\tau} - \alpha_1 - 1\right) - \frac{r}{\alpha_2}\left(\frac{\beta}{r}e^{-\mu\tau} - \alpha_1 - 1\right)$$

$$= 0.$$

This completes the proof.   □

Lemma 2.5. *Let $\sigma \in (0, \lambda_1)$ be sufficiently small. Then the functions $\underline{S}(t), \bar{I}(t)$ satisfy the inequality*

$$d_1\underline{S}''(t) - c\underline{S}'(t) + \mu(1 - \underline{S}(t)) - \beta f(\underline{S}, \bar{I})(t) \geq 0, \tag{2.6}$$

for all $t \neq t_2 := \frac{1}{\sigma}\ln\frac{\sigma\beta}{\mu\alpha_2 + \beta} < 0$.

Proof. If $t \geq t_2$, then $\underline{S}(t) = \frac{\mu\alpha_2}{\mu\alpha_2 + \beta}$. Hence,

$$d_1\underline{S}''(t) - c\underline{S}'(t) + \mu(1 - \underline{S}(t)) - \beta f(\underline{S}, \bar{I})(t)$$

$$\geq d_1\underline{S}''(t) - c\underline{S}'(t) + \mu(1 - \underline{S}(t)) - \frac{\beta}{\alpha_2}\underline{S}(t)$$

$$= \mu(1 - \underline{S}(t)) - \frac{\beta}{\alpha_2}\underline{S}(t)$$

$$= 0.$$

If $t < t_2$, then $\underline{S}(t) = 1 - \frac{1}{\sigma}e^{\sigma t}$. Note that $\bar{I}(t) \leq e^{\lambda_1 t}$ for all $t \in \mathbb{R}$, we have

$$d_1\underline{S}''(t) - c\underline{S}'(t) + \mu(1 - \underline{S}(t)) - \beta f(\underline{S}, \bar{I})(t)$$

$$\geq d_1\underline{S}''(t) - c\underline{S}'(t) + \mu(1 - \underline{S}(t)) - \beta\underline{S}(t)\bar{I}(t)$$

$$\geq \left(-d_1\sigma + c + \frac{\mu}{\sigma}\right)e^{\sigma t} - \beta e^{\lambda_1 t}$$

$$= \left(-d_1\sigma + c + \frac{\mu}{\sigma} - \beta e^{(\lambda_1 - \sigma)t}\right)e^{\sigma t}.$$

It follows from the fact that $e^{(\lambda_1 - \sigma)t} < \left(\frac{\sigma\beta}{\beta + \mu\alpha_2}\right)^{\frac{\lambda_1 - \sigma}{\sigma}}$ for $t < t_2$ that

$$d_1\underline{S}''(t) - c\underline{S}'(t) + \mu(1 - \underline{S}(t)) - \beta f(\underline{S}, \bar{I})(t) \geq e^{\sigma t}\left(-d_1\sigma + c + \frac{\mu}{\sigma} - \beta\left(\frac{\sigma\beta}{\beta + \mu\alpha_2}\right)^{\frac{\lambda_1 - \sigma}{\sigma}}\right).$$

Note that $\lim_{\sigma \to 0+0}\left(\frac{\sigma\beta}{\beta + \mu\alpha_2}\right)^{\frac{\lambda_1 - \sigma}{\sigma}} = 0$. Then, for sufficiently small $\sigma > 0$,

$$-d_1\sigma + c + \frac{\mu}{\sigma} - \beta\left(\frac{\sigma\beta}{\beta + \mu\alpha_2}\right)^{\frac{\lambda_1 - \sigma}{\sigma}} > 0, \qquad \forall t < t_2,$$

which implies (2.6) holds for $t < t_2$. This completes the proof.   □

Lemma 2.6. *Let* $0 < \varepsilon < \min\{\sigma, \lambda_1, \lambda_2 - \lambda_1\}$ *and* $M > 1$ *sufficiently large. Then functions* $\underline{S}(t), \underline{I}(t)$ *satisfy the inequality*

$$d_2 \underline{I}''(t) - c\underline{I}'(t) + \beta e^{-\mu\tau} f(\underline{S}, \underline{I})(t - c\tau) - r\underline{I}(t) \geq 0, \tag{2.7}$$

for all $t \neq t_3 := \frac{1}{\varepsilon} \ln \frac{1}{M}$.

Proof. If $t \geq t_3$, then inequality (2.7) holds immediately since $\underline{I}(t) = 0$ on $[t_3, \infty)$.

If $t < t_3$, then $\underline{I}(t) = e^{\lambda_1 t}(1 - Me^{\varepsilon t})$. In view of the facts

$$e^{\lambda_1 t}(1 - Me^{\varepsilon t}) \leq \underline{I}(t) \leq e^{\lambda_1 t}, \qquad 1 - \frac{1}{\sigma}e^{\sigma t} \leq \underline{S}(t) \leq 1, \qquad \forall \, t \in \mathbb{R},$$

then, for $t < t_3$,

$$
\begin{aligned}
f(\underline{S}, \underline{I})(t - c\tau) &\geq \frac{\underline{S}(t - c\tau)\underline{I}(t - c\tau)}{1 + \alpha_1 + \alpha_2 \underline{I}(t - c\tau)} \\
&\geq \frac{1}{1 + \alpha_1}\underline{S}(t - c\tau)\underline{I}(t - c\tau)\left(1 - \frac{\alpha_2}{1 + \alpha_1}\underline{I}(t - c\tau)\right) \\
&\geq \frac{1}{1 + \alpha_1}\left(1 - \frac{1}{\sigma}e^{\sigma(t - c\tau)}\right)\underline{I}(t - c\tau)\left(1 - \frac{\alpha_2}{1 + \alpha_1}\underline{I}(t - c\tau)\right) \\
&\geq \frac{1}{1 + \alpha_1}\left(\underline{I}(t - c\tau) - \frac{1}{\sigma}e^{\sigma(t - c\tau)}\underline{I}(t - c\tau) - \frac{\alpha_2}{1 + \alpha_1}\underline{I}^2(t - c\tau)\right).
\end{aligned}
$$

Hence, for $t < t_3$,

$$
\begin{aligned}
&d_2\underline{I}''(t) - c\underline{I}'(t) + \beta e^{-\mu\tau}f(\underline{S}, \underline{I})(t - c\tau) - r\underline{I}(t) \\
&\geq d_1\underline{I}''(t) - c\underline{I}'(t) + \frac{\beta}{1 + \alpha_1}e^{-\mu\tau}\underline{I}(t - c\tau) - r\underline{I}(t) \\
&\quad - \frac{\beta}{\sigma(1 + \alpha_1)}e^{-\mu\tau}e^{\sigma(t - c\tau)}\underline{I}(t - c\tau) - \frac{\beta\alpha_2}{(1 + \alpha_1)^2}e^{-\mu\tau}\underline{I}^2(t - c\tau) \\
&\geq -M\Delta(\lambda_1 + \varepsilon, c)e^{(\lambda_1 + \varepsilon)t} - \frac{\beta}{\sigma(1 + \alpha_1)}e^{\sigma(t - c\tau) - \mu\tau}e^{\lambda_1(t - c\tau)} \\
&\quad - \frac{\beta\alpha_2}{(1 + \alpha_1)^2}e^{2\lambda_1(t - c\tau) - \mu\tau} \\
&= e^{(\lambda_1 + \varepsilon)t}\left(-M\Delta(\lambda_1 + \varepsilon, c) - \frac{\beta}{\sigma(1 + \alpha_1)}e^{-(c(\lambda_1 + \sigma) + \mu)\tau}e^{(\sigma - \varepsilon)t} - \frac{\alpha_2}{(1 + \alpha_1)^2}e^{-(2c\lambda_1 + \mu)\tau}e^{(\lambda_1 - \varepsilon)t}\right).
\end{aligned}
$$

