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Differentiability in Fréchet spaces and delay differential equations

Hans-Otto Walther✉

Mathematisches Institut, Universität Gießen, Arndtstr. 2, 35392 Gießen, Germany

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Abstract. In infinite-dimensional spaces there are non-equivalent notions of continuous differentiability which can be used to derive the familiar results of calculus up to the Implicit Function Theorem and beyond. For autonomous differential equations with variable delay, not necessarily bounded, the search for a state space in which solutions are unique and differentiable with respect to initial data leads to smoothness hypotheses on the vector functional f in an equation of the general form

$$x'(t) = f(x_t) \in \mathbb{R}^n, \quad \text{with } x_t(s) = x(t+s) \quad \text{for } s \le 0,$$

which have implications (a) on the nature of the delay (which is hidden in f) and (b) on the type of continuous differentiability which is present. We find the appropriate *strong* kind of continuous differentiability and show that there is a continuous semiflow of continuously differentiable solution operators on a Fréchet manifold, with local invariant manifolds at equilibria.

Keywords: Fréchet space, Fréchet differentiability, delay differential equation, unbounded delay, semiflow, invariant manifolds

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1 Introduction

Consider an autonomous delay differential equation

$$x'(t) = f(x_t) \tag{1.1}$$

with $f : U \to \mathbb{R}^n$ defined on a set of maps $(-\infty, 0] \to \mathbb{R}^n$, and the segment, or history, x_t of the solution x at t defined by $x_t(s) = x(t+s)$ for all $s \le 0$. A solution on some interval $[t_0, t_e)$, $t_0 < t_e \le \infty$, is a map $x : (-\infty, t_e) \to \mathbb{R}^n$ with $x_t \in U$ for all $t \in [t_0, t_e)$ so that the restriction of x to $[t_0, t_e)$ is differentiable and Eq. (1.1) holds on this interval. Solutions on the whole real line are defined accordingly. A toy example which can be written in the form (1.1) is the equation

$$x'(t) = h(x(t-r)), \quad r = r(x(t)) \tag{1.2}$$

✉ Hans-Otto.Walther@math.uni-giessen.de

with functions $h : \mathbb{R} \to \mathbb{R}$ and $r : \mathbb{R} \to [0, \infty)$. Other examples arise from pantograph equations

$$x'(t) = a\,x(\lambda t) + b\,x(t) \tag{1.3}$$

with constants $a \in \mathbb{C}$, $b \in \mathbb{R}$ and $0 < \lambda < 1$, and from Volterra integro-differential equations

$$x'(t) = \int_0^t k(t,s)h(x(s))ds \tag{1.4}$$

with $k : \mathbb{R}^{n \times n} \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}^n$ continuous [25]. Eq. (1.3) is linear, and both equations (1.3) and (1.4) are non-autonomous. We shall come back to them in Section 9 below.

Building a theory of Eq. (1.1) which (a) covers examples with state-dependent delay like Eq. (1.2) and (b) results in solution operators $x_0 \mapsto x_t$, $t \geq 0$, which are continuously differentiable begins with the search for a suitable state space. For equations with *bounded* delay the basic steps of a solution theory were made in [20], starting from the observation that the reformulation of examples of the form (1.2) with a state-dependent delay bounded by some $R > 0$ as equations of the form (1.1) yields (vector-)functionals on the right hand side which are not even locally Lipschitz continuous on domains in the Banach space of continuous maps on the compact *initial interval* $[-R, 0]$. In order to obtain an equation with a continuously differentiable functional, so that there is hope for continuous differentiability of solutions with respect to initial data, one must restrict to the Banach space of continuously differentiable functions on the initial interval. This, in turn, means a restriction to solutions which are continuously differentiable everywhere, and not only on the interval where they satisfy Eq. (1.1), as in the by-now well established theory of retarded functional differential equations [2, 5]. A further observation in [20] is that for a continuously differentiable solution, say, $x : [-R, t_e) \to \mathbb{R}^n$, the differential equation at $t = 0$ becomes a compatibility condition on the continuously differentiable initial segment.

According to the preceding remarks the functional f in Eq. (1.1) above should be defined on a subset U of the vector space $C^1 = C^1((-\infty, 0], \mathbb{R}^n)$ of continuously differentiable maps $(-\infty, 0] \to \mathbb{R}^n$. Linearization as in [20] suggests that in the new theory autonomous linear equations with constant delay, like for example,

$$x'(t) = -\alpha\,x(t-1)$$

will appear, which have solutions on $\mathbb{R}$ with arbitrarily large exponential growth at $-\infty$. In order not to loose such solutions we stay with the full space C^1 and work with the topology of locally uniform convergence of maps and their derivatives, which makes C^1 a Fréchet space.

In infinite dimension there a different, non-equivalent generalizations of the canonical notion of continuous differentiability for maps in Euclidean spaces, all of which can be used as a basis for calculus, see for example [19]. For maps in Banach spaces, existence of the Fréchet derivatives and continuous dependence with respect to the norm topology on the Banach space of continuous linear mappings is convenient. Without norms, for maps in Fréchet spaces, often continuous differentiability in the sense of Michal [15] is chosen, which means for a continuous map $f : V \supset U \to W$, V and W topological vector spaces and $U \subset V$ open, that all directional derivatives

$$Df(u)v = \lim_{0 \neq t \to 0} \frac{1}{t}(f(u + tv) - f(u))$$

exist and that the map

$$U \times V \ni (u, v) \mapsto Df(u)v \in W$$

is continuous. In [23–26], this notion of continuous differentiability was called C^1_{MB}-smoothness, with reference to [15] and also to work of A. Bastiani [1]. In the present paper we prefer to speak of C^1_ζ-smoothness, for a reason which will become apparent below. Let us recommend Part I of Hamilton's paper [6] as an introduction into calculus based on C^1_ζ-smoothness.

We return to Eq. (1.1), now for $f : C^1 \supset U \to \mathbb{R}^n$ which is C^1_ζ-smooth and has the additional *extension property* that

(e) *each derivative* $Df(\phi) : C^1 \to \mathbb{R}^n$, $\phi \in U$, *has a linear extension* $D_e f(\phi) : C \to \mathbb{R}^n$, *and the map*

$$U \times C \ni (\phi, \chi) \mapsto D_e f(\phi)\chi \in \mathbb{R}^n$$

is continuous.

Here C is the Fréchet space of continuous maps $(-\infty, 0] \to \mathbb{R}^n$ with the topology of locally uniform convergence. Property (e) is closely related to the earlier notion of being *almost Fréchet differentiable* from [14], for maps on a Banach space of continuous functions, and it is often satisfied if f comes from an example of a differential equation with state-dependent delay.

The analogue of the compatibility condition from [20] defines the set

$$X_f = \{\phi \in U : \phi'(0) = f(\phi)\}.$$

In [23] we saw that under the hypotheses just mentioned X_f, if non-empty, is a C^1_ζ-submanifold of codimension n in C^1. Notice that X_f consists of the segments x_t, $0 \le t < t_e$, of all continuously differentiable solutions $x : (-\infty, t_e) \to \mathbb{R}^n$ on $[0, t_e)$, $0 < t_e \le \infty$, of Eq. (1.1). It is shown in [23] that these solutions constitute a continuous semiflow $(t, x_0) \mapsto x_t$ on X_f, with all solution operators $x_0 \mapsto x_t$, $t \ge 0$, C^1_ζ-smooth.

Let us call the set X_f the solution manifold associated with the map f.

The motivation for the present study of Eq. (1.1) is the fact that functionals $f : C^1 \supset U \to \mathbb{R}^n$ which are C^1_ζ-smooth amd have property (e) are in fact continuously differentiable in a stronger sense. This is the content of Proposition 8.3 below, which guarantees that such functionals f satisfy the conditions stated in the following definition.

Definition 1.1. A continuous map $f : V \supset U \to W$, V and W topological vector spaces and $U \subset V$ open, is said to be continuously differentiable in the sense of Fréchet if all directional derivatives exist, if each map $Df(u) : V \to W$, $u \in U$, is linear and continuous, and if the map $Df : U \ni u \mapsto Df(u) \in L_c(V, W)$ is continuous with respect to the topology β of uniform convergence on bounded sets, on the vector space $L_c(V, W)$ of continuous linear maps $V \to W$.

We abbreviate continuous differentiability in the sense of Fréchet by speaking of C^1_β-smoothness. In case V and W are Banach spaces C^1_β-smoothness is equivalent to the familiar notion of continuous differentiability with Fréchet derivatives, see e.g. Proposition 4.2 below. In case V and W are Fréchet spaces C^1_β-smoothness is equivalent to C^1_ζ-smoothness combined with the continuity of the derivative with respect to the topology β on $L_c(V, W)$, see e.g.

Corollary 3.2 below. For examples of maps $C^1 \to \mathbb{R}^n$ which are C^1_ζ-smooth but not C^1_β-smooth, see [26].

It seems that C^1_β-smoothness of maps in Fréchet spaces which are not Banach spaces has not attracted much attention, compared to C^1_ζ-smoothness and further notions of smoothness [19]. For possible reasons, see [1, Chapter II, Section 3]. In any case, for the study of Eq. (1.1) the notion of C^1_β-smoothness is useful – and yields, of course, slightly stronger results, compared to the theory based on C^1_ζ-smoothness in [23, 24]. The present paper shows how to obtain solution manifolds, solution operators, and local invariant manifolds at stationary points, all of them C^1_β-smooth, and discusses Eqs. (1.2)–(1.4) as examples. We mention in this context that we do not touch upon higher order derivatives, in light of the fact that solution manifolds are in general not better than C^1_β-smooth [13]. The same holds true for infinite-dimensional invariant manifolds in the solution manifold, like the local stable and center-stable manifolds at stationary points, whereas the finite-dimensional local unstable and center manifolds at stationary points may be k times continuously differentiable, $k \in \mathbb{N}$, under appropriate hypotheses on the map f in Eq. (1.1) [10, 12].

The present paper is divided into three parts. Part I with Sections 2–8 is about C^1_ζ- and C^1_β-maps in general. Many results in Part I, notably in Sections 2–4, are known in more general settings, see e.g. [9] and the references given there. We present this material in a form which is convenient for the purpose of this paper, and include proofs for convenience. Section 2 introduces topologies on spaces of continuous linear mappings, among them the topologies β and ζ of uniform convergence on bounded sets and on compact sets, respectively. Section 3 about maps in Fréchet spaces characterizes C^1_ζ-smoothness in terms of the topology ζ (therefore name and symbol) and compares it to C^1_β-smoothness. Section 4 provides elements of calculus for C^1_β-maps, including the chain rule. In order to keep Section 4 short we make extensive use of [6, Part I] on C^1_ζ-smoothness. Sections 5–6 deal with parametrized contractions and with the Implicit Function Theorem based on C^1_β-smoothness.. Section 7 contains simple transversality- and embedding results which yield C^1_β-submanifolds of finite dimension or finite codimension. The content of Sections 5–7 is familiar in the Banach space case, and most of it is well-known also in the C^1_ζ-setting [3, 4, 23, 24].

Part II with Sections 8–12 is about well-posedness of the initial value problem associated with Eq. (1.1). Section 8 introduces the Fréchet and Banach spaces of continuous and differentiable maps from intervals into Euclidean spaces which will be used in the sequel. We mentioned already Proposition 8.3 which establishes that functionals on the space C^1 with property (e) which are C^1_ζ-smooth also are C^1_β-smooth. Section 9 verifies that for examples of the form (1.2)–(1.4) the associated functionals f on the right hand side of Eq. (1.1) are C^1_ζ-smooth and have property (e) – so they are C^1_β-smooth and have property (e). Proposition 9.3 guarantees that the set $X_f \neq \emptyset$ is indeed a C^1_β-submanifold of codimension n in the space C^1. The proof is by Proposition 7.1 on transversality. Sections 10 about *segment evaluation maps* prepares the construction of solutions to Eq. (1.1) which start from initial data in X_f (Section 11). In Section 12 these solutions constitute a continuous semiflow on X_f whose solution operators are C^1_β-smooth. Sections 10–12 are analogous to parts of [23], and we only describe how to modify these parts of [23] in order to obtain the present result on the semiflow.

Part III on local invariant manifolds at equilibria is based on [24] about Eq. (1.1) with f only C^1_ζ-smooth. In Sections 13–17 below we explain how to modify constructions in [24], in order to obtain local invariant manifolds at stationary points of the semiflow which are C^1_β-smooth (and not only C^1_ζ-smooth), by means of the transversality and embedding results from Section 7.

The present approach also shows that a technical hypothesis on smoothness which was made in [24] is obsolete. Let us briefly expand on this. An important ingredient in [24] is [23, Proposition 1.2] which says for a map $f : C^1 \supset U \to \mathbb{R}^n$ that its smoothness has an implication on the nature of the delay: if f is C^1_ζ-smooth then it is of *locally bounded delay* in the sense that

(lbd) *for every $\phi \in U$ there are a neighbourhood $N(\phi) \subset U$ and $d > 0$ such that for all χ, ψ in $N(\phi)$ with*

$$\chi(t) = \psi(t) \quad \textit{for all} \quad t \in [-d, 0]$$

we have $f(\chi) = f(\psi)$.

Property (lbd) is used in [24] in combination with transversality in order to obtain a local stable manifold for Eq. (1.1), with its solution segments defined on $(-\infty, 0]$, from a local stable manifold for an associated equation

$$x'(t) = f_d(x_t) \tag{1.5}$$

with solution segments x_t defined on a *compact* interval. The local stable manifold for (1.5) stems from [7, Section 3.5], where it was found under the hypothesis that the functional on the right hand side of the delay differential equation considered is – in terms of the present paper – C^1_β-smooth and has an extension property analogous to property (e) above. Without knowing Proposition 8.3 of the present paper, the smoothness properties of f_d had to be *assumed* in [24] as property (d).

For other work on delay differential equations with states x_t in Fréchet spaces see [17, 18, 25].

Notation, preliminaries. The closure of a subset M of a topological space is denoted by $\overline{M}$ and its interior is denoted by $\mathring{M}$.

Part I

2 Preliminaries about topological vector spaces

This section provides a proposition about uniform continuity and introduces the two topologies which are relevant in the sequel. Nothing is new, proofs are included in order to make Part I of the paper more self-contained.

A topological vector space T is a vector space over the real or complex field together with a topology on T which makes addition and multiplication by scalars continuous, with respect to product topologies. We follow [16] and assume in addition that singletons in topological vector spaces are closed subsets. For elementary results about topological vector spaces which below are used without proof consult [16].

Recall that a subset B of a topological vector space T over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ is bounded if for every neighbourhood N of $0 \in T$ there exists a real $r_N \geq 0$ with $B \subset rN$ for all reals $r \geq r_N$. The points of convergent sequences form bounded sets, compact sets are bounded. A set $A \subset T$ is balanced if $zA \subset A$ for all $z \in \mathbb{K}$ with $|z| \leq 1$. If A is balanced and $|z| \geq 1$ then $A \subset zA$.

Continuous linear maps between topological vector spaces map bounded sets into bounded sets.

Products of topological vector spaces are always equipped with the product topology.

Proposition 2.1 ([25, Proposition 1.2]). *Suppose T is a topological space, W is a topological vector space, M is a metric space with metric d, $g : T \times M \supset U \to W$ is continuous, $U \supset \{t\} \times K$, $K \subset M$ compact. Then g is uniformly continuous on $\{t\} \times K$ in the following sense: For every neighbourhood N of 0 in W there exist a neighbourhood T_N of t in T and $\epsilon > 0$ such that for all $t' \in T_N$, all $\hat{t} \in T_N$, all $k \in K$, and all $m \in M$ with*

$$d(m,k) < \epsilon \quad \text{and} \quad (t',k) \in U, \quad (\hat{t},m) \in U$$

we have

$$g(t',k) - g(\hat{t},m) \in N.$$

Proof. Choose a neighbourhood N' of 0 in W with $N' + N' \subset N$. For every $k \in K$ there exist open neighbourhoods $T(k)$ of t in T and $\delta(k) > 0$ such that for all $t' \in T(k)$ and all $m \in M$ with $d(m,k) < \delta(k)$ and $(t',m) \in U$ we have

$$g(t',m) - g(t,k) \in N',$$

due to continuity. The compact set K is contained in a finite union of open neighbourhoods

$$\left\{m \in M : d(m,k_j) < \frac{\delta(k_j)}{2}\right\}, \quad j = 1,\dots,n,$$

with $k_1,\dots,k_n$ in K. Set

$$\epsilon = \min\left\{\frac{\delta(k_j)}{2} : j = 1,\dots,n\right\} \quad \text{and} \quad T_N = \cap_{j=1}^{n} T(k_j).$$

Let $t' \in T_N$, $\hat{t} \in T_N$, $k \in K$, and $m \in M$ with

$$d(m,k) < \epsilon, \quad (t',k) \in U, \quad (\hat{t},m) \in U$$

be given. For some $j \in \{1,\dots,n\}$, $d(k,k_j) < \frac{\delta(k_j)}{2}$. By the triangle inequality, $d(m,k_j) < \delta(k_j)$. It follows that

$$g(t',k) - g(\hat{t},m) = (g(t',k) - g(t,k_j)) + (g(t,k_j) - g(\hat{t},m)) \in N' + N' \subset N. \qquad \square$$

Let V, W be topological vector spaces over $\mathbb{R}$ or $\mathbb{C}$. The vector space of continuous linear maps $V \to W$ is denoted by $L_c = L_c(V,W)$. For a given family $\mathcal{F}$ of bounded subsets of V which is closed under finite union and contains all singletons $\{v\} \subset V$, $v \in V$, a topology $\tau = \tau_{\mathcal{F}}$ on $L_c(V,W)$ is defined as follows. For a neighbourhood N of 0 in W and a set $B \in \mathcal{F}$ consider the set

$$U_{N,B} = \{A \in L_c : AB \subset N\}.$$

Every finite intersection of such sets U_{N_j,B_j}, $j \in \{1,\dots,J\}$, contains a set of the same kind, because we have

$$\cap_{j=1}^{J} U_{N_j,B_j} \supset \left\{A \in L_c : A(\cup_{j=1}^{J} B_j) \subset \cap_{j=1}^{J} N_j\right\},$$

$\mathcal{F}$ is closed under finite union, and finite intersections of neighbourhoods of 0 in W are neighbourhoods of 0. Then the topology τ is the set of all $O \subset L_c$ which have the property that for each $A \in O$ there exist a neighbourhood N of 0 in W and a set $B \in \mathcal{F}$ with $A + U_{N,B} \subset O$. It is the easy to show that indeed τ is a topology, with the sets $U_{N,B}$ being neighbourhoods of $0 \in L_c(V,W)$.