Note that $0 < \varepsilon < \min\{\sigma, \lambda_1\}$ and $t_3 < 0$, it follows that $e^{(\sigma - \varepsilon)t} < 1$ and $e^{(\lambda_1 - \varepsilon)t} < 1$ for all $t < t_3$. Therefore,

$$
\begin{aligned}
&-M\Delta(\lambda_1 + \varepsilon, c) - \frac{\beta}{\sigma(1 + \alpha_1)}e^{-c\tau(\lambda_1 + \sigma) - \mu\tau}e^{(\sigma - \varepsilon)t} - \frac{\alpha_2}{(1 + \alpha_1)^2}e^{-(2c\lambda_1 + \mu)\tau}e^{(\lambda_1 - \varepsilon)t} \\
&\geq -M\Delta(\lambda_1 + \varepsilon, c) - \frac{\beta}{\sigma(1 + \alpha_1)} - \frac{\alpha_2}{(1 + \alpha_1)^2}.
\end{aligned}
$$

Consequently, we only choose

$$M > \max\left\{1, -\frac{1}{(1 + \alpha_1)\Delta(\lambda_1 + \varepsilon, c)}\left(\frac{\beta}{\sigma} + \frac{\alpha_2}{1 + \alpha_1}\right)\right\},$$

then (2.7) holds. $\qquad\square$

2.3   The verification of the Schauder fixed point theorem

In this subsection, we will use the upper and lower solutions $(\overline{S}(t), \overline{I}(t))$ and $(\underline{S}(t), \underline{I}(t))$ constructed in Section 2.2 to verify that the conditions of Schauder fixed point theorem hold. Denote

$$H_1(S, I)(t) := \beta_1 S(t) + \mu(1 - S(t)) - \beta f(S, I)(t),$$
$$H_2(S, I)(t) := \beta_2 I(t) + \beta e^{-\mu\tau} f(S, I)(t - c\tau) - rI(t).$$

Choose two constants $\beta_1 > \mu + \frac{\beta}{\alpha_2}$ and $\beta_2 > r$ such that H_1 is nondecreasing with respect to the first variable $S(t) \in [0, 1]$ and H_1 is nonincreasing with respect to the second variable $I(t) \in [0, \frac{1}{\alpha_2}(\frac{\beta}{r} e^{-\mu\tau} - \alpha_1 - 1)]$ for all $t \in \mathbb{R}$. H_2 is nondecreasing with respect to both $S(t) \in [0, 1]$ and $I(t) \in [0, \frac{1}{\alpha_2}(\frac{\beta}{r} e^{-\mu\tau} - \alpha_1 - 1)]$ for all $t \in \mathbb{R}$. Clearly, (2.3) is equal to

$$\begin{cases} d_1 S''(t) - c\,S'(t) - \beta_1 S(t) + H_1(S, I)(t) = 0, \\ d_2 I''(t) - c\,I'(t) - \beta_2 I(t) + H_2(S, I)(t) = 0. \end{cases} \tag{2.8}$$

Define the set

$$\Gamma = \left\{ (S, I) \in [\mathbf{0}, \mathbf{M}]_{\mathbb{C}} : \underline{S}(t) \le S(t) \le \overline{S}(t), \underline{I}(t) \le I(t) \le \overline{I}(t) \right\}.$$

Then the set Γ is nonempty, closed and convex in $[\mathbf{0}, \mathbf{M}]_{\mathbb{C}}$. Furthermore, define an operator $F : \Gamma \to C(\mathbb{R}, \mathbb{R}^2)$ by

$$F(S, I)(t) = (F_1(S, I), F_2(S, I))(t),$$

where

$$F_i(S, I)(t) = \frac{1}{\rho_i} \left(\int_{-\infty}^{t} e^{\lambda_{i1}(t-s)} + \int_{t}^{+\infty} e^{\lambda_{i2}(t-s)} \right) H_i(S, I)(s) \mathrm{d}s, \qquad i = 1, 2,$$

and

$$\lambda_{i1} = \frac{c - \sqrt{c^2 + 4d_i\beta_i}}{2d_i}, \quad \lambda_{i2} = \frac{c + \sqrt{c^2 + 4d_i\beta_i}}{2d_i}, \quad \rho_i = d_i(\lambda_{i2} - \lambda_{i1}), \qquad i = 1, 2.$$

Lemma 2.7. *The operator F maps Γ into Γ.*

Proof. For $(S, I) \in \Gamma$, we only need to prove the following inequalities hold.

$$\underline{S}(t) \le F_1(S, I)(t) \le \overline{S}(t), \quad \underline{I}(t) \le F_2(S, I)(t) \le \overline{I}(t), \qquad t \in \mathbb{R}.$$

We only prove the first inequality since the proof of the second inequality is similar to that of the first. Indeed, according to the monotonicity of H_1 with respect to S and I, we have

$$F_1(\underline{S}, \overline{I})(t) \le F_1(S, I)(t) \le F_1(\overline{S}, \underline{I})(t), \qquad t \in \mathbb{R}.$$

Thus it is sufficient to verify

$$\underline{S}(t) \le F_1(\underline{S}, \overline{I})(t) \le F_1(\overline{S}, \underline{I})(t) \le 1, \qquad t \in \mathbb{R}. \tag{2.9}$$

In fact, for $t \ne t_2$, by (2.6), we have

$$F_1(\underline{S}, \overline{I})(t) = \frac{1}{\rho_1} \left(\int_{-\infty}^{t} e^{\lambda_{11}(t-s)} + \int_{t}^{+\infty} e^{\lambda_{12}(t-s)} \right) H_1(\underline{S}, \overline{I})(s) \mathrm{d}s$$

$$\ge \frac{1}{\rho_1} \left(\int_{-\infty}^{t} e^{\lambda_{11}(t-s)} + \int_{t}^{+\infty} e^{\lambda_{12}(t-s)} \right) (\beta_1 \underline{S}(s) + c\underline{S}'(s) - d_1 \underline{S}''(s)) \mathrm{d}s.$$