For the introduction of the topology τ in case of Hausdorff locally convex spaces see for example [9, Section 0.1].

We call a map A from a topological space T into L_c τ-continuous at a point $t \in T$ if it is continuous at t with respect to the topology τ on L_c.

Remark 2.2.

(i) Convergence of a sequence in L_c with respect to τ is equivalent to uniform convergence on every set $B \in \mathcal{F}$. (Proof: By definition convergence $A_j \to A$ with respect to τ is equivalent to convergence $A_j - A \to 0$ with respect to τ. This means that for each neighbourhood N of 0 in W and for each set $B \in \mathcal{F}$ there exists $j_{N,B} \in \mathbb{N}$ so that for all integers $j \geq j_{N,B}$, $A_j - A \in U_{N,B}$. Or, for all integers $j \geq j_{N,B}$ and all $b \in B$, $(A_j - A)b \in N$. Now the assertion becomes obvious.)

(ii) If V and W are Banach spaces and if $\mathcal{F}$ consists of all bounded subsets of V then τ is the norm topology on $L_c(V,W)$ given by $|A| = \sup_{|v| \leq 1} |Av|$.

(iii) In order to verify τ-continuity of a map $A : T \to L_c$, T a topological space, at some $t \in T$ one has to show that, given a set $B \in \mathcal{F}$ and a neighbourhood N of 0 in W, there exists a neighbourhood N_t of t in T such that for all $s \in N_t$ we have $(A(s) - A(t))(B) \subset N$.

In case T has countable neighbourhood bases the map A is τ-continuous at $t \in T$ if and only if for any sequence $T \ni t_j \to t$ we have $A(t_j) \to A(t)$. For $A(t_j) \to A(t)$ we need that given a set $B \in \mathcal{F}$ and a neighbourhood N of 0 in W, there exists $J \in \mathbb{N}$ with

$$(A(t_j) - A(t))(B) \subset N \quad \text{for all integers } j \geq J.$$

In the sequel we shall use the previous statement frequently.

Proposition 2.3. *Singletons $\{A\} \subset L_c$ are closed with respect to the topology τ, and L_c equipped with the topology τ is a topological vector space.*

Proof. 1. (On singletons) Let $A \in L_c$ be given. We show that $L_c \setminus \{A\}$ is open. Let $S \in L_c \setminus \{A\}$. For some $b \in V$, $Ab \neq Sb$. For some neighbourhood N of 0 in W, $Ab \notin Sb + N$ [16, Theorem 1.12]. For all $S' \in U_{N,\{b\}}$ we have $S'b \in N$, hence $(S + S')b \in Sb + N$, and thereby, $S + S' \neq A$. Hence $S + U_{N,\{b\}} \subset L_c \setminus \{A\}$.

2. (Continuity of addition) Assume S, T in L_c and let $U_{N,B}$ be given, N a neighbourhood of 0 in W and $B \in \mathcal{F}$. We have to find neighbourhoods of S and T so that addition maps their Cartesian product into $S + T + U_{N,B}$. As W is a topological vector space there are neighbourhoods N_T, N_S of 0 in W with $N_T + N_S \subset N$. For every $T' \in T + U_{N_T,B}$ and for every $S' \in S + U_{N_S,B}$ and for every $b \in B$ we get $((T' + S') - (T + S))b = T'b - Tb + S'b - Sb \in N_T + N_S \subset N$, hence $((T' + S') - (T + S))B \subset N$, or $T' + S' \in T + S + U_{N,B}$.

3. (Continuity of multiplication with scalars, in case of vector spaces over $\mathbb{C}$) Let $c \in \mathbb{C}$, $T \in L_c$. Let a neighbourhood N of 0 in W and a set $B \in \mathcal{F}$ be given, and consider the neighbourhood $U_{N,B}$ of 0 in L_c. There is a neighbourhood $\hat{N}$ of 0 in W with $\hat{N} + \hat{N} + \hat{N} \subset N$, see e. g. [16, proof of Theorem 1.10]. We may assume that $\hat{N}$ is balanced [16, Theorem 1.14]. As TB is bounded there exists $r_N \geq 0$ such that for reals $r \geq r_N$, $TB \subset r\hat{N}$. We infer that for some $\epsilon > 0$, $(0, \epsilon)TB \subset \hat{N}$. As $\hat{N}$ is balanced we obtain that for all $z \in \mathbb{C}$ with $0 < |z| < \epsilon$,

$$zTB = z\frac{|z|}{|z|}TB = \frac{z}{|z|}|z|TB \subset \frac{z}{|z|}\hat{N} \subset \hat{N}.$$

For $z = 0$ we also have $zTB \subset \hat{N}$, since $0 \in \hat{N}$. Because of continuity of multiplication $\mathbb{C} \times W \to W$ there are neighbourhoods U_c of 0 in $\mathbb{C}$ and N_c of 0 in W with

$$U_c N_c \subset \hat{N}, \quad c\, N_c \subset \hat{N}, \quad U_c \subset \{z \in \mathbb{C} : |z| < \epsilon\}.$$

Consider $T' \in U_{N_c,B}$ and $c' \in U_c$. Observe

$$(c + c')(T + T') = cT + (c'T + cT' + c'T').$$

For every $b \in B$ we get

$$(c'T + cT' + c'T')b = c'Tb + cT'b + c'T'b \in c'TB + c\, N_c + U_c N_c \subset \hat{N} + \hat{N} + \hat{N} \subset N.$$

Therefore, $c'T + cT' + c'T' \in U_{N,B}$. It follows that

$$(c + U_c)(T + U_{N_c,B}) \subset c\,T + U_{N,B},$$

which yields the desired continuity at (c, T).

4. The arguments in Part 3 also work for vector spaces over $\mathbb{R}$, with a shorter derivation of the inclusion $zTB \subset \hat{N}$ for reals $z \in (-\epsilon, \epsilon)$. □

In case that $\mathcal{F}$ consists of all bounded subsets of V we write β instead of τ, and in case $\mathcal{F}$ consists of all compact subsets of V we write ζ instead of τ. Accordingly we speak of β-continuity and of ζ-continuity. Observe

$$\zeta \subset \beta$$

because $T + U_{N,K} \subset \mathcal{O} \in \zeta$ with K compact also means $T + U_{N,B} \subset \mathcal{O}$ with the bounded set $B = K$. It follows that for a map from a topological space into L_c,

$$\beta\text{-continuity implies } \zeta\text{-continuity.}$$

In case $\dim V < \infty$ we have

$$\zeta = \beta$$

because $T + U_{N,B} \subset \mathcal{O} \in \beta$ with $B \subset V$ bounded yields $T + U_{N,\overline{B}} \subset T + U_{N,B} \subset \mathcal{O}$, and $\overline{B}$ is compact due to $\dim V < \infty$.

3 C^1_ζ-smoothness versus C^1_β-smoothness in Fréchet spaces

A Fréchet space F is a locally convex topological vector space which is complete and metrizable. The topology is given by a sequence of seminorms $|\cdot|_j$, $j \in \mathbb{N}$, which are separating in the sense that $|v|_j = 0$ for all $j \in \mathbb{N}$ implies $v = 0$. The sets

$$N_{j,k} = \left\{ v \in F : |v|_j < \frac{1}{k} \right\}, \quad j \in \mathbb{N} \text{ and } k \in \mathbb{N},$$

form a neighbourhood base at the origin. If the sequence of seminorms is increasing then the sets

$$N_j = \left\{ v \in F : |v|_j < \frac{1}{j} \right\}, \quad j \in \mathbb{N},$$

form a neighbourhood base at the origin.

Products of Fréchet spaces, closed subspaces of Fréchet spaces, and Banach spaces are Fréchet spaces.

A curve in a Fréchet space F is a continuous map c from an interval $I \subset \mathbb{R}$ of positive length into F. For such a curve and for $t \in I$ the tangent vector at $t \in I$ is defined as

$$c'(t) = \lim_{0 \neq h \to 0} \frac{1}{h}(c(t+h) - c(t))$$

provided the limit exists. As in [6, Part I] the curve is said to be continuously differentiable if it has tangent vectors everywhere and if the map

$$c' : I \ni t \mapsto c'(t) \in F$$

is continuous.

For a continuous map $f : V \supset U \to F$, V and F Fréchet spaces and $U \subset V$ open, and for $u \in U$ and $v \in V$ the directional derivative is defined by

$$Df(u)v = \lim_{0 \neq h \to 0} \frac{1}{h}(f(u+hv) - f(u))$$

provided the limit exists. If for $u \in U$ all directional derivatives $Df(u)v$, $v \in V$, exist then the map $Df(u) : V \ni v \mapsto Df(u)v \in F$ is called the derivative of f at u.

Recall the notions of C^1_ζ- and C^1_β-smoothness from Section 1.

It is easy to see that continuously differentiable curves $c : I \to F$ on open intervals $I \subset \mathbb{R}$ are C^1_ζ-smooth and vice versa.

Next the notion of C^1_ζ-smoothness from Section 1 is expressed in terms of ζ-continuity.

Proposition 3.1. *Let a continuous map $f : V \supset U \to W$, V and W Fréchet spaces and $U \subset V$ open, be given. Suppose that all directional derivatives*

$$Df(u)v = \lim_{0 \neq t \to 0} \frac{1}{t}(f(u+tv) - f(u)), \quad u \in U, \quad v \in V,$$

exist, and that every derivative $Df(u) : V \ni v \mapsto Df(u)v \in W$, $u \in U$, is linear and continuous. Then f is C^1_ζ-smooth if and only if the map $Df : U \ni u \mapsto Df(u) \in L_c(V,W)$ is ζ-continuous.

For Part 2 of the following proof compare [9, Lemma 0.1.2].

Proof of Proposition 3.1.

1. Suppose f is C^1_ζ-smooth. Let $u \in U$ and let a neighbourhood $U_{N,K}$ of 0 in $L_c(V,W)$ be given, with a neighbourhood N of 0 in W and a compact set $K \subset V$. The map $g : U \times V \ni (u',v) \mapsto Df(u')v \in W$ is continuous. Proposition 2.1 applies to the set $\{u\} \times K$ and yields a neighbourhood N_u of u in U so that for all $u' \in N_u$ and all $v = v'$ in K,

$$Df(u')v - Df(u)v \in N.$$

Hence $Df(u') \in Df(u) + U_{N,K}$ for all $u' \in N_u$.

2. Suppose the map $Df : U \ni u \mapsto Df(u) \in L_c(V,W)$ is ζ-continuous. Assume $U \ni u_n \to u \in U$ and $V \ni v_n \to v \in V$ as $n \to \infty$. The set

$$K = \{v_n \in V : n \in \mathbb{N}\} \cup \{v\}$$

is compact. By uniform convergence on K,

$$Df(u_n)v_n - Df(u)v_n \to 0 \quad \text{as} \quad n \to \infty.$$

By the continuity of $Df(u)$,

$$Df(u)v_n - Df(u)v \to 0 \quad \text{as} \quad n \to \infty.$$

Together,

$$Df(u_n)v_n - Df(u)v = Df(u_n)v_n - Df(u)v_n + Df(u)v_n - Df(u)v \to 0 \quad \text{as} \quad n \to \infty. \qquad \square$$

The preceding proposition in combination with the relationship between the ζ- and β-topologies yields the following.

Corollary 3.2. *Let V and F be Fréchet spaces, and let a map $f : U \to F$ be given, with $U \subset V$ open.*

(i) f is C^1_β-smooth if and only if f is C^1_ζ-smooth with

$$U \ni u \mapsto Df(u) \in L_c(V,F) \quad \beta\text{-continuous.}$$

(ii) If f is C^1_ζ-smooth and $\dim V < \infty$ *then f is C^1_β-smooth.*

For maps $\mathbb{R}^k \supset U \to F$, $U \subset \mathbb{R}^k$ open and F a Fréchet space, C^1_β-smoothness and C^1_ζ-smoothness are equivalent, and we simply speak of continuously differentiable maps. For a curve $c : I \to F$ on an open interval $I \subset \mathbb{R}$ this notion of continuous differentiability coincides with the original one for curves on more general intervals.

For continuous maps $f : U \to F$, V, W, F Fréchet spaces and $U \subset V \times W$ open, partial derivatives are defined in the usual way. For example, $D_1 f(v,w) : V \to F$ is given by

$$D_1 f(v,w)\hat{v} = \lim_{0 \neq h \to 0} \frac{1}{h}(f(v + h\hat{v}, w) - f(v,w)).$$

4 C^1_β-maps in Fréchet spaces

In this section V, V_1, V_2, F, F_1, F_2 always denote Fréchet spaces. We begin with a few facts about C^1_ζ-maps $f : V \supset U \to F$. These involve the Riemann integral for continuous maps $[a, b] \to F$ into a Fréchet space and results from calculus based on C^1_ζ-smoothness which can be found in [6, Sections I.1–I.4].

Each derivative $Df(u) : V \to F$, $u \in U$, is linear and continuous. Differentiation of C^1_ζ-maps is linear, and the chain rule holds. We have

$$f(v) - f(u) = \int_0^1 Df(u + t(v-u))(v-u)dt \quad \text{for } u + [0,1]v \subset U. \tag{4.1}$$

Linear continuous maps $T \in L_c(V, F)$ are C^1_ζ-smooth with $DT(u) = T$ everywhere. If $f_1 : V \supset U \to F_1$ and $f_2 : V \supset U \to F_2$ are C^1_ζ-smooth then also $f_1 \times f_2 : V \supset U \ni u \mapsto (f_1(u), f_2(u)) \in F_1 \times F_2$ is C^1_ζ-smooth, with

$$D(f_1 \times f_2)(u)v = (Df_1(u)v, Df_2(u)v).$$

Proposition 4.1 (See [6, Part I]). *For continuous $f : V_1 \times V_2 \supset U \to F$, U open, the following statements are equivalent.*

(i) For every $(u_1, u_2) \in U$, $v_1 \in V_1$, $v_2 \in V_2$ the partial derivatives $D_k f(u_1, u_2)v_k$ exist and both maps

$$U \times V_k \ni (u_1, u_2, v_k) \mapsto D_k f(u_1, u_2)v_k \in F, \quad k \in \{1, 2\},$$

are continuous.

(ii) f is C^1_ζ-smooth.

In this case,

$$Df(u_1, u_2)(v_1, v_2) = D_1 f(u_1, u_2)v_1 + D_2 f(u_1, u_2)v_2$$

for all $(u_1, u_2) \in U$, $v_1 \in V_1$, $v_2 \in V_2$.

We turn to C^1_β-smoothness.

Proposition 4.2. *For Banach spaces V and F and $U \subset V$ open a map $f : V \supset U \to F$ is C^1_β-smooth if and only if there exists a continuous map $D_f : U \to L_c(V, F)$ such that for every $u \in U$ and*

(F) for every $\epsilon > 0$ there exists $\delta > 0$ with

$$|f(v) - f(u) - D_f(u)(v-u)| \le \epsilon |v-u| \quad \textit{for all } v \in U \textit{ with } |v-u| < \delta.$$

In this case, $D_f(u)v$ is the directional derivative $Df(u)v$, for every $u \in U, v \in V$.

Proof. We only show that for a C^1_β-map $f : V \supset U \to F$ and $u \in U$ the map $D_f(u) = Df(u) \in L_c(V, F)$ satisfies statement (F). Due to Corollary 3.2 (i) we may use the integral representation (4.1) for C^1_ζ-maps. For v in a convex neighbourhood $N \subset U$ of u this yields

$$\begin{aligned} |f(v) - f(u) - Df(u)(v-u)| &= \left| \int_0^1 Df(u + s(v-u))[v-u]ds - \int_0^1 Df(u)[v-u]ds \right| \\ &= \left| \int_0^1 [Df(u + s(v-u)) - Df(u)][v-u]du \right| \le \int_0^1 |\dots| ds \\ &\le \max_{0 \le s \le 1} |Df(u + s(v-u)) - Df(u)||v-u|. \end{aligned}$$

Now the continuity of Df at u completes the proof. □

Continuous linear maps $T : V \to F$ are C^1_β-smooth because they are C^1_ζ-smooth with constant derivative, $DT(u) = T$ for all $u \in V$. Using Corollary 3.2 (i) and continuity of addition and multiplication on $L_c(V, F)$ (with the topology β) one obtains from the properties of C^1_ζ-maps that linear combinations of C^1_β-maps are C^1_β-maps, that also for C^1_β-maps differentiation is linear, and the integral formula (4.1) holds. If $f_1 : V \supset U \to F_1$ and $f_2 : V \supset U \to F_2$ are C^1_β-smooth then also $f_1 \times f_2 : V \supset U \ni u \mapsto (f_1(u), f_2(u)) \in F_1 \times F_2$ is C^1_β-smooth, with

$$D(f_1 \times f_2)(u)v = (Df_1(u)v, Df_2(u)v).$$

This follows easily from the analogous property for C^1_ζ-maps, by means of the formula for the directional derivatives of $f_1 \times f_2$ and considering neighbourhoods of 0 in $F_1 \times F_2$ which are products of neighbourhoods of 0 in F_j, $j \in \{1, 2\}$.

Proposition 4.3 (Chain rule). *If $f : V \supset U \to F$ and $g : F \supset W \to G$ are C^1_β-maps, with $f(U) \subset W$, then also $g \circ f$ is a C^1_β-map.*

Compare [9, Corollary 1.3.2 in combination with Corollary 1.0.4].

Proof of Proposition 4.3. 1. The chain rule for C^1_ζ-maps yields that $g \circ f$ is C^1_ζ-smooth with $D(g \circ f)(u) = D(g(f(u))) \circ Df(u)$ for all $u \in U$. So it remains to prove that the map $U \ni u \mapsto Dg(f(u)) \circ Df(u) \in L_c(V, G)$ is β-continuous. As V has countable neighbourhood bases it is enough to show that, given a sequence $U \ni u_j \to u \in U$, a bounded set $B \subset V$, and a neighbourhood N of 0 in G, we have

$$[Dg(f(u_j)) \circ Df(u_j) - Dg(f(u)) \circ Df(u)]B \subset N \quad \text{for } j \in \mathbb{N} \text{ sufficiently large.}$$

So let a sequence $U \ni u_j \to u \in U$, a bounded set $B \subset V$, and a neighbourhood N of 0 in G be given.

2. There is a neighbourhood N_1 of 0 in G with $N_1 + N_1 + N_1 + N_1 \subset N$, see [16, proof of Theorem 1.10]. By linearity, for every $j \in \mathbb{N}$,

$$\begin{aligned} &Dg(f(u_j)) \circ Df(u_j) - Dg(f(u)) \circ Df(u) \\ &\quad = [Dg(f(u_j)) - Dg(f(u))] \circ Df(u_j)) + Dg(f(u)) \circ [Df(u_j) - Df(u)]. \end{aligned}$$

3. Consider the last term. By continuity of $Dg(f(u))$ at $0 \in F$, there is a neighbourhood N_2 of 0 in F with $Dg(f(u))N_2 \subset N_1$. By β-continuity of Df at $u \in U$, there is an integer j_s such that for all integers $j \geq j_2$,

$$[Df(u_j) - Df(u)]B \subset N_2.$$

Hence, for all integers $j \geq j_2$,

$$Dg(f(u)) \circ [Df(u_j) - Df(u)]B \subset N_1.$$

4. $Df(u)B$ is bounded. Using β-continuity of Dg at $f(u)$ and $\lim_{j\to\infty} f(u_j) = f(u)$ we find an integer $j_3 \geq j_2$ such that for all integers $j \geq j_3$ we have

$$[Dg(f(u_j)) - Dg(f(u))]Df(u)B \subset N_1.$$

5. Now we use the continuity of $W \times F \ni (w, h) \mapsto Dg(w)h \in G$ at $(f(u), 0)$. We find a neighbourhood N_3 of 0 in F and an integer $j_4 \geq j_3$ such that for all integers $j \geq j_4$ we have $Dg(f(u_j))N_3 \subset N_1$ and $-Dg(f(u))N_3 \subset N_1$. This yields

$$[Dg(f(u_j)) - Dg(f(u))]N_3 \subset N_1 + N_1 \quad \text{for all integers } j \geq j_4.$$

6. The β-continuity of Df at $u \in U$ yields an integer $j_N \geq j_4$ such that for all integers $j \geq j_N$ we have

$$[Df(u_j) - Df(u)]B \subset N_3,$$

hence

$$Df(u_j)B \subset Df(u)B + N_3.$$

7. For integers $j \geq j_N$ we obtain

$$\begin{aligned}
&[Dg(f(u_j)) \circ Df(u_j) - Dg(f(u)) \circ Df(u)]B \\
&\quad = [Dg(f(u_j)) - Dg(f(u))] \circ Df(u_j)) + Dg(f(u)) \circ [Df(u_j) - Df(u)]B \quad \text{(see Part 2)} \\
&\quad \subset [Dg(f(u_j)) - Dg(f(u))]Df(u_j))B + [Dg(f(u)) \circ [Df(u_j) - Df(u)]B \\
&\quad \subset [Dg(f(u_j)) - Dg(f(u))](Df(u)B + N_3) + N_1 \quad \text{(see Parts 6 and 3)} \\
&\quad \subset [Dg(f(u_j)) - Dg(f(u))]Df(u)B + [Dg(f(u_j)) - Dg(f(u))]N_3 + N_1 \\
&\quad \subset N_1 + (N_1 + N_1) + N_1 \quad \text{(see Parts 4 and 5)} \\
&\quad \subset N.
\end{aligned}$$

□

Proposition 4.4. *For a continuous map $f : V_1 \times V_2 \supset U \to F$, U open, the following statements are equivalent.*

(i) For all $(u_1, u_2) \in U$ and all $v_k \in V_k$, $k \in \{1,2\}$, f has a partial derivative $D_k f(u_1, u_2)v_k \in F$, all maps

$$D_k f(u_1, u_2) : V_k \to F, \quad (u_1, u_2) \in U, \quad k \in \{1,2\},$$

are linear and continuous, and the maps

$$U \ni (u_1, u_2) \mapsto D_k f(u_1, u_2) \in L_c(V_k, F), \quad k \in \{1,2\},$$

are β-continuous.

(ii) f is C^1_β-smooth.

In this case,

$$Df(u_1, u_2)(v_1, v_2) = D_1 f(u_1, u_2)v_1 + D_2 f(u_1, u_2)v_2$$

for all $(u_1, u_2) \in U$, $v_1 \in V_1$, $v_2 \in V_2$.

Proof. 1. Suppose (ii) holds. Then f is C^1_ζ-smooth, and all statements in (i) up to the last one follow from Proposition 4.1 on partial derivatives. In order to deduce the last statement in (i) for $k = 1$ let a sequence $((u_{1j}, u_{2j}))_{j=1}^\infty$ in U be given which converges to some $(u_1, u_2) \in U$. Let a neighbourhood N of 0 in F and a bounded set $B_1 \subset V_1$ be given, consider the neighbourhood U_{N,B_1} of 0 in $L_c(V_1, F)$. As $V_1 \ni v \mapsto (v, 0) \in V_1 \times V_2$ is linear and continuous, $B_1 \times \{0\}$ is a bounded subset of $V_1 \times V_2$. As f is C^1_β-smooth the map Df is β-continuous, and for j sufficiently large we get $(Df(u_{1j}, u_{2j}) - Df(u_1, u_2))(B_1 \times \{0\}) \subset N$ which yields $(D_1 f(u_{1j}, u_{2j}) - D_1 f(u_1, u_2))B_1 \subset N$. For $k = 2$ the proof is analogous.

2. Suppose (i) holds.

2.1. Claim: Both maps $U \times V_k \ni (u_1, u_2, v_k) \mapsto D_k f(u_1, u_2)v_k \in F$, $k \in \{1,2\}$, are continuous.

Proof for $k=1$: Let a sequence $(u_{1j},u_{2j},v_{1j})_1^\infty$ in $U\times V_1$ be given which converges to some $(u_1,u_2,v_1)\in U\times V_1$. Then $v_{1j}\to v_1$ in V_1, and $B_1=\{v_{1j}:j\in\mathbb{N}\}\cup\{v_1\}$ is a bounded subset of V_1. Let a neighbourhood N of 0 in F be given. By the β-continuity of D_1f,

$$(D_1f(u_{1j},u_{2j})-D_1f(u_1,u_2))B_1\subset N\quad \text{for } j \text{ sufficiently large.}$$

For each $j\in N$ we have

$$\begin{aligned} D_1f(u_{1j},u_{2j})v_{1j}-D_1f(u_1,u_2)v_1 \\ =(D_1f(u_{1j},u_{2j})-D_1f(u_1,u_2))v_{1j}+D_1f(u_1,u_2))(v_{1j}-v_1). \end{aligned}\tag{4.2}$$

Now it becomes obvious how to complete the proof, using the last equation, the statement right before it, and continuity of $D_1f(u_1,u_2)$.

2.2. Proposition 4.1 on partial derivatives applies and yields that f is C^1_ζ-smooth, with

$$Df(u_1,u_2)(v_1,v_2)=D_1f(u_1,u_2)v_1+D_2f(u_1,u_2)v_2$$

for all $(u_1,u_2)\in U$, $v_1\in V_1$, $v_2\in V_2$. According to Corollary 3.2 (i) it remains to prove that the map $Df:U\to L_c(V_1\times V_2,F)$ is β-continuous. The projections pr_k of $V_1\times V_2$ onto the factor V_k, for $k\in\{1,2\}$, are linear and continuous. For every $(u_1,u_2)\in U$ we have

$$Df(u_1,u_2)=D_1f(u_1,u_2)\circ pr_1+D_2f(u_1,u_2)\circ pr_2,$$

so it is sufficient to show that both maps

$$V_1\times V_2\supset U\ni(u_1,u_2)\mapsto D_kf(u_1,u_2)\circ pr_k\in L_c(V_1\times V_2,F),\quad k\in\{1,2\},$$

are β-continuous. We deduce this for $k=1$. Let a sequence $(u_{1j},u_{2j})_1^\infty$ in U be given which converges to some $(u_1,u_2)\in U$, as well as a bounded subset $B\subset V_1\times V_2$ and a neighbourhood N of 0 in F. We need to show

$$(D_1f(u_{1j},u_{2j})\circ pr_1-D_1f(u_1,u_2)\circ pr_1)B\subset N$$

for $j\in\mathbb{N}$ sufficiently large. $B_1=pr_1B$ is a bounded subset of V_1, and for every $j\in\mathbb{N}$ we have

$$(D_1f(u_{1j},u_{2j})\circ pr_1-D_1f(u_1,u_2)\circ pr_1)B\subset(D_1f(u_{1j},u_{2j})-D_1f(u_1,u_2))B_1.$$

The β-continuity of the map D_1f yields that the last set is contained in N for j sufficiently large. □

5 Contractions with parameters

The proof of Theorem 5.2 below employs twice the following basic uniform contraction principle.

Proposition 5.1 (See for example [2, Appendix VI, Proposition 1.2]). *Let a Hausdorff space T, a complete metric space M, and a map $f:T\times M\to M$ be given. Assume that f is a uniform contraction in the sense that there exists $k\in[0,1)$ so that*

$$d(f(t,x),f(t,y))\le k\,d(x,y)$$

for all $t\in T, x\in M, y\in M$, and $f(\cdot,x):T\to M$ is continuous for each $x\in M$. Then the map $g:T\to M$ given by $g(t)=f(t,g(t))$ is continuous.

Theorem 5.2. *Let a Fréchet space T, a Banach space B, open sets $V \subset T$ and $O_B \subset B$, and a C^1_β-map $A : V \times O_B \to B$ be given. Assume that for a closed set $M \subset O_B$ we have $A(V \times M) \subset M$, and A is a uniform contraction in the sense that there exists $k \in [0,1)$ so that*

$$|A(t,x) - A(t,y)| \leq k|x - y|$$

for all $t \in V, x \in O_B, y \in O_B$. Then the map $g : V \to B$ given by $g(t) = A(t, g(t)) \in M$ is C^1_β-smooth.

Notice that the derivative $\Gamma = Dg(t)\hat{t}$ of the map g satisfies the equation

$$\Gamma = D_1 A(t, g(t))\hat{t} + D_2 A(t, g(t))\Gamma. \tag{5.1}$$

Proof of Theorem 5.2. 1. A is continuous. So Proposition 5.1 applies to the restriction of A to $V \times M$ and yields a continuous map $g : V \to B$ with $g(t) = A(t, g(t)) \in M$ for all $t \in V$. Choose $\kappa \in (k, 1)$. Each linear map $D_2 A(t,x) : B \to B$, $(t,x) \in V \times O_B$, is continuous. The contraction property yields

$$|D_2 A(t,x)| = \sup_{|\hat{x}| \leq 1} |D_2 A(t,x)\hat{x}| \leq \kappa \quad \text{for all } (t,x) \in V \times O_B$$

since given $\epsilon = \kappa - k$ and $t \in V$, $x \in O_B$, and $\hat{x} \in B$ with $|\hat{x}| \leq 1$ there exists $\delta > 0$ such that for $h = \frac{\delta}{2}$,

$$x + h\hat{x} \in O_B \quad \text{and}$$

$$\begin{aligned} &|h^{-1}(A(t,x) - A(t, x + h\hat{x})) - D_2 A(t,x)\hat{x}| \\ &\quad = |h^{-1}(A(t,x) - A(t, x + h\hat{x})) - DA(t,x)(0, \hat{x})| \leq \epsilon, \end{aligned}$$

hence

$$\begin{aligned} |h||D_2 A(t,x)\hat{x}| &\leq \epsilon|h| + |A(t, x + h\hat{x}) - A(t,x)| \\ &\leq \epsilon|h| + k|h\hat{x}| \leq (\epsilon + k)|h| = \kappa|h|. \end{aligned}$$

Divide by $|h| = h$.

2. It follows that each map $id_B - D_2 A(t,x) \in L_c(B,B)$, $t \in V$ and $x \in O_B$, is a topological isomorphism. As A is C^1_β-smooth we get that the map

$$V \times O_B \ni (t,x) \mapsto D_2 A(t,x) \in L_c(B,B)$$

is β-continuous, or equivalently, continuous with respect to the usual norm-topology on $L_c(B,B)$. As inversion is continuous we see that also the map

$$V \times O_B \ni (t,x) \mapsto (id_B - D_2 A(t,x))^{-1} \in L_c(B,B)$$

is continuous.

3. For all $(t, x, \hat{t}) \in V \times O_B \times T$ and for all $\hat{x}, \hat{y}$ in B we have

$$\begin{aligned} |DA(t,x)(\hat{t}, \hat{x}) - DA(t,x)(\hat{t}, \hat{y})| &= |DA(t,x)(0, \hat{x} - \hat{y})| \\ &= |D_2 A(t,x)(\hat{x} - \hat{y})| \leq \kappa|\hat{x} - \hat{y}|. \end{aligned}$$

Hence Proposition 5.1 applies to the version

$$\Gamma = D_1A(t,x)\hat{t} + D_2A(t,x)\Gamma$$

of Eq. (5.1) with parameters $(t,x,\hat{t}) \in V \times O_B \times T$ and yields a continuous map $\gamma : V \times O_B \times T \to B$ with

$$\gamma(t,x,\hat{t}) = D_1A(t,x)\hat{t} + D_2A(t,x)\gamma(t,x,\hat{t}) \quad \text{for all } (t,x,\hat{t}) \in V \times O_B \times T,$$

or equivalently,

$$\gamma(t,x,\hat{t}) = (id_B - D_2A(t,x))^{-1}D_1A(t,x)\hat{t} \quad \text{for all } (t,x,\hat{t}) \in V \times O_B \times T.$$

This shows that each map $\gamma(t,x,\cdot)$, $(t,x) \in V \times O_B$, belongs to $L_c(T,B)$.

Claim: The map

$$\tilde{\gamma} : V \times O_B \ni (t,x) \mapsto \gamma(t,x,\cdot) \in L_c(T,B)$$

is β-continuous.

Proof. Let a sequence $(t_j,x_j)_1^\infty$ in $V \times O_B$ converge to a point $(t,x) \in V \times O_B$. Consider a neighbourhood N of 0 in B and a bounded set $T_b \subset T$. We have to show that for $j \in \mathbb{N}$ sufficiently large, $(\tilde{\gamma}(t_j,x_j) - \tilde{\gamma}(t,x))T_b \subset N$. For all $j \in \mathbb{N}$ and all $\hat{t} \in T_b$ we have

$$\begin{aligned}
&|(\tilde{\gamma}(t_j,x_j) - \tilde{\gamma}(t,x))\hat{t}| \\
&\quad = |((id_B - D_2A(t_j,x_j))^{-1}D_1A(t_j,x_j) - (id_B - D_2A(t,x))^{-1}D_1A(t,x))\hat{t}| \\
&\quad \le |(id_B - D_2A(t_j,x_j))^{-1} - (id_B - D_2A(t,x))^{-1})||D_1A(t_j,x_j)\hat{t}| \\
&\qquad + |(id_B - D_2A(t,x))^{-1})||(D_1A(t_j,x_j) - D_1A(t,x))\hat{t}| \\
&\quad \le |(id_B - D_2A(t_j,x_j))^{-1} - (id_B - D_2A(t,x))^{-1})|(|(D_1A(t_j,x_j) - D_1A(t,x))\hat{t}| \\
&\qquad + |D_1A(t,x)\hat{t}|) + |(id_B - D_2A(t,x))^{-1})||(D_1A(t_j,x_j) - D_1A(t,x))\hat{t}|.
\end{aligned}$$

Now it becomes obvious how to complete the proof, using

$$|(id_B - D_2A(t_j,x_j))^{-1} - (id_B - D_2A(t,x))^{-1})| \to 0 \quad \text{as } j \to \infty,$$

boundedness of $|D_1A(t,x)T_b|$, and β-continuity of the partial derivative

$$D_1A : V \times O_B \to L_c(T,B)$$

due to Proposition 4.4.

4. Consider the continuous map $\xi : V \times T \ni (t,\hat{t}) \mapsto \gamma(t,g(t),\hat{t}) \in B$. Using Part 3 we observe that the map $V \ni t \mapsto \xi(t,\cdot) \in L_c(T,B)$ is β-continuous. It remains to show that for all $t \in V$ and all $\hat{t} \in T$ we have

$$\lim_{0 \ne h \to 0} \frac{1}{h}(g(t+h\hat{t}) - g(t)) = \xi(t,\hat{t}),$$

which means that the directional derivative $Dg(t)\hat{t}$ exists and equals $\xi(t,\hat{t})$.