For $t > t_2$, since $\underline{S}'(t_2-) \leq 0$ and $\lambda_{12} > 0 > \lambda_{11}$, we have

$$
\begin{aligned}
F_1(\underline{S}, \bar{I})(t) &\geq \frac{1}{\rho_1} \left(\int_{-\infty}^{t_2} + \int_{t_2}^{t} \right) e^{\lambda_{11}(t-s)} \left(\beta_1 \underline{S}(s) + c\underline{S}'(s) - d_1 \underline{S}''(s) \right) ds \\
&\quad + \frac{1}{\rho_1} \int_{t}^{+\infty} e^{\lambda_{12}(t-s)} \left(\beta_1 \underline{S}(s) + c\underline{S}'(s) - d_1 \underline{S}''(s) \right) ds \\
&= \frac{\beta_1}{\rho_1} \left(\frac{1}{\lambda_{12}} - \frac{1}{\lambda_{11}} \right) \underline{S}(t) - \frac{d_1}{\rho_1} e^{\lambda_{11}(t-t_2)} \underline{S}'(t_2-) \\
&= \underline{S}(t) - \frac{d_1}{\rho_1} e^{\lambda_{11}(t-t_2)} \underline{S}'(t_2-) \\
&\geq \underline{S}(t).
\end{aligned}
$$

Similarly, we also have $F_1(\underline{S}, \bar{I})(t) \geq \underline{S}(t)$ for $t < t_2$. By the continuity of both $\underline{S}(t)$ and $F_1(\underline{S}, \bar{I})(t)$, we obtain $F_1(\underline{S}, \bar{I})(t) \geq \underline{S}(t)$ for all $t \in \mathbb{R}$.

On the other hand, note that H_1 is nondecreasing with respect to $S(t) \in [0, 1]$, we get

$$
H_1(\bar{S}, \underline{I})(t) \leq \beta_1 \bar{S}(t) + \mu(1 - \bar{S}(t)) = \beta_1
$$

for $t \in \mathbb{R}$, then

$$
\begin{aligned}
F_1(\bar{S}, \underline{I})(t) &= \frac{1}{\rho_1} \left(\int_{-\infty}^{t} e^{\lambda_{11}(t-s)} + \int_{t}^{+\infty} e^{\lambda_{12}(t-s)} \right) H_1(\bar{S}, \underline{I})(s) ds \\
&\leq \frac{\beta_1}{\rho_1} \left(\int_{-\infty}^{t} e^{\lambda_{11}(t-s)} + \int_{t}^{+\infty} e^{\lambda_{12}(t-s)} \right) ds \\
&= \frac{\beta_1}{\rho_1} \left(\frac{1}{\lambda_{12}} - \frac{1}{\lambda_{11}} \right) \\
&= 1.
\end{aligned}
$$

This completes the proof of (2.9). $\qquad\square$

Let $\nu > 0$ be a constant such that $\nu < \min\{-\lambda_{11}, -\lambda_{21}\}$. Define

$$
B_\nu(\mathbb{R}, \mathbb{R}^2) = \left\{ (S, I) \in [\mathbf{0}, \mathbf{M}]_{\mathbb{C}} : \sup_{t \in \mathbb{R}} |S(t)| e^{-\nu|t|} < +\infty, \ \sup_{t \in \mathbb{R}} |I(t)| e^{-\nu|t|} < +\infty \right\},
$$

with norm

$$
|(S, I)|_\nu = \max \left\{ \sup_{t \in \mathbb{R}} |S(t)| e^{-\nu|t|}, \sup_{t \in \mathbb{R}} |I(t)| e^{-\nu|t|} \right\}.
$$

It is easy to check that $B_\nu(\mathbb{R}, \mathbb{R}^2)$ is a Banach space with the decay norm $|\cdot|_\nu$.

Lemma 2.8. *The operator F is continuous with respect to the norm $|\cdot|_\nu$ in $B_\nu(\mathbb{R}, \mathbb{R}^2)$.*

Proof. For any $(S_1, I_1), (S_2, I_2) \in [\mathbf{0}, \mathbf{M}]_{\mathbb{C}}$, since

$$
\begin{aligned}
&|f(S_1, I_1)(t) - f(S_2, I_2)(t)| \\
&\quad = \left| \frac{I_1(t)(1 + \alpha_2 I_2(t))(S_1(t) - S_2(t)) + S_2(t)(1 + \alpha_1 S_1(t))(I_1(t) - I_2(t))}{(1 + \alpha_1 S_1(t) + \alpha_2 I_1(t))(1 + \alpha_1 S_2(t) + \alpha_2 I_2(t))} \right| \\
&\quad \leq \left(\frac{1}{\alpha_1} |I_1(t) - I_2(t)| + \frac{1}{\alpha_2} |S_1(t) - S_2(t)| \right),
\end{aligned}
$$

let $L = \max\{\beta_1 - \mu + \frac{\beta}{\alpha_2}, \frac{\beta}{\alpha_1}\} > 0$, then, for all $t \in \mathbb{R}$,

$$|H_1(S_1, I_1)(t) - H_1(S_2, I_2)(t)| \leq L(|S_1(t) - S_2(t)| + |I_1(t) - I_2(t)|).$$

For any $(S_1, I_1), (S_2, I_2) \in \Gamma$, then

$$
\begin{aligned}
&|F_1(S_1, I_1)(t) - F_1(S_2, I_2)(t)| \\
&\leq \frac{1}{\rho_1}\left(\int_{-\infty}^t e^{\lambda_{11}(t-s)} + \int_t^{+\infty} e^{\lambda_{12}(t-s)}\right)|H_1(S_1, I_1)(s) - H_1(S_2, I_2)(s)|\mathrm{d}s \\
&\leq \frac{L}{\rho_1}\left(\int_{-\infty}^t e^{\lambda_{11}(t-s)} + \int_t^{+\infty} e^{\lambda_{12}(t-s)}\right)(|S_1(s) - S_2(s)| + |I_1(s) - I_2(s)|)\mathrm{d}s.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&|F_1(S_1, I_1)(t) - F_1(S_2, I_2)(t)|e^{-\nu|t|} \\
&\leq \frac{L}{\rho_1}\left(\int_{-\infty}^t e^{\lambda_{11}(t-s)} + \int_t^{+\infty} e^{\lambda_{12}(t-s)}\right)(e^{\nu|s|-\nu|t|} + e^{\nu|s|-\nu|t|})\mathrm{d}s|S - I|_\nu \\
&= \frac{2L}{\rho_1}\left(\frac{1}{\lambda_{12} - \nu} - \frac{1}{\lambda_{11} + \nu}\right)|S - I|_\nu,
\end{aligned}
$$

which follows $F_1 : \Gamma \to \Gamma$ is continuous with respect to the norm $|\cdot|_\nu$. By the similar way, we also prove that $F_2 : \Gamma \to \Gamma$ is continuous with respect to the norm $|\cdot|_\nu$. $\qquad\square$

Lemma 2.9. *The operator F is compact with respect to the norm $|\cdot|_\nu$ in $B_\nu(\mathbb{R}, \mathbb{R}^2)$.*

Proof. For any $(S, I) \in \Gamma$, in view of $\lambda_{11} < 0 < \lambda_{12}$, we have, for all $t \in \mathbb{R}$,

$$
\begin{aligned}
|F_1'(S, I)(t)| &= \frac{1}{\rho_1}\left|\left(\int_{-\infty}^t \lambda_{11}e^{\lambda_{11}(t-s)} + \int_t^{+\infty} \lambda_{12}e^{\lambda_{12}(t-s)}\right)H_1(S, I)(s)\mathrm{d}s\right| \\
&\leq \frac{\alpha_2\beta_1 + \beta}{\alpha_2\rho_1}\left(\int_{-\infty}^t |\lambda_{11}|e^{\lambda_{11}(t-s)}\mathrm{d}s + \int_t^{+\infty} \lambda_{12}e^{\lambda_{12}(t-s)}\mathrm{d}s\right) \\
&\leq \frac{\alpha_2\beta_1 + \beta}{\alpha_2\rho_1}.
\end{aligned}
\tag{2.10}
$$

Similarly, we also get, for all $t \in \mathbb{R}$,

$$|F_2'(S, I)(t)| \leq \frac{\beta_2 + 2r}{\alpha_2\rho_2}\left(\frac{\beta}{r}e^{-\mu\tau} - \alpha_1 - 1\right). \tag{2.11}$$

It follows that $\{F_i(S, I)(t) : (S, I) \in \Gamma\}$ is a family of equicontinuous functions. Thus, $\{F(S, I)(t) : (S, I) \in \Gamma\}$ represents a family of equicontinuous functions.

On the other hand, for any $(S, I) \in \Gamma$, it is easy to see that $F : \Gamma \to \Gamma$ follows that

$$|F_1(S, I)(t)| \leq 1, \quad |F_2(S, I)(t)| \leq \frac{1}{\alpha_2}\left(\frac{\beta}{r}e^{-\mu\tau} - \alpha_1 - 1\right), \qquad \forall t \in \mathbb{R}.$$

Hence, for any $\varepsilon > 0$, we can find an $N > 0 (N \in \mathbb{N})$ satisfies

$$(|F_1(S, I)(t)| + |F_2(S, I)(t)|)e^{-\nu|t|} < \left(1 + \frac{1}{\alpha_2}\left(\frac{\beta}{r}e^{-\mu\tau} - \alpha_1 - 1\right)\right)e^{-\nu N} < \varepsilon, \qquad |t| > N. \tag{2.12}$$

By (2.10), (2.11) and the Arzelà–Ascoli theorem, we can choose finite elements in $F(\Gamma)$ such that there are a finite ε-net of $F(\Gamma)(t)$ in sense of supremum norm if we restrict them on $t \in [-N, N]$, which is also a finite ε-net of $F(\Gamma)(t) (t \in \mathbb{R})$ in sense of the norm $|\cdot|_\nu$ (by (2.12)) and implies F is compact with respect to the norm $|\cdot|_\nu$ in $B_\nu(\mathbb{R}, \mathbb{R}^2)$. $\qquad\square$

3 Existence and non-existence of the traveling wave solution

First, using the ideas in [7], we derive some boundedness property of the solution $(S(t), I(t))$ of system (2.1). That is, we give the following lemma.

Lemma 3.1. *Assume that (S, I) is a positive and bounded solution of (2.1). Then there exist positive constants L_i, $i = 1, 2, 3, 4$, such that*

$$- L_1 S(t) < S'(t) < L_2 S(t), \quad -L_3 I(t) < I'(t) < L_4 I(t), \qquad \forall\, t \geq 0. \tag{3.1}$$

Proof. We first show that $-L_1 S(t) < S'(t)$ for all $t \geq 0$, where L_1 is a positive constant sufficiently large such that both $-L_1 S(0) < S'(0)$ and $L_1 \geq \frac{2\beta}{c\alpha_2}$ hold. Let

$$\Phi(t) := S'(t) + L_1 S(t), \qquad \forall\, t \geq 0.$$

We next show that $\Phi(t) > 0$ for all $t \geq 0$ in the following. If not, by $\Phi(0) > 0$, then there exists $t_1 \geq 0$ such that $\Phi(t_1) = 0$, hence there exist two cases:

Case (i): $\Phi(t) \leq 0$, $\forall\, t \geq t_1$.

Case (ii): $\Phi(t)$ is an oscillatory function. i.e., there exist some $t_2 \geq t_1$ satisfy $\Phi(t_2) = 0$ and $\Phi'(t_2) \geq 0$.

For case (i), by the definition of $\Phi(t)$ and in view of $\Phi(t) \leq 0$, we get

$$cS'(t) \leq -\frac{2\beta}{\alpha_2} S(t), \qquad \forall\, t \geq t_1,$$

since $L_1 \geq \frac{2\beta}{c\alpha_2}$. Together with $0 < S \leq 1$ and $\frac{I}{1+\alpha_1 S+\alpha_2 I} \leq \frac{1}{\alpha_2}$, we deduce that

$$d_1 S''(t) = cS'(t) + \beta f(S, I)(t) + \mu(S(t) - 1) \leq -\frac{\beta}{\alpha_2} S(t) < 0, \qquad \forall\, t \geq t_1,$$

which implies that $S'(t)$ is decreasing on $[t_1, +\infty)$. Hence

$$S'(t) \leq S'(t_1) \leq -L_1 S(t_1) < 0, \qquad \forall\, t \geq t_1.$$

This implies that $S(t)$ is decreasing and convex, which contradicts the boundedness of $S(t)$.

For case (ii), since $\Phi(t_2) = 0$, $\Phi'(t_2) \geq 0$, we get

$$S'(t_2) = -L_1 S(t_2) < 0, \qquad S''(t_2) \geq -L_1 S'(t_2) > 0.$$

Hence, we obtain

$$0 = d_1 S''(t_2) - cS'(t_2) + \mu(1 - S(t_2)) - \beta f(S, I)(t_2)$$
$$\geq cL_1 S(t_2) - \frac{\beta}{\alpha_2} S(t_2) \geq \frac{\beta}{\alpha_2} S(t_2) > 0.$$

This is a contradiction. Similarly, we also can show that the other inequations of (3.1) hold for $t \geq 0$. $\qquad \square$

Now we are in a position to state and show our main results.

Theorem 3.2. *Assume that $R_0 > 1$ holds. Then there exists a constant $c^* > 0$ such that for every $c > c^*$, system (2.1) admits a nontrivial positive traveling wave solution $(S(x + ct), I(x + ct))$ satisfying the asymptotic boundary condition (2.4), and*

$$\lim_{t \to -\infty} e^{-\lambda_1 t} I(t) = 1, \qquad \lim_{t \to -\infty} e^{-\lambda_1 t} I'(t) = \lambda_1.$$

Proof. In view of Lemmas 2.6–2.9, it follows from Schauder's fixed point theorem that there exists a pair of $(S, I) \in \Gamma$, which is a fixed point of the operator F. Further, (S, I) is a solution of (2.1). Consequently, the solution $(S(x + ct), I(x + ct))$ is a traveling wave solution of system (2.1). Moreover, (S, I) satisfies the following inequalities

$$1 - \frac{1}{\sigma} e^{\sigma t} \leq S(t) \leq 1, \quad e^{\lambda_1 t}(1 - Me^{\varepsilon t}) \leq I(t) \leq e^{\lambda_1 t}, \quad \forall t \in \mathbb{R},$$

which follows that

$$S(-\infty) = 1, \quad I(-\infty) = 0, \quad \lim_{t \to -\infty} e^{-\lambda_1 t} I(t) = 1.$$