So let $t \in V$ and $\hat{t} \in T$ be given. Choose a convex neighbourhood $N_B \subset O_B$ of $g(t)$. There exists $\delta > 0$ such that for $-\delta \le h \le \delta$,

$$t + h\hat{t} \in V \quad \text{and} \quad g(t+h\hat{t}) \in N_B.$$

Notice that for all $h \in [-\delta, \delta]$ and for all $\theta \in [0,1]$,

$$g(t) + \theta(g(t + h\hat{t}) - g(t)) \in N_B.$$

With the abbreviation

$$\begin{aligned}\xi = \xi(t,\hat{t}) &= \gamma(t, g(t), \hat{t}) = D_1A(t, g(t))\hat{t} + D_2A(t, g(t))\gamma(t, g(t), \hat{t}) \\ &= D_1A(t, g(t))\hat{t} + D_2A(t, g(t))\xi\end{aligned}$$

one finds that

$$\begin{aligned}
&h^{-1}(g(t+h\hat{t}) - g(t)) - \xi = h^{-1}(A(t+h\hat{t}, g(t+h\hat{t})) - A(t, g(t)) - \xi, \quad \text{with } 0 < |h| < \delta, \\
&\quad = h^{-1}(A(t+h\hat{t}, g(t+h\hat{t})) - A(t+h\hat{t}, g(t)) - D_1A(t, g(t))\hat{t} - D_2A(t, g(t))\xi \\
&\qquad + h^{-1}(A(t+h\hat{t}, g(t)) - A(t, g(t))) \\
&\quad = h^{-1}(A(t+h\hat{t}, g(t)) - A(t, g(t))) - D_1A(t, g(t))\hat{t} \\
&\qquad + h^{-1}(A(t+h\hat{t}, g(t+h\hat{t})) - A(t+h\hat{t}, g(t)) \\
&\qquad - \int_0^1 D_2A(t+h\hat{t}, g(t) + \theta[g(t+h\hat{t}) - g(t)])\xi d\theta \\
&\qquad + \int_0^1 \{D_2A(t+h\hat{t}, g(t) + \theta[g(t+h\hat{t}) - g(t)]) - D_2A(t, g(t))\}\xi d\theta \\
&\quad = h^{-1}(A(t+h\hat{t}, g(t)) - A(t, g(t))) - D_1A(t, g(t))\hat{t} \\
&\qquad + \int_0^1 h^{-1}D_2A(t+h\hat{t}, g(t) + \theta[g(t+h\hat{t}) - g(t)])[g(t+h\hat{t}) - g(t)]d\theta \\
&\qquad - \int_0^1 D_2A(t+h\hat{t}, g(t) + \theta[g(t+h\hat{t}) - g(t)])\xi d\theta \\
&\qquad + \int_0^1 \{D_2A(t+h\hat{t}, g(t) + \theta[g(t+h\hat{t}) - g(t)]) - D_2A(t, g(t))\}\xi d\theta \\
&\quad = h^{-1}(A(t+h\hat{t}, g(t)) - A(t, g(t))) - D_1A(t, g(t))\hat{t} \\
&\qquad + \int_0^1 D_2A(t+h\hat{t}, g(t) + \theta[g(t+h\hat{t}) - g(t)])[h^{-1}(g(t+h\hat{t}) - g(t)) - \xi]d\theta \\
&\qquad + \int_0^1 \{D_2A(t+h\hat{t}, g(t) + \theta[g(t+h\hat{t}) - g(t)]) - D_2A(t, g(t))\}\xi d\theta.
\end{aligned}$$

Hence

$$|h^{-1}(g(t+h\hat{t}) - g(t)) - \xi|$$

is majorized by

$$\begin{aligned}&|h^{-1}(A(t+h\hat{t}, g(t)) - A(t, g(t))) - D_1A(t, g(t))\hat{t}| + \kappa|h^{-1}(g(t+h\hat{t}) - g(t)) - \xi| \\ &\quad + \left|\int_0^1 \{D_2A(t+h\hat{t}, g(t) + \theta[g(t+h\hat{t}) - g(t)]) - D_2A(t, g(t))\}\xi d\theta\right|,\end{aligned}$$

which yields

$$\begin{aligned}&(1-\kappa)|h^{-1}(g(t+h\hat{t}) - g(t)) - \xi| \\ &\quad \le |h^{-1}(A(t+h\hat{t}, g(t)) - A(t, g(t))) - D_1A(t, g(t))\hat{t}| \\ &\qquad + \left|\int_0^1 \{D_2A(t+h\hat{t}, g(t) + \theta[g(t+h\hat{t}) - g(t)]) - D_2A(t, g(t))\}\xi d\theta\right|.\end{aligned}$$

The first term in the last expression converges to 0 as $0 \neq h \to 0$. The map

$$[-\delta,\delta] \times [0,1] \ni (h,\theta) \mapsto \{D_2A(t+h\hat{t}, g(t)+\theta[g(t+h\hat{t})-g(t)]) - D_2A(t,g(t))\}\xi \in B$$

is uniformly continuous with value 0 on $\{0\} \times [0,1]$. This implies that for $0 \neq h \to 0$ the last integrand converges to 0 uniformly with respect to $\theta \in [0,1]$. Therefore the last integral tends to 0 as $0 \neq h \to 0$. □

6 The Implicit Function Theorem

From Theorem 5.2 one obtains the following Implicit Function Theorem.

Theorem 6.1. *Let a Fréchet space T, Banach spaces B and E, an open set $U \subset T \times B$, a C^1_β-map $f: U \to E$, and a zero $(t_0, x_0) \in U$ of f be given. Assume that $D_2f(t_0,x_0): B \to E$ is bijective. Then there are open neighbourhoods V of t_0 in T and W of x_0 in B with $V \times W \subset U$ and a C^1_β-map $g: V \to W$ with $g(t_0) = x_0$ and*

$$\{(t,x) \in V \times W : f(t,x) = 0\} = \{(t,x) \in V \times W : x = g(t)\}.$$

The proof follows the usual pattern, paying attention to C^1_β-smoothness.

Proof of Theorem 6.1. 1. (A fixed point problem) Choose an open neighbourhood $N_{T,1}$ of t_0 and a convex open neighbourhood N_B of x_0 in B with $N_{T,1} \times N_B \subset U$. The equation

$$f(t,x) = f(t,x_0) + D_2f(t_0,x_0)[x-x_0] + R(t,x)$$

defines a C^1_β-map $R: N_{T,1} \times N_B \to E$, with $R(t,x_0) = 0$ for all $t \in N_{T,1}$,

$$D_2R(t,x) = D_2f(t,x) - D_2f(t_0,x_0) \quad \text{for all} \quad t \in N_{T,1} \quad \text{and} \quad x \in N_B,$$

and in particular, $D_2R(t_0,x_0) = 0$. The map

$$N_{T,1} \times N_B \ni (t,x) \mapsto D_2R(t,x) \in L_c(B,E)$$

is β-continuous. In order to solve the equation $0 = f(t,x)$, $(t,x) \in N_{T,1} \times N_B$, for x as a function of t, observe that this equation is equivalent to

$$0 = f(t,x_0) + D_2f(t_0,x_0)[x-x_0] + R(t,x),$$

or,

$$\begin{aligned} x &= x_0 + (D_2f(t_0,x_0))^{-1}[-f(t,x_0) - R(t,x)] \\ &= x_0 - (D_2f(t_0,x_0))^{-1}f(t,x_0) - (D_2f(t_0,x_0))^{-1}R(t,x). \end{aligned}$$

The last expression defines a map

$$A: N_{T,1} \times N_B \to B$$

with $A(t_0,x_0) = x_0$, and for $(t,x) \in N_{T,1} \times N_B$,

$$0 = f(t,x) \quad \text{if and only if} \quad x = A(t,x).$$

The map A is C^1_β-smooth since the linear map $D_2f(t_0,x_0))^{-1}: E \to B$ is continuous, due to the Open Mapping Theorem.

2. (Contraction) For all $t \in N_{T,1}$ and for all $x, \hat{x}$ in N_B,

$$\begin{aligned}|A(t,\hat{x}) - A(t,x)| &= |-(D_2f(t_0,x_0))^{-1}R(t,\hat{x}) + (D_2f(t_0,x_0))^{-1}R(t,x)| \\ &\leq |(D_2f(t_0,x_0))^{-1}| \left| \int_0^1 D_2R(t, x+s[\hat{x}-x])[\hat{x}-x]ds \right|.\end{aligned}$$

Let

$$\epsilon = \frac{1}{2|(D_2f(t_0,x_0))^{-1}|}.$$

There are an open neighbourhood $N_{T,2} \subset N_{T,1}$ of t_0 and $\delta > 0$ such that for all $t \in N_{T,2}$ and all $x \in B$ with $|x - x_0| \leq \delta$,

$$x \in N_B \quad \text{and} \quad |D_2R(t,x)| = |D_2R(t,x) - D_2R(t_0,x_0)| < \epsilon.$$

For all $x \neq \hat{x}$ in B with $|x - x_0| \leq \delta$ and $|\hat{x} - x_0| \leq \delta$ and for all $t \in N_{T,2}$ and $s \in [0,1]$ it follows that $|x + s[\hat{x} - x] - x_0| \leq \delta$, hence

$$\left| D_2R(t, x+s[\hat{x}-x]) \frac{1}{|\hat{x}-x|}[\hat{x}-x] \right| < \epsilon,$$

and thereby

$$|A(t,\hat{x}) - A(t,x)| \leq \epsilon|\hat{x}-x||(D_2f(t_0,x_0))^{-1}| = \frac{1}{2}|\hat{x}-x|.$$

3. (Invariance) By continuity there is an open neighbourhood $N_{T,3} \subset N_{T,2}$ of t_0 such that

$$|A(t,x_0) - A(t_0,x_0)| < \frac{\delta}{4} \quad \text{for all } t \in N_{T,3}.$$

For all $t \in N_{T,3}$ and $x \in B$ with $|x - x_0| \leq \delta$ this yields

$$\begin{aligned}|A(t,x) - x_0| &= |A(t,x) - A(t_0,x_0)| \leq |A(t,x) - A(t,x_0)| + |A(t,x_0) - A(t_0,x_0)| \\ &< \frac{1}{2}|x - x_0| + \frac{\delta}{4} \leq \frac{\delta}{2} + \frac{\delta}{4} = \frac{3\delta}{4}.\end{aligned}$$

4. Set $V = N_{T,3}$, $O_B = \{x \in B : |x - x_0| < \delta\}$, and

$$M = \left\{ x \in B : |x - x_0| \leq \frac{3\delta}{4} \right\},$$

and apply Theorem 5.2 to the restriction of A to the set $V \times O_B$. This yields a C^1_β-map $g : V \to B$ with $g(t) = A(t, g(t)) \in O_B$ for all $t \in V$. Using Part 3 we get $|g(t) - x_0| < \frac{3\delta}{4}$ for all $t \in V$. Set

$$W = \left\{ x \in B : |x - x_0| < \frac{3\delta}{4} \right\}.$$

Then $g(V) \subset W$. From $g(t) = A(t, g(t))$ for all $t \in V$ we obtain $0 = f(t, g(t))$ for these t. Conversely, if $0 = f(t,x)$ for $(t,x) \in V \times W \subset V \times M$, then $x = A(t,x)$, hence $x = g(t)$. In particular, $x_0 = g(t_0)$. □

7 Submanifolds by transversality and embedding

C^1_β-submanifolds of a Fréchet space are defined in the same way as continuously differentiable submanifolds of a Banach space. We begin with the simple facts which are instrumental in the proofs of transversality and embedding results below, and in Parts II–III.

A C^1_β-diffeomorphism is an injective C^1_β-map from an open subset U of a Fréchet space F onto an open subset W of a Fréchet space V whose inverse defined on $W \subset V$ is a C^1_β-map.

Let $F = G \oplus H$ be a direct sum decomposition of a Fréchet space F into closed subspaces. A subset $M \subset F$ is a C^1_β-submanifold of F (modelled over the Fréchet space G) if for every point $m \in M$ there are an open neighbourhood U in F and a C^1_β-diffeomorphism $K : U \to F$ onto $W = K(U)$ with

$$K(M \cap U) = W \cap G.$$

For a subset M of a Fréchet space F the tangent cone of M at $x \in M$ is the set T_xM of all tangent vectors $v = c'(0)$ of continuously differentiable curves $c : I \to F$ with I open, $0 \in I$, $c(0) = x$, $c(I) \subset M$. If M is a C^1_β-submanifold then the tangent cones of M are closed subspaces of F. For a direct sum decomposition $F = G \oplus H$ and a C^1_β-diffeomorphism K as before in the definition of a C^1_β-submanifold the map $(DK(m))^{-1}$ defines a topological isomorphism from G onto T_mM, and K^{-1} defines an injective map P from the open neighbourhood $V \cap G$ of $K(m)$ in G onto the open neighbourhood $U \cap M$ of m in M.

Open subsets of C^1_β-submanifolds are C^1_β-submanifolds.

A C^1_β-map $h : M \to H$, M a C^1_β-submanifold of F and H a Fréchet space, is defined by the property that for all local parametrizations P as above the composition $f \circ P$ is a C^1_β-map.

For h as before and $m \in M$ the derivative $T_mh : T_mM \to H$ is defined by $T_mh(t) = (h \circ c)'(0)$, for any continuously differentiable curve $c : I \to F$ with $c(0) = m$, $c(I) \subset M$, $c'(0) = t$. The map T_mh is linear and continuous.

In case $h(M)$ is contained in a C^1_β-submanifold M_H of H and $z : M_H \to Z$ is C^1_β-smooth the chain rule holds, with $T_mh(T_mM) \subset T_{h(m)}M_H$ and $T_m(z \circ h)(t) = T_{h(m)}zT_mh(t)$.

The restriction of a C^1_β-map on an open subset of F to a C^1_β-submanifold M of F, with range in a Fréchet space H, is a C^1_β-map from M into the target space.

Proposition 7.1. *Let a C^1_β-map $g : F \supset U \to G$ and a C^1_β-submanifold $M \subset G$ of finite codimension m be given. Assume that g and M are transversal at a point $x \in g^{-1}(M)$ in the sense that*

$$G = Dg(x)F + T_{g(x)}M.$$

Then there is an open neighbourhood V of x in U so that $V \cap g^{-1}(M)$ is a C^1_β-submanifold of codimension m in F, and $T_x(g^{-1}(M) \cap V) = Dg(x)^{-1}T_{g(x)}M$.

In case dim $G = m$*, $M = \{g(x)\}$, and $Dg(x)$ surjective the assertion holds with $T_{g(x)}M = \{0\}$.*

Proof for $M \neq \{g(x)\}$. 1. There are an open neighbourhood N_G of $\gamma = g(x)$ in G and a C^1_β-diffeomorphism $K : N_G \to G$ onto an open set $U_G \subset G$ such that $K(\gamma) = 0$, $K(N_G \cap M) = U_G \cap T_\gamma M$. We may assume $DK(\gamma) = id$ since otherwise we can replace K with $DK(\gamma)^{-1} \circ K$. Then $DK(\gamma) = id$ maps $T_\gamma M$ onto itself.

2. By transversality and codim $M = m$ we find a subspace $Q \subset Dg(x)F$ of dimension m which complements $T_\gamma M$ in G,

$$G = T_\gamma M \oplus Q.$$

The projection $P : G \to Q$ along $T_\gamma M$ onto Q is linear and continuous (see [16, Theorem 5.16]), and $PDK(\gamma)Dg(x) = PDg(x)$ is surjective. The preimage $U_F = g^{-1}(N_G)$ is open, with $x \in U_F \subset U$. For $z \in U_F$ we have

$$z \in g^{-1}(M) \cap U_F \Leftrightarrow g(z) \in M \cap N_G \Leftrightarrow PK(g(z)) = 0.$$

For the C^1_β-map $h = P \circ K \circ (g|_{U_F})$ we infer $g^{-1}(M) \cap U_F = h^{-1}(0)$. The derivative $Dh(x) : F \to Q$ is surjective. It follows that there is a subspace R of F with dim $R =$ dim $Q = m$ and

$$F = Dh(x)^{-1}(0) \oplus R.$$

The restriction $Dh(x)|_R$ is an isomorphism.

3. The C^1_β-map

$$H : \{(z,r) \in Dh(x)^{-1}(0) \times R : x + z + r \in U_F\} \ni (z,r) \mapsto h(x+z+r) \in Q$$

satisfies $H(0,0) = 0$. Because of $D_2H(0,0)\hat{r} = Dh(x)\hat{r}$ for all $\hat{r} \in R$ and dim $R =$ dim Q the map $D_2H(0,0)$ is an isomorphism. Theorem 6.1 yields convex open neighbourhoods V_H of 0 in $Dh(x)^{-1}(0)$ and V_R of 0 in R, with $x + V_H + V_R \subset U_F$, and a C^1_β-map $w : V_H \to V_R$ with $w(0) = 0$ and

$$(V_H \times V_R) \cap H^{-1}(0) = \{(z,r) \in V_H \times V_R : r = w(z)\}.$$

For every $y \in x + V_H + V_R$, $y = x + z + r$ with $z \in V_H$ and $r \in V_R$, we have

$$y \in g^{-1}(M) \cap U_F \Leftrightarrow h(y) = 0 \Leftrightarrow h(x+z+r) = 0 \Leftrightarrow H(z,r) = 0 \Leftrightarrow r = w(z).$$

Hence $g^{-1}(M) \cap (x + V_H + V_R) = \{x + z + w(z) : z \in V_H\}$, which implies that $g^{-1}(M) \cap (x + V_H + V_R)$ is a C^1_β-submanifold of F, with codimension equal to dim $R =$ dim $Q = m$. Set $V = x + V_H + V_R$.

4. (On tangent spaces) From $g^{-1}(M) \cap U_F = h^{-1}(0)$ and $h(x) = 0$ we get $h(g^{-1}(M) \cap V) = \{0\}$, hence $Dh(x)T_x(g^{-1}(M) \cap V) = \{0\}$, or

$$T_x(g^{-1}(M) \cap V) \subset Dh(x)^{-1}(0).$$

As both spaces have the same codimension m they are equal. For every $v \in F$ we have

$$\begin{aligned} v \in Dh(x)^{-1}(0) &\Leftrightarrow Dh(x)v = 0 \Leftrightarrow PDg(x)v = 0 \\ &\Leftrightarrow Dg(x)v \in P^{-1}(0) = T_{g(x)}M \Leftrightarrow v \in Dg(x)^{-1}T_xM. \end{aligned}$$

Using this we obtain

$$T_x(g^{-1}(M) \cap V) = Dh(x)^{-1}(0) = Dg(x)^{-1}T_xM.$$ □

Proposition 7.2. *Suppose W is an open subset of a finite-dimensional normed space V, $b \in W$, F is a Fréchet space, $j : V \supset W \to F$ is a C^1_β-map, and $Dj(b)$ is injective. Then there is an open neighbourhood N of $j(b)$ in F such that $N \cap j(W)$ is a C^1_β-submanifold of F, with $T_{j(b)}(N \cap j(W)) = Dj(b)V$ (hence* dim $(N \cap j(W)) =$ dim V*).*

Proof. 1. The topology induced by F on the finite-dimensional subspace $Y = Dj(b)V$ of F is given by a norm [16, Section 1.19], and Y has a closed complementary space $Z \subset F$, see [16, Lemma 4.21]. The projection $P : F \to F$ along Z onto Y is linear and continuous ([16, Theorem 5.16]). The map $P \circ j$ is C^1_β-smooth and defines a C^1_β-map $W \to Y$. Its derivative at b is an isomorphism $V \to Y$ (use $Py = y$ on Y and the injectivity of $Dj(b)$). The Inverse Mapping Theorem (for maps between finite-dimensional normed spaces) yields a C^1_β-map $g : Y \cap U \to V$, U open in F and $P(j(b)) \in Y \cap U$, such that $g(P(j(b))) = b$, and an open neighbourhood $W_1 \subset W$ of b in V such that $g(Y \cap U) = W_1$, $(P \circ j)(W_1) = Y \cap U$, $(g \circ (P \circ j))(v) = v$ on W_1, and $((P \circ j) \circ g)(y) = y$ on $Y \cap U$. It follows that the map $h : Y \cap U \to Z$ given by

$$h(y) = ((id_F - P) \circ j \circ g)(y)$$

is C^1_β-smooth.