Note that $(S, I) \in \Gamma$ is a fixed point of the operator of F. Applying L'Hospital's rule to the maps F_1 and F_2, it is easy to see that $S'(-\infty) = 0$ and $I'(-\infty) = 0$. Integrating both sides of the second equation of (2.3) from $-\infty$ to t gives

$$d_2 I'(t) = cI(t) - \beta e^{-\mu \tau} \int_{-\infty}^{t} f(S, I)(s - c\tau) ds + r \int_{-\infty}^{t} I(s) ds.$$

Hence, by $\lim_{t \to -\infty} e^{-\lambda_1 t} I(t) = 1$,

$$
\begin{aligned}
\lim_{t \to -\infty} e^{-\lambda_1 t} I'(t) &= \frac{c}{d_2} - \frac{\beta}{d_2} e^{-\mu \tau} \lim_{t \to -\infty} e^{-\lambda_1 t} \int_{-\infty}^{t} f(S, I)(s - c\tau) ds \\
&\quad + \frac{r}{d_2} \lim_{t \to -\infty} e^{-\lambda_1 t} \int_{-\infty}^{t} I(s) ds \\
&= \frac{c}{d_2} - \frac{\beta}{d_2 \lambda_1} e^{-\mu \tau} \lim_{t \to -\infty} e^{-\lambda_1 t} f(S, I)(t - c\tau) + \frac{r}{d_2 \lambda_1} \\
&= \frac{1}{d_2 \lambda_1} \left(c\lambda_1 + r - \frac{\beta}{1 + \alpha_1} e^{-c\tau \lambda_1 - \mu \tau} \right) \\
&= \lambda_1.
\end{aligned}
$$

Next we claim that, for all $t \in \mathbb{R}$,

$$\frac{\mu \alpha_2}{\mu \alpha_2 + \beta} < S(t) < 1, \quad 0 < I(t) < \frac{1}{\alpha_2} \left(\frac{\beta e^{-\mu \tau}}{r} - \alpha_1 - 1 \right). \tag{3.2}$$

That is, the traveling wave solution of (2.1) is nontrivial positive. Indeed,

$$
\begin{aligned}
S(t) &= F_1(S, I)(t) \geq F_1(\underline{S}, \bar{I})(t) \\
&= \frac{1}{\rho_1} \left(\int_{-\infty}^{t} e^{\lambda_{11}(t-s)} + \int_{t}^{+\infty} e^{\lambda_{12}(t-s)} \right) H_1(\underline{S}, \bar{I})(s) ds \\
&\geq \frac{\beta_1 - \mu - \frac{\beta}{\alpha_2}}{\rho_1} \left(\int_{-\infty}^{t} e^{\lambda_{11}(t-s)} + \int_{t}^{+\infty} e^{\lambda_{12}(t-s)} \right) \underline{S}(s) ds \\
&> \frac{\mu \alpha_2}{\mu \alpha_2 + \beta}.
\end{aligned}
$$

Similarly, we can prove another inequality is also true.

In the following, motivated by the ideas [3–5, 7, 10], we construct the Lyapunov functional to show that the obtained positive traveling wave solutions of model (2.1) connect the endemic equilibrium $E^* = (S^*, I^*)$. That is, we shall show $\lim_{t \to +\infty}(S(t), I(t)) = (S^*, I^*)$ holds.

To simplify the notation, let

$$g(x) = x - 1 - \ln x, \qquad x > 0,$$

it is easy to see that

$$g(x) \geq 0, \ x > 0, \quad \text{and} \quad g(x) = 0 \quad \text{if and only if } x = 1.$$

By (3.2), we see that (S, I) is a positive and bounded solution of (2.1). Define

$$\mathcal{D} = \left\{ (S, I) : \frac{\mu \alpha_2}{\mu \alpha_2 + \beta} < S(t) < 1, 0 < I(t) < \frac{1}{\alpha_2} \left(\frac{\beta e^{-\mu \tau}}{\mu + \alpha + \gamma} - \alpha_1 - 1 \right), \right.$$

$$\left. - L_1 S(t) < S'(t) < L_2 S(t), -L_3 I(t) < I'(t) < L_4 I(t), t \geq 0 \right\}.$$

By Lemma 3.1, we see that $\mathcal{D} \neq \emptyset$. For each $(S, I) \in \mathcal{D}$, we consider the following Lyapunov functional $V(S, I) : \mathbb{R}^+ \to \mathbb{R}$ as follows

$$V(S, I)(t) = cV_1(S, I)(t) + crI^* V_2(S, I)(t) + V_3(S, I)(t),$$

where

$$V_1(S, I)(t) = e^{-\mu \tau} \left(S - S^* - \int_{S^*}^S \frac{1 + \alpha_1 \theta + \alpha_2 I^*}{1 + \alpha_1 S^* + \alpha_2 I^*} \frac{S^*}{\theta} d\theta \right) + I - I^* - I^* \ln \frac{I}{I^*},$$

$$V_2(S, I)(t) = \int_{t-c\tau}^t g \left(\frac{\beta}{rI^*} e^{-\mu \tau} f(S, I)(y) \right) dy,$$

$$V_3(S, I)(t) = d_1 e^{-\mu \tau} S' \left(\frac{1 + \alpha_1 S + \alpha_2 I^*}{1 + \alpha_1 S^* + \alpha_2 I^*} \frac{S^*}{S} - 1 \right) + d_2 I' \left(\frac{I^*}{I} - 1 \right).$$

By a direct calculation, we have

$$c \frac{dV_1}{dt} = e^{-\mu \tau} \left(1 - \frac{S^*}{S} \frac{1 + \alpha_1 S + \alpha_2 I^*}{1 + \alpha_1 S^* + \alpha_2 I^*} \right) (d_1 S'' + \mu(1 - S) - \beta f(S, I)(t))$$

$$+ \left(1 - \frac{I^*}{I} \right) (d_2 I'' + \beta e^{-\mu \tau} f(S, I)(t - c\tau) - rI).$$

From the fact that

$$\mu(1 - S^*) = e^{\mu \tau} rI^*, \qquad \beta e^{-\mu \tau} S^* I^* = rI^* (1 + \alpha_1 S^* + \alpha_2 I^*),$$

we get

$$c \frac{dV_1}{dt} = e^{-\mu \tau} \left(1 - \frac{S^*}{S} \frac{1 + \alpha_1 S + \alpha_2 I^*}{1 + \alpha_1 S^* + \alpha_2 I^*} \right) d_1 S'' + \left(1 - \frac{I^*}{I} \right) d_2 I''$$

$$+ \mu e^{-\mu \tau} (S^* - S) + r(I^* - I) + \beta e^{-\mu \tau} (f(S, I)(t - c\tau) - f(S, I)(t))$$

$$- rI^* \frac{S^*}{S} \frac{1 + \alpha_1 S + \alpha_2 I^*}{1 + \alpha_1 S^* + \alpha_2 I^*} \frac{1 - S}{1 - S^*} + rI \frac{1 + \alpha_1 S + \alpha_2 I^*}{1 + \alpha_1 S + \alpha_2 I}$$

$$- rI^* \frac{1 + \alpha_1 S^* + \alpha_2 I^*}{S^* I} f(S, I)(t - c\tau) + rI^*, \tag{3.3}$$

$$crI^* \frac{dV_2}{dt} = rI^* \frac{d}{dt} \int_{t-c\tau}^t \left(\frac{\beta e^{-\mu \tau}}{rI^*} f(S, I)(y) - 1 - \ln \frac{\beta e^{-\mu \tau}}{rI^*} f(S, I)(y) \right) dy$$

$$= \beta e^{-\mu \tau} f(S, I)(t) - \beta e^{-\mu \tau} f(S, I)(t - c\tau) + rI^* \ln \frac{f(S, I)(t - c\tau)}{f(S, I)(t)}, \tag{3.4}$$

and

$$\frac{dV_3}{dt} = d_1 e^{-\mu\tau}\left(\frac{1+\alpha_1 S + \alpha_2 I^*}{1+\alpha_1 S^* + \alpha_2 I^*}\frac{S^*}{S}-1)\right)S'' - d_1 e^{-\mu\tau}\frac{1+\alpha_2 I^*}{1+\alpha_1 S^* + \alpha_2 I^*}\frac{S^*(S')^2}{S^2}$$
$$+ d_2\left(\frac{I^*}{I}-1\right)I'' - d_2\frac{I^*(I')^2}{I^2}. \tag{3.5}$$

Combining (3.3)–(3.5), we obtain

$$\frac{dV}{dt} = -d_1 e^{-\mu\tau}\frac{1+\alpha_2 I^*}{1+\alpha_1 S^* + \alpha_2 I^*}\frac{S^*(S')^2}{S^2} - d_2\frac{I^*(I')^2}{I^2} + \mu e^{-\mu\tau}(S^* - S) + r(I^* - I)$$
$$- rI^*\frac{S^*}{S}\frac{1+\alpha_1 S + \alpha_2 I^*}{1+\alpha_1 S^* + \alpha_2 I^*}\frac{1-S}{1-S^*} + rI\frac{1+\alpha_1 S + \alpha_2 I^*}{1+\alpha_1 S^* + \alpha_2 I} + rI^*\ln\frac{f(S,I)(t-c\tau)}{f(S,I)(t)}$$
$$- rI^*\frac{1+\alpha_1 S^* + \alpha_2 I^*}{S^* I}f(S,I)(t-c\tau) + rI^*$$
$$= -d_1 e^{-\mu\tau}\frac{1+\alpha_2 I^*}{1+\alpha_1 S^* + \alpha_2 I^*}\frac{S^*(S')^2}{S^2} - d_2\frac{I^*(I')^2}{I^2} - \frac{\mu e^{-\mu\tau}(1+\alpha_2 I^*)(S-S^*)^2}{S(1+\alpha_1 S^* + \alpha_2 I^*)}$$
$$+ rI^*\left(2 - \frac{S^*(1+\alpha_1 S + \alpha_2 I^*)}{S(1+\alpha_1 S^* + \alpha_2 I^*)} - \frac{1+\alpha_1 S^* + \alpha_2 I^*}{S^* I}f(S,I)(t-c\tau)\right)$$
$$+ rI^*\left(-\frac{I}{I^*} + \frac{I(1+\alpha_1 S + \alpha_2 I^*)}{I^*(1+\alpha_1 S + \alpha_2 I)}\right) + rI^*\ln\frac{f(S,I)(t-c\tau)}{f(S,I)(t)}.$$

Note that

$$\ln\frac{f(S,I)(t-c\tau)}{f(S,I)(t)} = \ln\frac{S^*(1+\alpha_1 S + \alpha_2 I^*)}{S(1+\alpha_1 S^* + \alpha_2 I^*)} + \ln\frac{1+\alpha_1 S^* + \alpha_2 I^*}{S^* I}f(S,I)(t-c\tau)$$
$$+ \ln\frac{1+\alpha_1 S + \alpha_2 I}{1+\alpha_1 S + \alpha_2 I^*}.$$

Hence, we get

$$\frac{dV}{dt} = -d_1 e^{-\mu\tau}\frac{1+\alpha 1 I^*}{1+\alpha_1 S^* + \alpha_2 I^*}\frac{S^*(S')^2}{S^2} - d_2\frac{I^*(I')^2}{I^2} - \frac{\mu e^{-\mu\tau}(1+\alpha_2 I^*)(S-S^*)^2}{S(1+\alpha_1 S^* + \alpha_I^*)}$$
$$+ rI^*\left(1 - \frac{S^*(1+\alpha_1 S + \alpha_2 I^*)}{S(1+\alpha_1 S^* + \alpha_2 I^*)} + \ln\frac{S^*(1+\alpha_1 S + \alpha_2 I^*)}{S(1+\alpha_1 S^* + \alpha_2 I^*)}\right)$$
$$+ rI^*\left(1 - \frac{1+\alpha_1 S^* + \alpha I^*}{S^* I}f(S,I)(t-c\tau) + \ln\frac{1+\alpha_1 S^* + \alpha_2 I^*}{S^* I}f(S,I)(t-c\tau)\right)$$
$$+ rI^*\left(1 - \frac{1+\alpha_1 S + \alpha_2 I}{1+\alpha_1 S + \alpha_2 I^*} + \ln\frac{1+\alpha_1 S + \alpha_2 I}{1+\alpha_1 S + \alpha_2 I^*}\right)$$
$$+ rI^*\left(-1 - \frac{I}{I^*} + \frac{1+\alpha_1 S + \alpha_2 I}{1+\alpha_1 S + \alpha_2 I^*} + \frac{I(1+\alpha_1 S + \alpha_2 I^*)}{I^*(1+\alpha_1 S + \alpha_2 I)}\right)$$
$$= -d_1 e^{-\mu\tau}\frac{1+\alpha_2 I^*}{1+\alpha_1 S^* + \alpha_2 I^*}\frac{S^*(S')^2}{S^2} - d_2\frac{I^*(I')^2}{I^2} - \frac{\mu e^{-\mu\tau}(1+\alpha_2 I^*)(S-S^*)^2}{S(1+\alpha_1 S^* + \alpha_2 I^*)}$$
$$- rI^* g\left(\frac{S^*(1+\alpha_1 S + \alpha_2 I^*)}{S(1+\alpha_1 S^* + \alpha_2 I^*)}\right) - rI^* g\left(\frac{1+\alpha_1 S^* + \alpha_2 I^*}{S^* I}f(S,I)(t-c\tau)\right)$$
$$- rI^* g\left(\frac{1+\alpha_1 S + \alpha_2 I}{1+\alpha_1 S + \alpha_2 I^*}\right) - \frac{r\alpha_2(1+\alpha_1 S)}{(1+\alpha_1 S + \alpha_2 I)(1+\alpha_1 S + \alpha_2 I^*)}(I-I^*)^2$$
$$\leq 0.$$

Thus, $V(t)$ is non-increasing in $t \in \mathbb{R}^+$. Furthermore, by Lemma 3.1, one could show that $V(S, I)(t)$ is bounded below on $\mathbb{R}^+$. And also, it is clear that

$$\frac{dV(t)}{dt} = 0 \quad \text{if and only if} \quad S(t) \equiv S^*, \ I(t) \equiv I^*, \ S'(t) = 0, \ I'(t) = 0 \quad \text{for } t \in \mathbb{R},$$

and the maximal invariant set of

$$\left\{ (S, I) : \frac{dV(t)}{dt} = 0 \right\}$$

consists of only point, i.e., the equilibrium (S^*, I^*). Then LaSalle's invariance principle [8, Theorem 4.3.4] implies that $(S, I) \to (S^*, I^*)$ as $t \to +\infty$. Therefore, $\lim_{t \to +\infty}(S(t), I(t)) = (S^*, I^*)$. This completes the proof. $\qquad\square$

Finally, we apply the ideas of [10] to establish the non-existence of traveling wave solutions of system (2.1).