2. Proof of $j(W_1) = \{y + h(y) : y \in Y \cap U\}$: (a) For $y \in Y \cap U$,

$$\begin{aligned} y + h(y) &= y + ((id_F - P) \circ j \circ g)(y) \\ &= ((P \circ j) \circ g)(y) + (j \circ g)(y) - ((P \circ j) \circ g)(y) = j(g(y)) \in j(W_1). \end{aligned}$$

(b) For $x \in j(W_1)$ there exists $y \in Y \cap U$ with

$$\begin{aligned} x &= j(g(y)) = ((P \circ j) \circ g)(y) + j(g(y)) - (P \circ j)(g(y)) \\ &= y + ((id_F - P) \circ j \circ g)(y) = y + h(y). \end{aligned}$$

The graph representation of $j(W_1)$ now yields that it is a C^1_β-submanifold of F. □

Part II

8 Spaces of continuous and differentiable maps

We begin with the Fréchet spaces which are used in the sequel. For $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$ and $T \in \mathbb{R}$, $C^k_T = C^k((-\infty, T], \mathbb{R}^n)$ denotes the Fréchet space of k-times continuously differentiable maps $\phi : (-\infty, T] \to \mathbb{R}^n$ with the seminorms given by

$$|\phi|_{k,T,j} = \sum_{\kappa=0}^{k} \max_{T-j \le t \le T} |\phi^{(\kappa)}(t)|, \quad j \in \mathbb{N},$$

which define the topology of uniform convergence of maps and their derivatives on compact sets. Analogously we consider the space $C^k_\infty = C^k(\mathbb{R}, \mathbb{R}^n)$, with

$$|\phi|_{k,\infty,j} = \sum_{\kappa=0}^{k} \max_{-j \le t \le j} |\phi^{(\kappa)}(t)|.$$

In case $T = 0$ we abbreviate $C^k = C^k_0$, $|\cdot|_{k,j} = |\cdot|_{k,0,j}$, and also $C = C^0 = C^0_0$, $|\cdot|_j = |\cdot|_{0,j} = |\cdot|_{0,0,j}$. In case $T = \infty$ we abbreviate $C_\infty = C^0_\infty$, $|\cdot|_{\infty,j} = |\cdot|_{0,\infty,j}$.

The space C^1 is dense in the space C because each neighbourhood of a point $\chi \in C$ contains a set

$$N = \{\eta \in C : |\eta - \chi|_j < \epsilon\} \quad \text{for some } j \in \mathbb{N}, \quad \epsilon > 0$$

and there is an $\eta \in N$ with polynomial components due to the Weierstraß Approximation Theorem.

Proposition 8.1 (Ascoli–Arzelà). *Let $B \subset C$ be pointwise bounded and equicontinuous at every $t \le 0$. Then $\overline{B} \subset C$ is compact.*

Proof. It is enough to show that each sequence $(\phi_m)_1^\infty$ in B has a subsequence which converges with respect to the topology on C. For every $j \in \mathbb{N}$ the set of restrictions $\phi_m|_{[-j,0]}$, $m \in \mathbb{N}$, is pointwise bounded and equicontinuous. The Theorem of Ascoli and Arzelà yields a subsequence which is uniformly convergent on $[-j, 0]$. Beginning with $j = 1$ one chooses successively subsequences $(\phi_{\lambda_1(m)})_1^\infty$, $(\phi_{\lambda_1 \circ \lambda_2(m)})_1^\infty, \ldots, (\phi_{\lambda_1 \circ \cdots \circ \lambda_j(m)})_1^\infty$, so that for every $j \in \mathbb{N}$ the sequence $(\phi_{\lambda_1 \circ \cdots \circ \lambda_j(m)})_1^\infty$ uniformly converges on $[-j, 0]$ to a continuous function $\chi_j : [-j, 0] \to \mathbb{R}^n$. Clearly, $\chi_{j+1}(t) = \chi_j(t)$ on $[-j, 0]$. The diagonal sequence $(\phi_{\lambda(m)})_1^\infty$ given by $\lambda(m) = \lambda_1 \circ \cdots \circ \lambda_m(m)$ is a subsequence of each of the former subsequences (up to finitely many indices) and converges uniformly on every interval $[-j, 0]$, $j \in \mathbb{N}$, to the continuous map $\chi : (-\infty, 0] \to \mathbb{R}^n$ given by $\chi(t) = \chi_j(t)$ for $-j \le t \le 0$ and $j \in \mathbb{N}$. Or, for every $j \in N$, $|\phi_{\lambda(m)} - \chi|_j \to 0$ as $m \to \infty$. □

Proposition 8.2. *The inclusion map $C^1 \to C$ maps bounded sets into sets with compact closure.*

Proof. Let $B \subset C^1$ be bounded. Then each set $\{\phi(t) : \phi \in B\} \subset \mathbb{R}^n$, $t \le 0$, is bounded. For every $j \in \mathbb{N}$, $c_j = \sup\{|\phi'|_j : \phi \in B\} < \infty$, and c_j is a Lipschitz constant for all restrictions $\phi|_{[-j,0]}$, $\phi \in B$. It follows that B is equicontinuous at every $t \le 0$. Proposition 8.1 yields that the closure of B as a subset of C is compact. □

Proposition 8.3. *Every C^1_ζ-smooth map $f : C^1 \supset U \to \mathbb{R}^n$ with property (e) is C^1_β-smooth.*

Proof. Let $\phi \in U \subset C^1$ and a neighbourhood of 0 in $L_c(C^1, \mathbb{R}^n)$ with respect to the topology β be given, say, a set $N_{W,B}$ as in Section 2, with a neighbourhood W of 0 in $\mathbb{R}^n$ and a bounded set $B \subset C^1$. The closure K of B with respect to the topology of C is compact, due to Proposition 8.2, and due to condition (e) the map

$$g : U \times C \ni (\phi, \chi) \mapsto D_e f(\phi)\chi \in \mathbb{R}^n$$

is continuous. Proposition 2.1 yields a neighbourhood N of ϕ in $U \subset C^1$ such that for all $\psi \in N$ and for all $\chi \in K$,

$$D_e f(\psi)\chi - D_e f(\phi)\chi \in W.$$

For all $\psi \in N$ and for all $\chi \in B \subset K$ this gives

$$(Df(\psi) - Df(\phi))\chi = D_e f(\psi)\chi - D_e f(\phi)\chi \in W,$$

which means $Df(\psi) - Df(\phi) \in N_{W,B}$. This is the desired continuity of $Df : U \to L_c(C^1, \mathbb{R}^n)$ at $u \in U$, with respect to the topology β on $L_c(C^1, \mathbb{R}^n)$. □

Remark 8.4. See [26] for examples of maps $C \to \mathbb{R}^n$ and $C^1 \to \mathbb{R}^n$ which are C^1_ζ-smooth but not C^1_β-smooth (and thereby must violate condition (e)).

We turn to other spaces and maps which occur in the sequel. The vector space $C^\infty = \cap_{k=0}^\infty C^k$ will be used without a topology on it.

The differentiation map $\partial_{k,T} : C^k_T \ni \phi \mapsto \phi' \in C^{k-1}_T$, $k \in \mathbb{N}$ and $T \in \mathbb{R}$ or $T = \infty$, is linear and continuous. We abbreviate $\partial_T = \partial_{1,T}$ and $\partial = \partial_0 = \partial_{1,0}$.

The following Banach spaces occur in Parts II and III: For $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$ and reals $a < b$, $C^k([a,b], \mathbb{R}^n)$ denotes the Banach space of k-times continuously differentiable maps $[a,b] \to \mathbb{R}^n$ with the norm given by

$$|\phi|_{[a,b],k} = \sum_{\kappa=0}^{k} \max_{a \le t \le b} |\phi^{(\kappa)}(t)|.$$

We use various abbreviations, for $S < T$ and $d > 0$ and $k \in \mathbb{N}_0$:

$$\begin{aligned} C_{ST} &= C^0([S,T], \mathbb{R}^n), & |\cdot|_{ST} &= |\cdot|_{[S,T],0} \\ C^1_{ST} &= C^1([S,T], \mathbb{R}^n), & |\cdot|_{1,ST} &= |\cdot|_{[S,T],1} \\ C^k_d &= C^k([-d,0], \mathbb{R}^n), & |\cdot|_{d,k} &= |\cdot|_{[-d,0],k} \end{aligned}$$

It is easy to see that the linear restriction maps

$$R_{d,k} : C^k \to C^k_d, \quad d > 0 \text{ and } k \in \mathbb{N}_0,$$

and the linear prolongation maps

$$P_{d,k} : C^k_d \to C^k, \quad d > 0 \text{ and } k \in \mathbb{N}_0,$$

given by $(P_{d,k}\phi)(s) = \phi(s)$ for $-d \le s \le 0$ and

$$(P_{d,k}\phi)(s) = \sum_{\kappa=0}^{k} \frac{\phi^{(\kappa)}(-d)}{\kappa!}(s+d)^\kappa \quad \text{for } s < -d$$

are continuous, and for all $d > 0$ and $k \in \mathbb{N}_0$,

$$R_{d,k} \circ P_{d,k} = \mathrm{id}_{C_d^k}.$$

In Part II we also need the closed subspaces

$$C_{0T,0} = \{\phi \in C_{0T} : \phi(0) = 0\} \quad \text{and} \quad C^1_{0T,0} = \{\phi \in C^1_{0T} : \phi(0) = 0 = \phi'(0)\}.$$

In Part III we make use of the Banach space B_a, for $a > 0$ given, of all $\phi \in C$ with

$$\sup_{t\le 0} |\phi(t)|e^{at} < \infty, \quad |\phi|_a = \sup_{t\le 0} |\phi(t)|e^{at},$$

and of the Banach space B^1_a of all $\phi \in C^1$ with

$$\phi \in B_a, \quad \phi' \in B_a, \quad |\phi|_{a,1} = |\phi|_a + |\phi'|_a.$$

Solutions of equations

$$x'(t) = g(x_t), \quad \text{with} \quad g : C^1_d \supset U \to \mathbb{R}^n \quad \text{or} \quad g : B^1_a \supset U \to \mathbb{R}^n,$$

on some interval $I \subset \mathbb{R}$ are defined as in case of Eq. (1.1): With $J = [-d, 0]$ or $J = (-\infty, 0]$, respectively, they are continuously differentiable maps $x : J + I \to \mathbb{R}^n$ so that $x_t \in U$ for all $t \in I$ and the differential equation holds for all $t \in I$. Observe that x_t may denote a map on $[-d, 0]$ or on $(-\infty, 0]$, depending on the context.

For results on strongly continuous semigroups given by solutions of linear autonomous retarded functional differential equations

$$x'(t) = \Lambda x_t$$

with $\Lambda : C_d \to \mathbb{R}^n$ linear and continuous, see [2,5].

9 Examples, and the solution manifold

We begin with the toy example (1.2),

$$x'(t) = h(x(t - r(x(t))))$$

with continuously differentiable functions $h : \mathbb{R} \to \mathbb{R}$ and $r : \mathbb{R} \to [0, \infty) \subset \mathbb{R}$. For continuously differentiable functions $(-\infty, t_e)$, $0 < t_e \le \infty$, which satisfy Eq. (1.2) for $0 \le t < t_e$ this delay differential equation has the form (1.1) for $U = C^1$ with $n = 1$ and $f = f_{h,r}$ given by

$$f_{h,r}(\phi) = h(\phi(-r(\phi(0)))).$$

In order to see that $f_{h,r}$ is a composition of C^1_β-maps all defined on open sets of Fréchet spaces it is convenient to introduce the odd prolongation maps $P_{\mathrm{odd}} : C \to C_\infty$ (with $n = 1$) and $P_{\mathrm{odd},1} : C^1 \to C^1_\infty$ (with $n = 1$) which are defined by the relations

$$(P_{\mathrm{odd}}\phi)(t) = \phi(t) \quad \text{for } t \le 0, \quad (P_{\mathrm{odd}}\phi)(t) = -\phi(-t) + 2\phi(0) \quad \text{for } t > 0,$$

and $P_{\mathrm{odd},1}\phi = P_{\mathrm{odd}}\phi$ for $\phi \in C^1$. Both maps are linear and continuous. With the evaluation map

$$ev_{\infty,1} : C^1_\infty \times \mathbb{R} \to \mathbb{R}, \quad ev_{\infty,1}(\phi, t) = \phi(t),$$

we have

$$f_{h,r}(\phi) = h \circ ev_{\infty,1}(P_{\mathrm{odd},1}\phi, -r(\phi(0)))$$

for all $\phi \in C^1$. We also need the evaluation map $ev_\infty : C_\infty \times \mathbb{R} \to \mathbb{R}$ given by $ev_\infty(\phi, t) = \phi(t)$.

Proposition 9.1. *The map ev_∞ is continuous and the map $ev_{\infty,1}$ is C^1_β-smooth with*

$$D\,ev_{\infty,1}(\phi,t)(\chi,s) = D_1 ev_{\infty,1}(\phi,s)\chi + D_2 ev_{\infty,1}(\phi,s)t = \chi(t) + s\phi'(t)$$

Proof. Arguing as in the proof of [23, Proposition 2.1] one shows that ev_∞ is continuous and that $ev_{\infty,1}$ is C^1_ζ-smooth, and that the partial derivatives satisfy the equations in the proposition. It remains to prove that the map $C^1_\infty \times \mathbb{R} \ni (\phi,t) \mapsto D\,ev_{\infty,1}(\phi,t) \in L_c(C^1_\infty \times \mathbb{R},\mathbb{R})$ is β-continuous. As $C^1_\infty \times \mathbb{R}$ has countable neighbourhood bases it is enough to show that, given a sequence $C^1_\infty \times \mathbb{R} \ni (\phi_k,t_k) \to (\phi,t) \in C^1_\infty \times \mathbb{R}$ for $k\to\infty$, a neighbourhood N of 0 in $\mathbb{R}$ and a bounded subset $B \subset C^1_\infty \times \mathbb{R}$, we have

$$(D\,ev_{\infty,1}(\phi_k,t_k) - D\,ev_{\infty,1}(\phi,t))B \subset N \quad \text{for } k \text{ sufficiently large.}$$

In order to prove this, choose $j\in\mathbb{N}$ with $|t|<j$ and $|t_k|<j$ for all $k\in\mathbb{N}$. By [16, Theorem 1.37], $c_j = \sup_{(\chi,s)\in B}(|\chi|_{1,\infty,j} + |s|) < \infty$. For every $k\in\mathbb{N}$ and $(\chi,s)\in B$,

$$\begin{aligned}
|(D\,ev_{\infty,1}(\phi_k,t_k) - D\,ev_{\infty,1}(\phi,t))(\chi,s)| &= |\chi(t_k)-\chi(t) + s[\phi_k'(t_k)-\phi'(t)]| \\
&\le \max_{-j\le u\le j}|\chi'(u)||t_k-t| + c_j\left(\max_{-j\le u\le j}|\phi_k'(u)-\phi'(u)| + |\phi'(t_k)-\phi'(t)|\right) \\
&\le c_j(|t_k-t)| + |\phi_k-\phi|_{\infty,1,j} + |\phi'(t_k)-\phi'(t)|),
\end{aligned}$$

and it becomes obvious how to complete the proof. □

The map $ev_1(\cdot,0) : C^1 \ni \phi \mapsto \phi(0) \in \mathbb{R}$ is linear and continuous, and the evaluation $ev : C\times(-\infty,0] \ni (\phi,t) \mapsto \phi(t) \in \mathbb{R}$ is continuous, see [23, Proposition 2.1].

The next result says that $f_{h,r}$ satisfies the hypotheses for the results on semiflows and local invariant manifolds in the subsequent sections.

Corollary 9.2. *For $r:\mathbb{R}\to[0,\infty)\subset\mathbb{R}$ and $h:\mathbb{R}\to\mathbb{R}$ continuously differentiable the map $f_{h,r}$ is C^1_β-smooth and has property (e).*

Proof. The functions r and h are C^1_β-smooth. The map $ev_1(\cdot,0)$ is linear and continuous, hence C^1_β-smooth. It follows that $P_{\text{odd},1}\times(-r\circ ev_1(\cdot,0)) : C^1 \to C^1_\infty\times\mathbb{R}$ is C^1_β-smooth, by the chain rule (Proposition 4.3) and by C^1_β-smoothness of maps into product spaces. Now use that $ev_{\infty,1}$ is C^1_β-smooth, due to Proposition 9.1, and apply the chain rule to the composition

$$f_{h,r} = h\circ ev_{\infty,1}\circ(P_{\text{odd},1}\times(-r\circ ev_1(\cdot,0))).$$

It follows that $f_{h,r}$ is C^1_β-smooth with

$$Df_{h,r}(\phi)\chi = h'(\phi(-r(\phi(0))))[\chi(-r(\phi(0))) + \phi'(-r(\phi(0)))\{-r'(\phi(0))\}\chi(0)].$$

For each $\phi\in C^1$ the term on the right hand side of this equation defines a linear continuation $D_e f_{h,r}(\phi) : C\to\mathbb{R}$ of $Df_{h,r}(\phi)$. Using that the evaluation ev and differentiation $C^1\to C$ are continuous one finds that the map

$$C^1\times C \ni (\phi,\chi) \mapsto D_e f_{h,r}(\phi)\chi \in \mathbb{R}$$

is continuous. □

It is almost obvious that restrictions $f : C^1 \supset U \to \mathbb{R}^n$ of C^1_ζ-smooth maps $\hat{f} : C \supset \hat{U} \to \mathbb{R}^n$, with $U = \hat{U} \cap C^1$, are C^1_ζ-smooth and have property (e); by Proposition 8.3 they are in fact C^1_β-smooth.

Let us mention that the C^1_β-smooth maps $f = f_{h,r}$ with property (e) from Corollary 9.2, which arise from differential equations with state-dependent delay, are in general *not* of this kind. In order to see this consider the case $h(\xi) = \xi$ and, say, $r(\xi) = \xi^2$, for which

$$f(\phi) = f_{h,r}(\phi) = \phi(-(\phi(0))^2),$$

and assume there exists a C^1_ζ-smooth functional $\hat{f} : C \to \mathbb{R}$ with $f(\phi) = \hat{f}(\phi)$ for all $\phi \in C^1$. The equation

$$\hat{f}(\phi) = f(\phi) = [ev \circ (id \times ((-r) \circ ev(\cdot, 0))](\phi) \quad \text{for all } \phi \in C^1$$

in combination with the continuity of the evaluation ev and with the fact that C^1 is dense in C yields

$$\hat{f}(\chi) = \chi(-(\chi(0))^2) \quad \text{for all } \chi \in C.$$

Choose $\chi \in C$ with $\chi(0) = 1$ which is not differentiable at $t = -1$. By assumption, the directional derivative

$$\begin{aligned}
D\hat{f}(\chi)\chi &= \lim_{0 \neq t \to 0} \frac{1}{t}[\hat{f}(\chi + t\chi) - \hat{f}(\chi)] \\
&= \lim_{0 \neq t \to 0} \left(\chi(-(\chi(0) + t\,\chi(0))^2) + \frac{1}{t}[\chi(-(\chi(0) + t\,\chi(0))^2) - \chi(-(\chi(0))^2)] \right) \\
&= \lim_{0 \neq t \to 0} \left(\chi(-(1+t)^2) + \frac{1}{t}[\chi(-(1+t)^2) - \chi(-1)] \right) \\
&= \lim_{0 \neq t \to 0} \left(\chi(-(1+t)^2) - \frac{2t+t^2}{t} \frac{1}{-2t-t^2}[\chi(-1-2t-t^2) - \chi(-1)] \right)
\end{aligned}$$

exists, which leads to a contradiction to the non-differentiability of χ at -1.