Theorem 3.3. *Assume that $R_0 > 1$ holds. Then there exists a constant $c^* > 0$ such that for $c \in (0, c^*)$, system (2.1) does not admit a traveling wave solution $(S(x + ct), I(x + ct))$ satisfying (2.4).*

Proof. For some $c_1 \in (0, c^*)$, assume there exists a traveling wave solution $(S(x + ct), I(x + ct))$ of system (2.1) satisfying (2.4). Let $\epsilon > 0$ such that equation

$$d_2 \lambda^2 - c\lambda + \frac{\beta}{1 + \alpha_1}(1 - 2\epsilon)e^{-(\lambda c + \mu)\tau} - r = 0$$

has no real solution for $c \in (0, \frac{c_1 + 2c^*}{3})$, which is admissible by Lemma 2.3. By (2.4), we can take $T(\epsilon) < 0$ large enough such that

$$1 - \epsilon \le S(t) < 1 \quad \text{for any } t < T(\epsilon).$$

Thus, for $t < T(\epsilon)$, we have

$$c_1 I'(t) \ge d_2 I''(t) + \frac{\beta e^{-\mu\tau}(1 - \epsilon)I(t - c_1\tau)}{1 + \alpha_1 + \alpha_2 I(t - c_1\tau)} - rI(t). \tag{3.6}$$

According to (3.2), there exists a constant $h > 0$ large enough, such that

$$\frac{\beta e^{-\mu\tau}I(t - c_1\tau)}{[1 + \alpha_1 + \alpha_2 I(t - c_1\tau)]^{h+1}} \le \frac{\beta e^{-\mu\tau}S(t - c_1\tau)I(t - c_1\tau)}{1 + \alpha_1 + \alpha_2 I(t - c_1\tau)}, \qquad t \ge T(\epsilon).$$

In fact, it is equivalent to the following inequality

$$\frac{1}{[1 + \alpha_1 + \alpha_2 I(t - c_1\tau)]^h} \le S(t - c_1\tau), \qquad t \ge T(\epsilon),$$

which is available for h large enough. Then, by (3.6),

$$c_1 I'(t) \ge d_2 I''(t) + \frac{\beta e^{-\mu\tau}I(t - c_1\tau)}{[1 + \alpha_1 + \alpha_2 I(t - c_1\tau)]^{h+1}} - rI(t), \qquad t \ge T(\epsilon). \tag{3.7}$$

Define

$$b(u) = \inf_{v \in \left(u, \frac{1}{\alpha_2}\left(\frac{\beta}{r}e^{-\mu\tau} - \alpha_1 - 1\right)\right)} \frac{\beta e^{-\mu\tau}(1 - \epsilon)v}{[1 + \alpha_1 + \alpha_2 v]^{h+1}}. \tag{3.8}$$

Combining (3.7) and (3.8), we can obtain that $u(x,t) = I(x + c_1 t) > 0$ satisfies

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} \geq d_2 \dfrac{\partial^2 u(x,t)}{\partial x^2} + b(u(x,t-\tau)) - ru(x,t), & x \in \mathbb{R}, \ t > 0, \\ u(x,s) = I(x + c_1 s) > 0, & x \in \mathbb{R}, \ s \in (-\tau, 0). \end{cases}$$

By the comparison principle (see [12, Theorem 2.2]), $u(x,t)$ is an upper solution of the following initial value problem

$$\begin{cases} \dfrac{\partial \omega(x,t)}{\partial t} \geq d_2 \dfrac{\partial^2 \omega(x,t)}{\partial x^2} + b(\omega(x,t-\tau)) - r\omega(x,t), & x \in \mathbb{R}, \ t > 0, \\ \omega(x,s) = I(x + c_1 s) > 0, & x \in \mathbb{R}, \ s \in (-\tau, 0). \end{cases}$$

By the theory of asymptotic spreading (see [14, Theorem 2.5]), we obtain that

$$\liminf_{t \to \infty} \omega(x,t) > 0, \qquad |x| \leq \frac{c_1 + c^*}{2} t.$$

Hence,

$$\liminf_{t \to \infty} u(x,t) \geq \liminf_{t \to \infty} \omega(x,t) > 0, \qquad |x| \leq \frac{c_1 + c^*}{2} t. \tag{3.9}$$

Let $-x = \frac{c_1 + c^*}{2} t$, then $t \to \infty$ implies that $x + c_1 t \to -\infty$. Consequently,

$$\lim_{t \to \infty} I(x,t) = 0,$$

which contradicts (3.9). This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

4 Numerical simulations

In this section, we carry out numerical simulations to illustrate the theoretical results obtained in Sections 3. For simplify, we use the following trivial functions as initial conditions

$$S(x,\theta) = \begin{cases} 0.1, & x = 0, \\ 0, & x > 0, \end{cases} \qquad I(x,\theta) = \begin{cases} 0.0000001, & x = 0, \\ 0, & x > 0 \end{cases} \tag{4.1}$$

for $\theta \in [-\tau, 0]$.

In view of [3], for system (2.1), we set

$$d_1 = 0.2, \quad d_2 = 0.4, \quad \mu = 0.25, \quad \beta = 20, \quad r = 0.95, \quad \alpha_1 = 0.9, \quad \alpha_2 = 0.2, \quad \tau = 0.875.$$

Thus, system (2.1) with above coefficients has a disease-free steady state $E_0 = (1,0)$ and a unique endemic $E^* = (0.0649, 0.1977)$. By the direct computation, one gets $R_0 = 8.9033 > 1$. It follows from Theorem 3.2 that system (2.1) always has a traveling wave solution with speed $c \geq c^*$ connecting E_0 and E^*. The fact is illustrated by the numerical simulation in Figure 4.1.
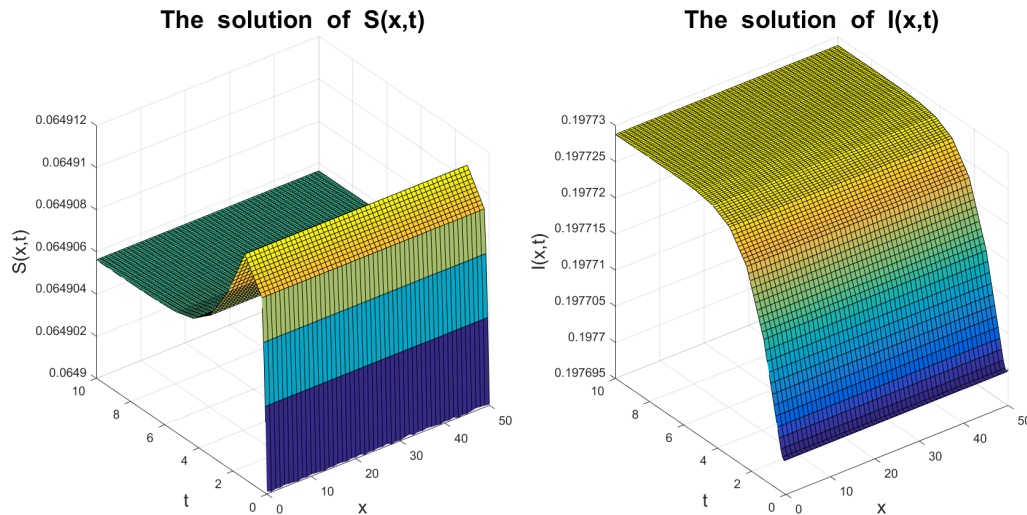
Acknowledgments

Figure 4.1: The traveling wave is observed in the system (2.1) with initial conditions (4.1).

References

[1] Z. Bai, S. L. Wu, Traveling waves in a delayed SIR epidemic model with nonlinear incidence, *Appl. Math. Comput.* **263**(2015), 221–232. MR3348538; https://doi.org/10.1016/j.amc.2015.04.048

[2] Z. Bai, S. Zhang, Traveling waves of a diffusive SIR epidemic model with a class of nonlinear incidence rates and distributed delay, *Commun. Nonlinear Sci. Numer. Simul.* **22**(2015), 1370–1381. MR3282507; https://doi.org/10.1016/j.cnsns.2014.07.005

[3] A. Abta, A. Kaddar, H. T. Alaoui, Global stability for delay SIR and SEIR epidemic models with saturated incidence rates, *Electron. J. Differential Equations* **2012**, No. 23, 1–13. MR2889629

[4] A. Ducrot, M. Langlais, P. Magal, Qualitative analysis and travelling wave solutions for the SI model with vertical transmission, *Commun. Pure. Appl. Anal.* **11**(2012), 97–113. MR2833340; https://doi.org/10.3934/cpaa.2012.11.97

[5] A. Ducrot, P. Magal, Travelling wave solutions for an infection-age structured epidemic model with external supplies, *Nonlinearity.* **24**(2011), 2891–2911. MR2842101; https://doi.org/10.1088/0951-7715/24/10/012

[6] D. Dances, P. K. Medina, *Abstract evolution equations, periodic problems and applications*, Pitman Research Notes in Mathematics, Vol. 729, Longman Scientific & Technical, Harlow, 1992. MR1204883

[7] S. Fu, Traveling waves for a diffusive SIR model with delay, *J. Math. Anal. Appl.* **435**(2016), 20–37. MR3423382; https://doi.org/10.1016/j.jmaa.2015.09.069

[8] D. Henry, *Geometric theory of semilinear parabolic equations*, Springer, New York, 1981. MR0610244

[9] C. Hsu, T. Yang, Existence, uniqueness, monotonicity and asymptotic behaviour of travelling waves for epidemic models, *Nonlinearity.* **26**(2013), 121–139. MR3001765; `https://doi.org/10.1088/0951-7715/26/1/121`

[10] Y. Li, W. Li, G. Lin, Traveling waves of a delayed diffusive SIR epidemic model, *Comm. Pure. Appl. Anal.* **14**(2015), 1001–1022. MR3320163; `https://doi.org/10.3934/cpaa.2015.14.1001`

[11] R. Martin, H. Smith, Abstract functional differential equations and reaction-diffusion systems, *Trans. Amer. Math. Soc.* **321**(1990), 1–44. MR0967316; `https://doi.org/10.2307/2001590`

[12] H. Smith, X. Zhao, Global asymptotic stability of traveling waves in delayed reaction-diffusion equations, *SIAM J. Math. Anal.* **31**(2000), 514–534. MR1740724; `https://doi.org/10.1137/S0036141098346785`

[13] J. W.-H. So, J. Wu, X. Zou, A reaction–diffusion model for a single species with age structure, I. Travelling wavefronts on the unbounded domains, *Proc. R. Soc. Lond. Ser. A.* **457**(2001), 1841–1853. MR1852431; `https://doi.org/10.1098/rspa.2001.0789`

[14] H. R. Thieme, X. Zhao, Asymptotic speeds of spread and traveling waves for integral equations and delayed reaction–diffusion models, *J. Differential Equations* **195**(2003), 430–470. MR2016819; `https://doi.org/10.1016/S0022-0396(03)00175-X`

[15] X. Wang, H. Wang, J. Wu, Traveling waves of diffusive predator–prey systems: disease outbreak propagation, *Discrete Contin. Dyn. Syst.* **32**(2012), 3303–3324. MR2912077; `https://doi.org/10.3934/dcds.2012.32.3303`

[16] Z. Wang, J. Wu, Traveling waves of a diffusive Kermack–McKendrick epidemic model with nonlocal delayed transmission, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **466**(2010), 237–261. MR2564926 ; `https://doi.org/10.1098/rspa.2009.0377`

[17] Z. Wang, J. Wu, Travelling waves in a bio-reactor model with stage-structure, *J. Math. Anal. Appl.* **385**(2012), 683–692. MR2834844; `https://doi.org/10.1016/j.jmaa.2011.06.084`

[18] Z. Wang, J. Wu, R. Liu, Traveling waves of the spread of avian influenza, *Proc. Amer. Math. Soc.* **140**(2012), No. 11, 3931–3946. MR2944733; `https://doi.org/10.1090/S0002-9939-2012-11246-8`

[19] P. Weng, X. Zhao, Spreading speed and traveling waves for a multi-type SIS epidemic model, *J. Differential Equations* **229**(2006), 270–296. MR2265628; `https://doi.org/10.1016/j.jde.2006.01.020`

[20] J. Wu, *Theory and applications of partial functional differential equations*, Springer-Verlag, New York, 1996. MR1415838; `https://doi.org/10.1007/978-1-4612-4050-1`

[21] Z. Xu, Traveling waves in a Kermack–Mckendrick epidemic model with diffusion and latent period, *Nonlinear Anal.* **111**(2014), 66–81. MR3263504; `https://doi.org/10.1016/j.na.2014.08.012`

[22] Z. Xu, Wave propagation in an infectious disease model, *J. Math. Anal. Appl.* **449**(2017), 853–871. MR3595237; `https://doi.org/10.1016/j.jmaa.2016.11.051`

[23] F. Yang, Y. Li, W. Li, Z. Wang, Traveling waves in a nonlocal dispersal Kermack–McKendrick epidmic model, *Discrete Contin. Dyn. Sys. Ser. B* **18**(2013), 1969–1993. MR3066330; https://doi.org/10.3934/dcdsb.2013.18.1969

[24] Y. Zhang, Z. Xu, Dynamics of a diffusive HBV model with delayed Beddington–DeAngelis response, *Nonlinear Anal. Real World Appl.* **15**(2014), 118–139. MR3110559; https://doi.org/10.1016/j.nonrwa.2013.06.005

[25] L. Zhao, Z. Wang, Traveling wave fronts in a diffusive epidemic model with multiple parallel infectious stages, *IMA J. Appl. Math.* **81**(2016), 795–823. MR3556384; https://doi.org/10.1093/imamat/hxw033