The pantograph equation (1.3), namely,

$$x'(t) = a\,x(\lambda t) + b\,x(t)$$

with constants $a \in \mathbb{C}$, $b \in \mathbb{R}$ and $0 < \lambda < 1$, was extensively studied in [8]. For real parameters a, b and arguments $t > 0$ this is a nonautonomous linear equation with unbounded delay $\tau(t) = (1-\lambda)t > 0$ since $\lambda t = t - \tau(t)$. Define $F : \mathbb{R} \times C^1 \to \mathbb{R}$ by

$$F(t, \phi) = a(P_{\text{odd},1}\phi)(-\tau(t)) + b\,\phi(0),$$

or,

$$F = a\,ev_{\infty,1} \circ ((P_{\text{odd},1} \circ pr_2) \times (-\tau \circ pr_1))) + b\,ev_{\infty,1}(\cdot, 0) \circ P_{\text{odd},1} \circ pr_2,$$

with the projections pr_j onto the first and second component, respectively. The map F is C^1_β-smooth, and every continuously differentiable function $x : (-\infty, t_e) \to \mathbb{R}$, $0 < t_e \leq \infty$, which satisfies the pantograph equation for $0 \leq t < t_e$ also solves the nonautonomous equation

$$x'(t) = F(t, x_t) \tag{9.1}$$

for $0 \leq t < t_e$.

The role of the odd prolongation map in the definition of F is more essential than for $f_{h,d}$ above. Here it helps to overcome the obstacle that on one hand F should be defined on an open set containing $\{0\} \times C^1$ while on the other hand for every $t < 0$ the term $-\tau(t) > 0$ is not in the domain of data $\phi \in C^1$.

The solutions of Eq. (9.1) can be obtained from the autonomous equation (1.1) with $n = 2$ and $f : C^1 \to \mathbb{R}^2$ given by

$$f_1(\phi_1, \phi_2) = 1, \quad f_2(\phi_1, \phi_2) = F(\phi_1(0), \phi_2)$$

in the familiar way: If the continuously differentiable map $x : (-\infty, t_e) \to \mathbb{R}$ satisfies Eq. (9.1) for $t_0 \le t < t_e \le \infty$ then $(s, z) : (-\infty, t_e - t_0) \to \mathbb{R}^2$ given by

$$s(t) = t + t_0 \quad \text{and} \quad z(t) = x(t + t_0)$$

satisfies the system

$$s'(t) = 1 = f_1(s_t, z_t)$$
$$z'(t) = F(s(t), z_t) = f_2(s_t, z_t)$$

for $0 \le t < t_e$ and $s(0) = t_0$. The map f is C^1_β-smooth and has the extension property (e).

For the Volterra integro-differential equation (1.4),

$$x'(t) = \int_0^t k(t,s) h(x(s)) ds$$

with $k : \mathbb{R}^{n \times n} \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^n$ continuously differentiable the scenario is simpler than in both cases above where delays are discrete. In [25] it is shown that every continuous function $(-\infty, t_e) \to \mathbb{R}^n$, $0 < t_e \le \infty$, which for $0 < t < t_e$ is differentiable and satisfies Eq. (1.4), also satisfies an equation of the form (9.1) for $0 < t < t_e$, with the C^1_β-map $F = F_{k,h}$ in Eq. (9.1) defined on the space $\mathbb{R} \times C$. The associated autonomous equation of the form (1.1) is given by the C^1_β-map $f_{k,h} : C((-\infty, 0], \mathbb{R}^{n+1}) \to \mathbb{R}^{n+1}$ with

$$f_{k,h} = f_1 \times \hat{f}, \quad f_1(\psi) = 1, \quad \psi = (\psi_1, \phi), \quad \hat{f}(\psi) = F(\psi_1(0), \phi).$$

It follows that the restriction of $f_{k,h}$ to $C^1((-\infty, 0], \mathbb{R}^{n+1})$ is C^1_β-smooth and has property (e), which means that the hypotheses for the theory of Eq. (1.1) in the following sections, with a semiflow on the solution manifold in $C^1((-\infty, 0], \mathbb{R}^{n+1})$, are satisfied. However, in the present case we also get a nice semiflow without recourse to this theory. A result in [25] for Eq. (1.1) with a map $f : C \supset U \to \mathbb{R}^n$ which is C^1_β-smooth establishes a continuous semiflow on U, with all solution operators C^1_β-smooth. In the present case, with $f = f_{k,h}$, the semiflow yields a process of solution operators for the nonautonomous equation (9.1), all of them defined on open subsets of $C((-\infty, 0], \mathbb{R}^n)$ and C^1_β-smooth. The process incorporates all solutions of the Volterra integro-differential equation.

We return to the general equation (1.1) and begin with the solution manifold

$$X_f = \{\phi \in U : \phi'(0) = f(\phi)\}.$$

Proposition 9.3. *For a C^1_β-map $f : C^1 \supset U \to \mathbb{R}^n$ with property (e) and $X_f \neq \emptyset$ the set X_f is a C^1_β-submanifold of codimension n in the space C^1, with tangent spaces*

$$T_\phi X_f = \{\chi \in C^1 : \chi'(0) = Df(\phi)\chi\} \quad \text{for all } \phi \in X_f.$$

.

Proof. X_f is the preimage of $0 \in \mathbb{R}^n$ under the map C^1_β-map $g : C^1 \supset U \ni \phi \mapsto \phi'(0) - f(\phi) \in \mathbb{R}^n$. [23, Proposition 2.2] applies as f also is C^1_ζ-smooth. It follows that all derivatives $Dg(u)$, $u \in U$, are surjective. Apply Proposition 7.1 to g and $M = \{0\}$. □

10 Segment evaluation maps

For the construction of solutions to Eq. (1.1) we need a few facts about the *segment evaluation maps*

$$E_T : C_T \times (-\infty, T] \ni (\phi, t) \mapsto \phi_t \in C,$$
$$E^1_T : C^1_T \times (-\infty, T] \ni (\phi, t) \mapsto \phi_t \in C^1$$

and

$$E^{10}_T : C^1_T \times (-\infty, T] \ni (\phi, t) \mapsto \phi_t \in C$$

with $T \in \mathbb{R}$ and about their analogues E_∞, E^1_∞ with $T = \infty$. All of these maps are linear in the first argument.

Proposition 10.1. *Let $T \leq \infty$.*

(i) The maps E_T and E^1_T are continuous.

(ii) For every $\phi \in C^1_T$ the curve $\Phi : (-\infty, T) \ni t \mapsto \phi_t \in C$ is continuously differentiable, with $\Phi'(t) = E_T(\partial_T \phi, t)$.

(iii) The map $E^{10}_T|_{C^1_T \times (-\infty,T)}$ is C^1_β-smooth, with

$$D_1 E^{10}_T(\phi, t)\hat{\phi} = E^{10}_T(\hat{\phi}, t) = \hat{\phi}_t \quad \textit{and}$$
$$D_2 E^{10}_T(\phi, t)s = s\, E_T(\partial_T \phi, t) = s(\partial_T \phi)_t = s(\phi')_t.$$

Proof. 1. For assertions (i) and (ii), for the fact that the map $E^{10}_T|_{C^1_T \times (-\infty,T)}$ is C^1_ζ-smooth, and for the formulae for the partial derivatives in assertion (iii) see the proof of [23, Proposition 3.1]. It remains to show that the map

$$C^1_T \times (-\infty, T) \ni (\phi, t) \mapsto DE^{10}_T(\phi, t) \in L_c(C^1_T \times \mathbb{R}, C)$$

is β-continuous. Let a sequence $(\phi_j, t_j)_1^\infty$ in $C^1_T \times (-\infty, T)$ be given which converges to some $(\phi, t) \in C^1_T \times (-\infty, T)$. Let a bounded subset $B \subset C^1_T \times \mathbb{R}$ and a neighbourhood V of 0 in C be given. We may assume

$$V = \left\{ \chi \in C : |\chi|_l < \frac{1}{l} \right\} \quad \text{for some integer } l > 0$$

and have to show that for j sufficiently large,

$$(DE^{10}_T(\phi_j, t_j) - DE^{10}_T(\phi, t))B \subset V.$$

For every $(\hat{\phi}, \hat{t}) \in C^1_T \times \mathbb{R}$ and for all $j \in \mathbb{N}$,

$$(DE^{10}_T(\phi_j, t_j) - DE^{10}_T(\phi, t))(\hat{\phi}, \hat{t}) = \hat{\phi}_{t_j} - \hat{\phi}_t + \hat{t}[(\phi'_j)_{t_j} - (\phi')_t].$$

2. As the projections from $C^1_T\times\mathbb{R}$ onto C^1_T and onto $\mathbb{R}$ are continuous and linear they map the bounded set B into bounded sets, and we obtain that for some real $r>0$ and for all $k\in\mathbb{N}$,

$$\{\hat{t}\in\mathbb{R}:\text{for some }\hat{\phi}\in C^1_T,(\hat{\phi},\hat{t})\in B\}\subset[-r,r]$$

and

$$\sigma_{1,T,k}=\sup\{|\hat{\phi}|_{1,T,k}\in\mathbb{R}:\text{for some }\hat{t}\in\mathbb{R},(\hat{\phi},\hat{t})\in B\}<\infty.$$

3. Choose $k\in\mathbb{N}$ so large that for all $s\in[-l,0]$ and for all $j\in\mathbb{N}$,

$$T-k<s+t_j<T\quad\text{and}\quad T-k<s+t<T.$$

Consider $(\hat{\phi},\hat{t})\in B$. For each $j\in\mathbb{N}$ we have

$$\begin{aligned}|\hat{\phi}_{t_j}-\hat{\phi}_t|_l&=\max_{-l\le s\le 0}|\hat{\phi}(t_j+s)-\hat{\phi}(t+s)|\\&\le\max_{T-k\le u\le T}|\hat{\phi}'(u)||t_j-t|\le\sigma_{1,T,k}|t_j-t|\end{aligned}$$

and

$$\begin{aligned}|\hat{t}[(\phi_j')_{t_j}-(\phi')_t]|_l&\le r[|(\phi_j')_{t_j}-(\phi')_{t_j}|_l+|(\phi')_{t_j}-(\phi')_t|_l]\\&\le r[\max_{-l\le s\le 0}|\phi_j'(s+t_j)-\phi'(s+t_j)|+\max_{-l\le s\le 0}|(\phi')(s+t_j)-(\phi')(s+t)|]\\&\le r[\max_{T-k\le u\le T}|\phi_j'(u)-\phi'(u)|+\max_{-l\le s\le 0}|(\phi')(s+t_j)-(\phi')(s+t)|].\end{aligned}$$

Altogether, for every $j\in\mathbb{N}$ and for all $(\hat{\phi},\hat{t})\in B$,

$$\begin{aligned}&|(DE^{10}_T(\phi_j,t_j)-DE^{10}_T(\phi,t))(\hat{\phi},\hat{t})|_l\\&\quad\le\sigma_{1,T,k}|t_j-t|+r\left[|\phi_j-\phi|_{1,T,k}+\max_{-l\le s\le 0}|(\phi')(s+t_j)-(\phi')(s+t)|\right].\end{aligned}$$

Using $t_j\to t$ and $|\phi_j-\phi|_{1,T,k}\to 0$ as $j\to\infty$ and the uniform continuity of ϕ' on $[T-k,T]$ one finds $J\in\mathbb{N}$ such that for all integers $j\ge J$ and for all $(\hat{\phi},\hat{t})\in B$,

$$|(DE^{10}_T(\phi_j,t_j)-DE^{10}_T(\phi,t))(\hat{\phi},\hat{t})|_l<\frac{1}{l}$$

It follows that $(DE^{10}_T(\phi_j,t_j)-DE^{10}_T(\phi,t))B\subset V$ for all integers $j\ge J$. □

11 Solutions of the delay differential equation

In the sequel we always assume that $U\subset C^1$ is open and that $f:U\to\mathbb{R}^n$ is C^1_β-smooth and has the property (e).

Following [23] we rewrite the initial value problem

$$x'(t)=f(x_t)\quad\text{for }t\ge 0,\quad x_0=\phi\in X_f\tag{11.1}$$

as a fixed point equation: Suppose $x:(-\infty,T]\to\mathbb{R}^n$, $T>0$, is a solution of Eq. (1.1) on $[0,T]$ with $x_0=\phi$. Extend ϕ by $\phi(t)=\phi(0)+t\phi'(0)$ to a continuously differentiable function $\hat{\phi}:(-\infty,T]\to\mathbb{R}^n$. Then $y=x-\hat{\phi}$ satisfies $y(t)=0$ for $t\le 0$, the curve $(-\infty,T]\ni s\mapsto$

$x_s \in C^1$ is continuous (use $x_s = E^1_T(x, s)$ and apply Proposition 10.1 (i)), as well as the curves $(-\infty, T] \ni s \mapsto y_s \in C^1$ and $(-\infty, T] \ni s \mapsto \hat{\phi}_s \in C^1$. For $0 \le t \le T$ we get

$$\begin{aligned} y(t) = x(t) - \hat{\phi}(t) &= x(0) + \int_0^t f(x_s)ds - \phi(0) - t\phi'(0) \\ &= \int_0^t f(y_s + \hat{\phi}_s)ds - t\,f(\phi) \\ &= \int_0^t f(y_s + \hat{\phi}_s) - f(\phi))ds. \end{aligned}$$

Obviously, $y(0) = 0 = y'(0)$. So $\eta = y|_{[0,T]} \in C^1_{0T,0}$ satisfies the fixed point equation

$$\eta(t) = \int_0^t (f(\bar{\eta}_s + \hat{\phi}_s) - f(\phi))ds, \quad 0 \le t \le T, \tag{11.2}$$

where $\bar{\eta} \in C^1_T$ is the prolongation of η given by $\bar{\eta}(t) = 0$ for all $t < 0$. In order to find a solution of the initial value problem (11.1) one solves the fixed point equation (11.2) by means of a parametrized contraction on a subset of the Banach space $C^1_{0T,0}$ with the parameter $\phi \in U$ in the Fréchet space C^1. For $\phi \in X_f$ the associated fixed point $\eta = \eta_\phi$ yields a solution $x = \bar{\eta} + \hat{\phi}$ of the initial value problem (11.1).

The application of a suitable contraction mapping theorem, namely, Theorem 5.2, requires some preparation. We begin with the substitution operator

$$F_T : \mathrm{dom}_T \to C_{0T}$$

which for $0 < T < \infty$ is given by

$$\mathrm{dom}_T = \{\phi \in C^1_T : \text{for } 0 \le s \le T, \phi_s \in U\}$$

and

$$F_T(\phi)(t) = f(\phi_t) = f(E^1_T(\phi, t)) \in \mathbb{R}^n.$$

[23, Proposition 3.2] guarantees that for $0 < T < \infty$ the domain dom_T is open and that F_T is a C^1_ζ-map with

$$(DF_T(\phi)\chi)(s) = D_e f(E^1_T(\phi, s))E^{10}_T(\chi, s).$$

(Notice that in order to obtain that F_T is C^1_ζ-smooth the chain rule can not be applied, due to lack of smoothness of the map E^1_T.)

Proposition 11.1. *The map F_T, $0 < T < \infty$, is C^1_β-smooth.*

Proof. 1. Let $\phi \in \mathrm{dom}_T \subset C^1_T$, $\epsilon > 0$, and a bounded set $B \subset C^1_T$ be given. Using the norm on C_{0T} we have to find a neighbourhood N of ϕ in C^1_T so that for every $\psi \in N$ and for all $\chi \in B$,

$$\max_{0 \le s \le T} |((DF_T(\psi) - DF_T(\phi))\chi)(s)| < \epsilon.$$

Define

$$B_T = \{E^1_T(\chi, s) \in C^1 : 0 \le s \le T, \chi \in B\}.$$

Claim: $B_T \subset C^1$ is bounded.

Proof: Consider a seminorm $|\cdot|_{1,j}$, $j \in \mathbb{N}$. Choose an integer $k \geq j+T$. The seminorm $|\cdot|_{1,T,k}$ is bounded on B. For every $\chi \in B$ and for all $s \in [0,T]$ we see from

$$\begin{aligned}|E_T^1(\chi,s)|_{1,j} &= \max_{-j\leq u\leq 0}|\chi(s+u)| + \max_{-j\leq u\leq 0}|\chi'(s+u)|\\ &\leq \max_{-j\leq w\leq T}|\chi(w)| + \max_{-j\leq w\leq T}|\chi'(w)| \leq |\chi|_{1,T,k}\end{aligned}$$

that $|\cdot|_{1,j}$ is bounded on B_T.

2. For every $\psi \in \mathrm{dom}_T$, $\chi \in B$, $s \in [0,T]$ we have

$$\begin{aligned}((DF_T(\psi) - DF_T(\phi))\chi)(s) &= (D_e f(E_T^1(\psi,s)) - D_e f(E_T^1(\phi,s))E_T^{10}(\chi,s)\\ &\quad \text{(see [23, Proposition 3.2])}\\ &= (Df(E_T^1(\psi,s)) - Df(E_T^1(\phi,s))E_T^1(\chi,s)\\ &\quad \text{(with } \ E_T^{10}(\chi,s) \in C^1),\end{aligned}$$

where $E_T^1(\chi,s)$ is in the bounded set B_T. As f is C^1_β-smooth the composition

$$C_T^1 \times \mathbb{R} \supset \mathrm{dom}_T \times [0,T] \ni (\psi,s) \mapsto Df(E_T^1(\psi,s)) \in L_c(C^1,\mathbb{R}^n)$$

is β-continuous, hence uniformly β-continuous on the compact set $\{\phi\} \times [0,T]$ (see Proposition 2.1). It follows that there is a neighbourhood N of ϕ in C^1_T such that for every $\psi \in N$ and for all $s \in [0,T]$ the difference

$$Df(E_T^1(\psi,s)) - Df(E_T^1(\phi,s))$$

is contained in the neighbourhood for all $N_{U_\epsilon(0),B_T}$ of 0 in $L_c(C^1,\mathbb{R}^n)$, with $U_\epsilon(0) = \{x \in \mathbb{R}^n : |x| < \epsilon\}$. Finally, we obtain for each $\psi \in N$, $s \in [0,T]$, $\chi \in B$,

$$\begin{aligned}|((DF_T(\psi) - DF_T(\phi))\chi)(s)| &= |(Df(E_T^1(\psi,s)) - Df(E_T^1(\phi,s))E_T^1(\chi,s)|\\ &< \epsilon.\end{aligned}$$

□

The *prolongation maps*

$$P_T : C^1 \to C^1_T, \quad 0 < T \leq \infty,$$

given by

$$P_T\phi(t) = \phi(t) \quad \text{for } t \leq 0, \quad P_T\phi(t) = \phi(0) + t\phi'(0) \quad \text{for } 0 < t \leq T,$$

and

$$P_{ST} : C^1_{0S} \to C^1_{0T}, \quad 0 < S < T < \infty,$$

given by

$$P_{ST}\phi(t) = \phi(t) \quad \text{for } 0 \leq t \leq S, \quad P_{ST}\phi(t) = \phi(S) + (t-S)\phi'(S) \quad \text{for } S < t \leq T,$$

and

$$Z_T : C_{0T,0} \to C_T, \quad 0 < T < \infty$$

given by

$$Z_T\phi(t) = \phi(t) \quad \text{for } 0 \leq t \leq T, \quad Z_T(\phi)(t) = 0 \quad \text{for } t < 0,$$

and the integration operators

$$I_T : C_{0T,0} \to C^1_{0T,0}, \quad 0 < T < \infty, \quad \text{given by} \quad I_T\phi(t) = \int_0^t \phi(s)ds$$

are all linear and continuous. We have $Z_T C^1_{0T,0} \subset C^1_T$, and the induced map $C^1_{0T,0} \stackrel{Z_T}{\to} C^1_T$ is continuous, too. For P_{ST}, $0 < S < T$,

$$P_{ST} C^1_{0S,0} \subset C^1_{0T,0},$$

and

$$|P_{ST}\phi|_{1,0T} \leq (2+T)|\phi|_{1,0S} \quad \text{for all } \phi \in C^1_{0S}$$

because of the estimate

$$\begin{aligned} |P_{ST}\phi|_{1,0T} &= \max_{0\leq t\leq T} |P_{ST}\phi(t)| + \max_{0\leq t\leq T} |(P_{ST}\phi)'(t)| \\ &\leq \max_{0\leq t\leq S} |\phi(t)| + |\phi(S)| + |\phi'(S)|T + \max_{0\leq t\leq S} |\phi'(t)|. \end{aligned}$$

It follows that for every $T > 0$ the set

$$D_T = \{(\phi,\eta) \in U \times C^1_{0T,0} : P_T\phi + Z_T\eta \in \mathrm{dom}_T\}$$

is open. Let pr_1 and pr_2 denote the projections from $C^1 \times C^1_{0T,0}$ onto the first and second factor, respectively. Define $\tau : \mathbb{R}^n \mapsto C_{0T}$ by $\tau(\xi)(t) = \xi$. Both projections and τ are continuous linear maps. Using Proposition 11.1, the chain rule, and linearity of differentiation we infer that the map

$$G_T : C^1 \times C^1_{0T,0} \supset U \times C^1_{0T,0} \supset D_T \to C_{0T,0} \subset C_{0T}$$

given by

$$G_T(\phi,\eta) = F_T(P_T pr_1(\phi,\eta) + Z_T pr_2(\phi,\eta)) - \tau \circ f \circ pr_1(\phi,\eta)$$

$$(\text{notice that} \quad G_T(\phi,\eta)(0) = f((P_T\phi + Z_T\eta)_0) - f(\phi) = f(\phi + 0) - f(\phi) = 0)$$

is C^1_β-smooth. For the derivatives we obtain the following result.

Corollary 11.2. *Let $0 < T < \infty$. For $(\phi,\eta) \in D_T$ and $\tilde{\phi} \in C^1, \tilde{\eta} \in C^1_{0T,0}$,*

$$DG_T(\phi,\eta)(\tilde{\phi},\tilde{\eta}) = DF_T(P_T\phi + Z_T\eta)(P_T\tilde{\phi} + Z_T\tilde{\eta}) - \tau(Df(\phi)\tilde{\phi}),$$

and for $0 \leq t \leq T$,

$$\begin{aligned} DG_T(\phi,\eta)(\tilde{\phi},\tilde{\eta})(t) &= (D_e f(E^1_T(P_T\phi + Z_T\eta, t)E^{10}_T(P_T\tilde{\phi} + Z_T\tilde{\eta}, t) - \tau(Df(\phi)\tilde{\phi})(t) \\ &= D_e f((P_T\phi)_t + (Z_T\eta)_t)((P_T\tilde{\phi})_t + (Z_T\tilde{\eta})_t) - Df(\phi)\tilde{\phi}. \end{aligned}$$

The map $A_T = I_T \circ G_T$ is C^1_β-smooth. We now restate [23, Proposition 3.4], which prepares the proof that A_T with $T > 0$ sufficiently small defines a uniform contraction on a small ball in $C^1_{0T,0}$.

Proposition 11.3. *Let $\phi \in U$ be given. There exist $T = T_\phi > 0$, a neighbourhood $V = V_\phi$ of ϕ in U, $\epsilon = \epsilon_\phi > 0$, and $j = j_\phi \in \mathbb{N}$ such that for all $S \in (0,T)$, $\chi \in V$, η and $\tilde{\eta}$ in $C^1_{0S,0}$ with $|\eta|_{1,0S} < \epsilon$ and $|\tilde{\eta}|_{1,0S} < \epsilon$, $w \in [0,S]$, and $\theta \in [0,1]$,*

$$(P_S\chi)_w + (Z_S\eta)_w + \theta[(Z_S\tilde{\eta})_w - (Z_S\eta)_w] \in U$$

and

$$|D_e f((P_S\chi)_w + (Z_S\eta)_w + \theta[(Z_S\tilde{\eta})_w - (Z_S\eta)_w])[(Z_S\tilde{\eta})_w - (Z_S\eta)_w)]| \leq 2j\,|\tilde{\eta} - \eta|_{0S}.$$

Proof. See the proof of [23, Proposition 3.4] □

Let $\phi \in U$, and let $T = T_\phi > 0$, a convex neighbourhood $V = V_\phi$ of ϕ in U, $\epsilon = \epsilon_\phi > 0$, and $j = j_\phi \in \mathbb{N}$ be given as in Proposition 11.3.

Then Propositions 4.1, 4.2, 4.3 from [23] hold, with verbatim the same proofs. We restate these propositions as follows.

Proposition 11.4. *For every $S \in (0, T)$, $\chi \in V$, η and $\tilde{\eta}$ in $C^1_{0S,0}$ with $|\eta|_{1,0S} < \epsilon$ and $|\tilde{\eta}|_{1,0S} < \epsilon$, we have*

$$(\chi, \eta) \in D_S,\ (\chi, \tilde{\eta}) \in D_S, \quad \textit{and} \quad |A_S(\chi, \tilde{\eta}) - A_S(\chi, \eta)|_{1,0S} \leq 2jS(S+1)|\tilde{\eta} - \eta|_{1,0S}.$$

Proposition 11.5. $\lim_{S \searrow 0} A_S(\phi, 0) = 0$.

Proposition 11.6. *There exist $S_\phi \in (0, T_\phi)$ and an open neighbourhood W_ϕ of ϕ in V_ϕ such that for all $\chi \in W_\phi$, for all $S \in (0, S_\phi]$, and all $\eta \in C^1_{0S,0}$ and $\tilde{\eta} \in C^1_{0S,0}$ with $|\eta|_{1,0S} \leq \frac{\epsilon_\phi}{2}$ and $|\tilde{\eta}|_{1,0S} \leq \frac{\epsilon_\phi}{2}$, we have*

$$(\chi, \eta) \in D_S,\ (\chi, \tilde{\eta}) \in D_S,$$
$$|A_S(\chi, \eta)|_{1,0S} < \frac{\epsilon_\phi}{2} \quad \textit{and} \quad |A_S(\chi, \tilde{\eta}) - A_S(\chi, \eta)|_{1,0S} \leq \frac{1}{2}|\tilde{\eta} - \eta|_{1,0S}.$$

For each $S \in (0, S_\phi]$ now the uniform contraction result Theorem 5.2 applies to the map

$$W_\phi \times \{\eta \in C^1_{0S,0} : |\eta|_{1,0S} < \epsilon_\phi\} \ni (\chi, \eta) \mapsto A_S(\chi, \eta) \in C^1_{0S,0},$$

with $M = M_\phi = \{\eta \in C^1_{0S,0} : |\eta|_{1,0S} \leq \frac{\epsilon_\phi}{2}\}$, and yields a C^1_β-map

$$W_\phi \ni \chi \mapsto \eta_\chi \in C^1_{0S,0}$$

given by $\eta_\chi \in M_\phi$ and $A_S(\chi, \eta_\chi) = \eta_\chi$. As the maps P_S and $C^1_{0S,0} \stackrel{Z_S}{\to} C^1_S$ are linear and continuous it follows from linearity of differentiation and by means of the chain rule that also the map

$$\Sigma_\phi : W_\phi \ni \chi \mapsto P_S\chi + Z_S\eta_\chi \in C^1_S$$

is C^1_β-smooth. An application of the chain rule to the compositions of this map with the continuous linear maps $E^1_S(\cdot, t) : C^1_S \to C^1$, $0 \leq t \leq S$, yields that all maps

$$W_\phi \ni \chi \mapsto E^1_S(\Sigma_\phi(\chi), t) \in C^1, \quad 0 \leq t \leq S,$$

are C^1_β-smooth. As E^1_S is continuous we obtain that the composition

$$[0, S] \times W_\phi \ni (t, \chi) \mapsto E^1_S(\Sigma_\phi(\chi), t) \in C^1$$

is continuous.

[23, Proposition 4.4] showed that the restriction of the map Σ_ϕ to the solution manifold provides us with solutions of the initial value problem (11.1). It remains valid, with the same proof, and is restated as follows.

Proposition 11.7. *For every $S \in (0, S_\phi]$ and for every $\chi \in W_\phi \cap X_f$ the function $x = x^{(\chi)} = \Sigma_\phi(\chi)$ is a solution of Eq. (1.1) on $[0, S]$, with $x_0 = \chi$ and $x_t \in X_f$ for $0 \leq t \leq S$.*

The restriction

$$[0, S] \times (W_\phi \cap X_f) \ni (t, \chi) \mapsto E^1_S(\Sigma_\phi(\chi), t) \in C^1$$

is continuous, and the maps

$$W_\phi \cap X_f \ni \chi \mapsto E^1_S(\Sigma_\phi(\chi), t) \in C^1, \quad 0 \leq t \leq S,$$

are C^1_β-smooth from their domains in the C^1_β-submanifold X_f into C^1.

12 The semiflow on the solution manifold

The uniqueness results [23, Propositions 4.5 and 5.1] remain valid, with the same proofs. As in [23, Section 5] we find maximal solutions $x^{\phi}:(-\infty,t_{\phi})\to\mathbb{R}^n$, $0<t_{\phi}\le\infty$, of the initial value problems

$$x'(t)=f(x_t)\quad\text{for } t>0,\ x_0=\phi\in X_f,$$

which are solutions on $[0,t_{\phi})$ and have the property that any other solution on some interval with left endpoint 0, of the same initial value problem, is a restriction of x^{ϕ}. The relations

$$\Omega_f=\{(t,\phi)\in[0,\infty)\times X_f:t<t_{\phi}\},\quad \Sigma_f(t,\phi)=x_t^{\phi}$$

define a semiflow $\Sigma_f:\Omega_f\to X_f$ on X_f, compare [23, Proposition 5.2]. In [23, Proposition 5.3] and in its proof the words *continuously differentiable* can everywhere be replaced by the expression C^1_{β}-smooth. Thus Σ_f is continuous, with each domain

$$\Omega_{f,t}=\{\phi\in X_f:(t,\phi)\in\Omega_f\},\quad t\ge0,$$

an open subset of X_f and the time-t-map

$$\Sigma_{f,t}:\Omega_{f,t}\to X_f,\quad \Sigma_{f,t}(\phi)=\Sigma_f(t,\phi),\quad t\ge0,$$

C^1_{β}-smooth in case $\Omega_{f,t}\ne\emptyset$.

[23, Proposition 5.5] and its proof remain valid. In the proof of [23, Proposition 6.1] the words *continuously differentiable* can everywhere be replaced by the expression C^1_{β}-smooth. This yields

$$D_2\Sigma_f(t,\phi)\chi=D\Sigma_{f,t}(\phi)\chi=v_t^{\phi,\chi}$$

with the unique maximal continuously differentiable solution $v=v^{\phi,\chi}$ of the initial value problem

$$v'(t)=Df(\Sigma_f(t,\phi))v_t\quad\text{for } t>0,\quad v_0=\chi\in T_{\phi}X_f.$$

Part III

13 Locally bounded delay

Assume as in Part II that $f : C^1 \supset U \to \mathbb{R}^n$ is C^1_β-smooth and that f has property (e). It is convenient from here on to abbreviate $X = X_f$, $\Omega = \Omega_f$, and $\Sigma = \Sigma_f$. Let a stationary point $\bar{\phi} \in X$ of Σ be given, $\Sigma(t, \bar{\phi}) = \bar{\phi}$ for all $t \geq 0$. Then $\bar{\phi}$ is constant. (Proof of this: The solution x of Eq. (1.1) on $[0, \infty)$ with $x_0 = \phi$ satisfies $x(t) = x_t(0) = \Sigma(t, \phi)(0) = \phi(0)$ for all $t \geq 0$. For all $s < 0$ we have $x(s) = \phi(s) = \Sigma(-s, \phi)(s) = x_{-s}(s) = x(0) = \phi(0)$.)

Choose an open neighbourhood N of $\bar{\phi}$ in U and $d > 0$ according to property (lbd) from Section 1. We restate [24, Proposition 2.1] as follows.

Proposition 13.1. *For every $\phi \in N$ we have*

$$Df(\phi)\psi = 0 \quad \text{for all } \psi \in C^1 \quad \text{with } \psi(s) = 0 \quad \text{on } [-d, 0],$$

and

$$D_e f(\phi)\chi = 0 \quad \text{for all } \chi \in C \quad \text{with } \chi(s) = 0 \quad \text{on } [-d, 0].$$

Recall the restriction and prolongation maps $R_{d,1}$ and $P_{d,1}$ from Section 8, respectively. Set $\bar{\phi}_d = R_{d,1}\bar{\phi} = \bar{\phi}|_{[-d,0]}$. As $\bar{\phi}$ is constant we have

$$P_{d,1}\bar{\phi}_d = \bar{\phi} \in N,$$

and it follows that there exists a neighbourhood U_d of $\bar{\phi}_d$ in C^1_d with $P_{d,1}U_d \subset N$. Due to the chain rule the map

$$f_d : C^1_d \supset U_d \to \mathbb{R}^n, \quad f_d(\phi) = f(P_{d,1}\phi),$$

is C^1_β-smooth, with

$$Df_d(\phi)\chi = Df(P_{d,1}\phi)P_{d,1}\chi.$$

According to [24, Proposition 2.2] f_d has property (e). Results from [20,21] apply and show that the equation

$$x'(t) = f_d(x_t) \tag{13.1}$$

(with segments $x_t : [-d, 0] \ni s \mapsto x(t+s) \in \mathbb{R}^n$) defines a continuous semiflow $\Sigma_d : \Omega_d \to X_d$ on the submanifold

$$X_d = \{\phi \in U_d : \phi'(0) = f_d(\phi)\}, \quad \operatorname{codim} X_d = n,$$

of the Banach space C^1_d. In the terminology of the present paper, the manifold X_d and all solution operators $\Sigma_d(t, \cdot)$, $t \geq 0$, with non-empty domain are C^1_β-smooth.

The proofs of [24, Propositions 2.3–2.5] remain valid without change. We restate the result as follows.

Proposition 13.2.

(i) *$X_d = R_{d,1}(X \cap N \cap R^{-1}_{d,1}(U_d))$.*

(ii) *For every $\phi \in X \cap N \cap R^{-1}_{d,1}(U_d)$,*

$$T_{R_{d,1}\phi}X_d = R_{d,1}T_\phi X.$$

(iii) *For* $(t,\phi)\in\Omega$ *with* $\Sigma([0,t]\times\{\phi\})\subset N\cap R_{d,1}^{-1}(U_d)$,

$$(t,R_{d,1}\phi)\in\Omega_d \quad \textit{and} \quad \Sigma_d(t,R_{d,1}\phi)=R_{d,1}\Sigma(t,\phi).$$

(iv) *If* $(t,\chi)\in\Omega_d$ *and if* $x:(-\infty,t]\to\mathbb{R}^n$ *given by* $x(s)=x^\chi(s)$ *on* $[-d,t]$ *and by* $x(s)=(P_{d,1}\chi)(s)$ *for* $s<-d$ *satisfies* $\{x_s:0\le s\le t\}\subset N$ *then*

$$(t,P_{d,1}\chi)\in\Omega \quad \textit{and} \quad R_{d,1}\Sigma(t,P_{d,1}\chi)=\Sigma_d(t,\chi).$$

Proposition 13.2 (iii) shows that $\bar{\phi}_d$ is a stationary point of the semiflow Σ_d.

For $t\ge 0$ consider the operators $T_t=D_2\Sigma(t,\bar{\phi})$ on $T_{\bar{\phi}}X$ and $T_{d,t}=D_2\Sigma_d(t,\bar{\phi}_d)$ on $T_{\bar{\phi}_d}X_d$. The proof of [24, Corollary 2.6] remains valid. We state the result as follows.

Corollary 13.3.

(i) *For* $(t,\phi)\in\Omega$ *as in Proposition 13.2 (iii) and for all* $\chi\in T_\phi X$,

$$R_{d,1}\chi\in T_{R_{d,1}\phi}X_d \quad \textit{and} \quad R_{d,1}D_2\Sigma(t,\phi)\chi=D_2\Sigma_d(t,R_{d,1}\phi)R_{d,1}\chi.$$

(ii) *For all* $\chi\in T_{\bar{\phi}}X$ *and for all* $t\ge 0$,

$$R_{d,1}\chi\in T_{\bar{\phi}_d}X_d \quad \textit{and} \quad R_{d,1}T_t\chi=T_{d,t}R_{d,1}\chi.$$

From [7, Sections 3.5 and 4.1-4.3] and from [11] we get local stable, center, and unstable manifolds of Σ_d at $\bar{\phi}_d\in X_d\subset C_d^1$, all of them C_β^1-smooth.

14 Spectral decomposition of the tangent space

Let $Y=T_{\bar{\phi}}X$. In this section we recall from [24, Section 3] the definitions of the linear stable, center, and unstable spaces of the operators $T_t:Y\to Y$, $t\ge 0$.

The linear stable space in Y is defined by

$$Y_s=R_{d,1}^{-1}Y_{d,s}$$

with the linear stable space $Y_{d,s}$ of the strongly continuous semigroup $(T_{d,t})_{t\ge 0}$ on the tangent space $Y_d=T_{\bar{\phi}_d}X_d\subset C_d^1$. We have $Y_{d,s}=Y_d\cap C_{d,s}$ with the linear stable space $C_{d,s}$ of the strongly continuous semigroup of solution operators $T_{d,e,t}:C_d\to C_d$, $t\ge 0$, which is defined by the equation

$$v'(t)=D_e f_d(\bar{\phi}_d)v_t. \tag{14.1}$$

Let $C_{d,c}$ and $C_{d,u}$ denote the finite-dimensional linear center and unstable spaces of the semigroup on C_d. Each $\chi\in C_{d,c}\oplus C_{d,u}$ uniquely defines an analytic solution $v=v^\chi$ on $\mathbb{R}$ of Eq. (14.1). The injective map

$$I:C_{d,c}\oplus C_{d,u}\ni\chi\mapsto v^\chi|_{(-\infty,0]}\in C^1$$

is linear, and continuous (as its domain is finite-dimensional). The center and unstable spaces in Y are defined as

$$Y_c=IC_{d,c} \quad \text{and} \quad Y_u=IC_{d,u},$$

respectively. They are finite-dimensional and the maps T_t, $t \geq 0$, act as isomorphisms on each of them. The stable space Y_s is closed and positively invariant under each map T_t, $t \geq 0$, and we have the decomposition

$$Y = Y_s \oplus Y_c \oplus Y_u.$$

Finally, recall the Banach space B^1_a from Section 8 and observe

$$Y_u \subset B^1_a$$

since each v^χ, $\chi \in C_{d,u}$, and its derivative both have limit 0 at $-\infty$.

15 The local stable manifold

We begin with the local stable manifold $W^s_d \subset X_d$ of the semiflow Σ_d at the stationary point $\bar{\phi}_d \in X_d \subset C^1_d$ as it was obtained in [7]. Recall Proposition 4.2. It is easy to see that W^s_d is a C^1_β-smooth submanifold of the Banach space C^1_d which is locally positively invariant under S_d, with tangent space

$$T_{\bar{\phi}_d} W^s_d = Y_{d,s}$$

at $\bar{\phi}_d$, and that it has the following properties (I) and (II), for some $\beta > 0$ chosen with

$$\Re z < -\beta < 0$$

for all z with $\Re z < 0$ in the spectrum of the generator of the semigroup on C_d.

(I) There are $\gamma > \beta$, an open neighbourhood $\tilde{W}^s_d$ of $\bar{\phi}_d$ in W^s_d, and $\tilde{c} > 0$ such that $[0,\infty) \times \tilde{W}^s_d \subset \Omega_d$, and $\Sigma_d([0,\infty) \times \tilde{W}^s_d) \subset W^s_d$, and for all $\psi \in \tilde{W}^s_d$ and all $t \geq 0$,

$$|\Sigma_d(t,\psi) - \bar{\phi}_d|_{d,1} \leq \tilde{c}\, e^{-\gamma t} |\psi - \bar{\phi}_d|_{d,1}.$$

(II) There exists a constant $\bar{c} > 0$ such that each $\psi \in X_d$ with $[0,\infty) \times \{\psi\} \subset \Omega_d$ and

$$e^{\beta t} |\Sigma_d(t,\psi) - \bar{\phi}_d|_{d,1} < \bar{c} \quad \text{for all } t \geq 0$$

belongs to W^s_d.

The codimension of W^s_d in C^1_d is equal to

$$n + \dim Y_{d,c} + \dim Y_{d,u} = n + \dim C_{d,c} + \dim C_{d,u}.$$

As the continuous linear map $R_{d,1} : C^1 \to C^1_d$ is surjective we can apply Proposition 7.1 and obtain an open neighbourhood V of $\bar{\phi}$ in $N \subset U \subset C^1$ so that

$$W^s = W^s(\bar{\phi}) = V \cap R^{-1}_{d,1}(W^s_d)$$

is a C^1_β-submanifold of C^1 with codimension $n + \dim C_{d,c} + \dim C_{d,u}$ and tangent space

$$T_{\bar{\phi}} W^s = R^{-1}_{d,1}(T_{\bar{\phi}_d} W^s_d) = R^{-1}_{d,1}(Y_{d,s}) = Y_s.$$

The next proposition shows that W^s is the desired local stable manifold of Σ at $\bar{\phi}$.

Proposition 15.1.

(i) $W^s \subset X$, *and* W^s *is locally positively invariant.*

(ii) There are an open neighbourhood $\tilde{V}$ *of* $\bar{\phi}$ *in* V *with* $[0,\infty) \times (\tilde{V} \cap W^s) \subset \Omega$ *and a constant* $\tilde{c} > 0$ *such that for all* $\phi \in \tilde{V} \cap W^s$ *the solution* $x : \mathbb{R} \to \mathbb{R}^n$ *on* $[0,\infty)$ *of Eq. (1.1) with* $x_0 = \phi$ *satisfies*

$$|x(t) - \bar{\phi}(0)| + |x'(t)| \le \tilde{c}e^{-\gamma t}|R_{d,1}\phi - \bar{\phi}_d|_{d,1} \quad \text{for all } t \ge 0.$$

(iii) There are an open neighbourhood $\hat{V}$ *of* $\bar{\phi}$ *in* V *and a constant* $\hat{c} > 0$ *such that for every solution* $x : \mathbb{R} \to \mathbb{R}^n$ *on* $[0,\infty)$ *of Eq. (1.1) with* $x_0 \in \hat{V} \cap X$ *and*

$$|x(t) - \bar{\phi}(0)| + |x'(t)| \le \hat{c}\, e^{-\beta t} \quad \text{for all } t \ge 0$$

we have $x_0 \in W^s$.

Proposition 15.1 is proved exactly as [24, Propositions 4.1, 4.2], using the properties of W^s_d stated above.

16 The local unstable manifold

In this section all segments x_t are defined on $(-\infty, 0]$. Fix some $a > 0$ and consider the Banach spaces $B_a \subset C$ and $B^1_a \subset C^1$ introduced in Section 1. It is easy to see that the linear inclusion maps

$$j_0 : B_a \to C \quad \text{and} \quad j_1 : B^1_a \to C^1$$

are continuous, as well as the restriction and prolongation maps

$$R_{a,d,1} : B^1_a \ni \phi \mapsto R_{d,1}\phi \in C^1_d \quad \text{and} \quad P_{a,d,1} : C^1_d \ni \chi \mapsto P_{d,1}\chi \in B^1_a.$$

The set $U_a = j_1^{-1}(N) \cap R^{-1}_{a,d,1}(U_d) \subset B^1_a$ is open and contains $\bar{\phi}$, and the C^1_β-map

$$f_a : U_a \to \mathbb{R}^n, \quad f_a(\phi) = f(j_1\phi),$$

satisfies $f_a(\bar{\phi}) = 0$. Notice that every solution of the equation

$$x'(t) = f_a(x_t) \tag{16.1}$$

on some interval also is a solution of Eq. (1.1) on this interval. The proof of [24, Proposition 5.1] remains valid. Therefore we have

$$f_a(\phi) = f_a(\psi) \text{ for all } \phi \in U_a \text{ and } \psi \in U_a \text{ with } \phi(s) = \psi(s) \text{ on } [-d, 0], \tag{16.2}$$

each derivative $Df_a(\phi) : B^1_a \to \mathbb{R}^n$, $\phi \in U_a$, has a linear extension $D_e f_a(\phi) : B_a \to \mathbb{R}^n$, and the map

$$U_a \times B_a \ni (\phi, \chi) \mapsto D_e f_a(\phi)\chi \in \mathbb{R}^n$$

is continuous. Now results from [22] show that $X_a = \{\phi \in U_a : \phi'(0) = f_a(\phi)\}$ is a C^1_β-submanifold of B^1_a, that the solutions of Eq. (16.1) define a continuous semiflow $\Sigma_a : \Omega_a \to X_a$ on X_a, and that there is a local unstable manifold $W^u_a \subset X_a$ at the stationary point $\bar{\phi} \in W^u_a$. W^u_a is a C^1_β-submanifold of B^1_a consisting of data $\phi \in X_a$ which are solutions of Eq. (16.1) on $(-\infty, 0]$ with $\phi_s \to \bar{\phi}$ as $s \to -\infty$, and there exist $\overline{\beta} > \overline{\gamma} > 0$ and $c_u > 0$ so that

(I) $|\phi_s - \bar{\phi}|_{a,1} \le c_u e^{\bar{\beta}s} |\phi - \bar{\phi}|_{a,1}$ for all $\phi \in W^u_a$ and $s \le 0$,

and

(II) for every solution $\psi \in B^1_a$ of Eq. (16.1) on $(-\infty, 0]$ with

$$\sup_{s \le 0} |\psi_s - \bar{\phi}|_{a,1} e^{-\bar{\gamma}s} < \infty$$

there exists $s_\psi \le 0$ with $\psi_s \in W^u_a$ for all $s \le s_\psi$.

In [22] the tangent space $T_{\bar{\phi}} W^u_a$ of W^u_a at $\bar{\phi}$ is obtained as the vector space T of all maps $\hat{\chi} : (-\infty, 0] \to \mathbb{R}^n$ with $\hat{\chi}_0 = \chi \in C_{d,u}$ which for some $t > 0$ and for all integers $j < 0$ satisfy

$$\hat{\chi}_{jt} = \Lambda^{-j} \chi$$

where $\Lambda : C_{d,u} \to C_{d,u}$ is the isomorphism whose inverse is given by $T_{d,e,t}$. We have

$$T_{\bar{\phi}} W^u_a = Y_u.$$

because the maps in the vector space $Y_u = I C_{d,u}$ share the property defining the space T and $\dim Y_u = \dim C_{d,u} = \dim T$.

From a manifold chart for W^u_a at $\bar{\phi}$ we obtain $\epsilon > 0$ and a C^1_β-map

$$w^u_a : Y_u(\epsilon) \to B^1_a, \quad Y_u(\epsilon) = \{\phi \in Y_u : |\phi|_{a,1} < \epsilon\},$$

with $w^u_a(0) = \bar{\phi}$, $w^u_a(Y_u(\epsilon))$ an open subset of W^u_a, and $Dw^u_a(0)\eta = \eta$ for all $\eta \in Y_u$. Proposition 7.2 applies to the C^1_β-map $j_1 \circ w^u_a$. So we may assume that

$$W^u = W^u(\bar{\phi}) = j_1 w^u_a(Y_u(\epsilon))$$

is a C^1_β-submanifold of the Fréchet space C^1 with

$$T_{\bar{\phi}} W^u = j_1 D w^u_a(0) Y_u = Y_u.$$

The proof of [24, Proposition 5.2] remains valid in the present setting. We state the result about the properties of the local unstable manifold W^u as follows.

Proposition 16.1.

(i) Every $\phi \in W^u$ is a solution of Eq. (1.1) on $(-\infty, 0]$, with $\phi_s \to \bar{\phi}$ as $s \to -\infty$, and for all $s \le 0$,

$$|\phi(s) - \bar{\phi}(0)| \le c_u e^{\bar{\beta}s} |\phi - \bar{\phi}|_{a,1} \quad \text{and} \quad |\phi'(s)| \le c_u e^{\bar{\beta}s} |\phi - \bar{\phi}|_{a,1}.$$

(ii) For every $\psi \in X$ which is a solution of Eq. (1.1) on $(-\infty, 0]$ with

$$\sup_{s \le 0} e^{-\bar{\gamma}s} |\psi(s) - \bar{\phi}(0)| < \infty \quad \text{and} \quad \sup_{s \le 0} e^{-\bar{\gamma}s} |\psi'(s)| < \infty$$

there exists $s(\psi) \le 0$ with $\psi_s \in W^u$ for all $s \le s(\psi)$.

17 Local center manifolds

In this section we assume
$$\{0\} \neq Y_c$$
which is equivalent to
$$\{0\} \neq C_{d,c}.$$
In the sequel we recall the steps which in [24, Section 6] led to a local center manifold at $\bar{\phi}$ which is C^1_ζ-smooth, and point out the observation which yields C^1_β-smoothness.

The approach from [24, Section 6] first follows constructions from the proof of [11, Theorem 2.1] which were done for the case $\bar{\phi}_d = 0$. Therefore we introduce $V_d = U_d - \bar{\phi}_d \subset C^1_d$ and the C^1_β-map
$$g_d : V_d \ni \phi \mapsto f_d(\phi + \bar{\phi}_d) \in \mathbb{R}^n.$$
Then $g_d(0) = 0$ and $Dg_d(0) = Df_d(\bar{\phi}_d)$.

There is a decomposition
$$C^1_d = C^1_{d,s} \oplus C_{d,c} \oplus C_{d,u}, \quad C^1_{d,s} = C^1_d \cap C_{d,s},$$
into closed subspaces which defines a projection $P^1_{d,c} : C^1_d \to C^1_d$ onto $C_{d,c}$, and there is a norm $\|\cdot\|_{d,1}$ on C^1_d which is equivalent to $|\cdot|_{d,1}$ and whose restriction to $C_{d,c} \setminus \{0\}$ is C^∞-smooth.

Next there exists $\Delta > 0$ with
$$N_\Delta = \{\phi \in C^1_d : \|\phi\|_{d,1} < \Delta\}$$
contained in V_d so that the restricted remainder map
$$N_\Delta \ni \phi \mapsto g_d(\phi) - Dg_d(0)\phi \in \mathbb{R}^n$$
has a global continuation
$$r_{d,\Delta} : C^1_d \to \mathbb{R}^n$$
with Lipschitz constant
$$\lambda = \sup_{\phi \neq \psi} \frac{\|r_{d,\Delta}(\phi) - r_{d,\Delta}(\psi)\|_{d,1}}{\|\phi - \psi\|_{d,1}} < 1.$$

The desired local center manifold at $\bar{\phi} \in X$ will be given, up to translation, by segments $(-\infty, 0] \to \mathbb{R}^n$ of solutions on $\mathbb{R}$ of the equation
$$x'(t) = Dg_d(0)x_t + r_{d,\Delta}(x_t) \quad \text{(with segments in } C^1_d) \tag{17.1}$$
which do not grow too much at $\pm\infty$.

For $\eta > 0$ let $C^1_{d,\eta}$ denote the Banach space of all continuous maps $u : \mathbb{R} \to C^1_d$ with
$$\sup_{t \in \mathbb{R}} e^{-\eta|t|} |u(t)|_{d,1} < \infty$$
and the norm given by the preceding supremum. There exists $\eta_1 > 0$ so that for every $\phi \in C_{d,c}$ there is a unique continuously differentiable map
$$x^{[\phi]} : \mathbb{R} \to \mathbb{R}^n$$

with the curve $\mathbb{R} \ni t \mapsto x_t^{[\phi]} \in C_d^1$ contained in C_{d,η_1}^1 and Eq. (17.1) satisfied for all $t \in \mathbb{R}$ and $P_{d,c}^1 x_0^{[\phi]} = \phi$. Observe that we have

$$x^{[0]}(t) = 0 \quad \text{for all } t \in \mathbb{R}.$$

Incidentally, from here on the proof in [24, Section 6] deviates from the approach in [11].

Now consider the map

$$J : C_{d,c} \ni \phi \mapsto \bar{\phi} + x^{[\phi]}|_{(-\infty,0]} \in C^1.$$

Observe that the proof of [24, Corollary 6.2] shows that the map J is in fact C_β^1-smooth, not only C_ζ^1-smooth, and

$$DJ(0)\phi = I\phi \quad \text{for all } \phi \in C_{d,c},$$

with the injectice map I from Section 13. As $C_{d,c}$ is finite-dimensional Proposition 7.2 yields an open neighbourhood $N_{d,c}$ of 0 in $C_{d,c}$ so that the image

$$W^c = J(N_{d,c})$$

is a C_β^1-submanifold of the Fréchet space C^1, with

$$T_{\bar{\phi}} W^c = IC_{d,c} = Y_c.$$

We may assume $J(N_{d,c}) \subset N \subset U$ since J is continuous and $J(0) = \bar{\phi}$. By continuity of the map

$$C_{d,c} \ni \phi \mapsto R_{d,1}(J(\phi) - \bar{\phi}) \in C_d^1$$

at $0 \in C_{d,c}$ we also may assume that for all $\phi \in N_{d,c}$ we have

$$\|x_0^{[\phi]}\|_{d,1} < \Delta \quad \text{for all } \phi \in N_{d,c}$$

or, $x_0^{[\phi]} \in N_\Delta$ for all $\phi \in N_{d,c}$, with segments $x_0^{[\phi]}$ defined on $[-d, 0]$.

We take W^c, with tangent space Y_c at the stationary point $\bar{\phi} \in X$, as the desired local center manifold of the semiflow Σ and verify that it has the appropriate properties. Following the proof of [24, Proposition 6.3] we get

$$W^c \subset X.$$

Next, choose an open neighbourhood U_* of $\bar{\phi}$ in $N \subset U$ so small that

$$R_{d,1} U_* \subset U_d \cap (N_\Delta + \bar{\phi}_d)$$

and for all $\psi \in U_*$,

$$P_{d,c}^1 R_{d,1}(\psi - \bar{\phi}) \in N_{d,c}.$$

Then the proofs of [24, Proposition 6.4, Proposition 6.5] remain valid. We state the result as follows.

Proposition 17.1.

(i) (Local positive invariance) For every $(t, \psi) \in \Omega$ with $\psi \in W^c \subset X$ and $\Sigma([0, t] \times \{\psi\}) \subset U_*$ we have $\Sigma([0, t] \times \{\psi\}) \subset W^c$.

(ii) For every solution $y : \mathbb{R} \to \mathbb{R}^n$ of Eq. (1.1) on $\mathbb{R}$ with $y_t \in U_*$ for all $t \in \mathbb{R}$ we have $y_t \in W^c$ for all $t \in \mathbb{R}$.

Observe that the proofs of both parts of Proposition 17.1 make use of [24, Lemma 7.1] on uniqueness for an initial value problem with data in C_d^1.

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