Multiplicity of positive solutions for a class of singular elliptic equations with critical Sobolev exponent and Kirchhoff-type nonlocal term

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Abstract. We study a class of singular elliptic equations involving critical Sobolev exponent and Kirchhoff-type nonlocal term $-\left(a + b \int_{\Omega} |\nabla u|^2 dx\right)\Delta u = u^5 + g(x, u) + \lambda u^{-\gamma}$, $x \in \Omega, u > 0$, $x \in \Omega$, $u = 0$, $x \in \partial\Omega$, where $\Omega \subset \mathbb{R}^3$ is a bounded domain, $a, b, \lambda > 0$, $0 < \gamma < 1$ and $g \in C(\overline{\Omega} \times \mathbb{R})$ satisfies some conditions. By the perturbation method, variational method and some analysis techniques, we establish a multiplicity theorem.

Keywords: singular elliptic equation, Kirchhoff-type nonlocal term, critical Sobolev exponent, positive solutions, perturbation method.

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1 Introduction

In this paper, we consider the following singular elliptic equation with critical Sobolev exponent and Kirchhoff-type nonlocal term

$$\begin{cases} -\left(a + b \int_{\Omega} |\nabla u|^2 dx\right) \Delta u = u^5 + g(x, u) + \lambda u^{-\gamma}, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain, $a, b, \lambda > 0$, $0 < \gamma < 1$ and $2^* = 6$ is the well-known critical Sobolev exponent. The nonlinear term $g(x, s) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions.

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(g_1) $g \in C(\overline{\Omega} \times \mathbb{R})$, $g(x,s) \geq 0$ if $s \geq 0$ and $g(x,s) = 0$ if $s \leq 0$ for all $x \in \Omega$.

(g_2) $\lim_{s \to 0^+} \frac{g(x,s)}{s} = 0$ and $\lim_{s \to +\infty} \frac{g(x,s)}{s^5} = 0$ uniformly for all $x \in \Omega$.

(g_3) There exists $0 < \kappa \ll a$ such that $g(x,s)s - 4G(x,s) \geq -(a-\kappa)\lambda_1 s^2$ for all $x \in \Omega$ and $s \geq 0$, where $G(x,s) = \int_0^s g(x,t)dt$ and $\lambda_1 > 0$ is the first eigenvalue of the operator $-\Delta$.

(g_4) There exists a nonempty open set $\omega \subset \Omega$ with $0 \in \omega$ such that $\lim_{s \to +\infty} \frac{g(x,s)}{s^3} = +\infty$ uniformly for $x \in \omega$.

Because of the presence of the term $b \int_\Omega |\nabla u|^2 dx$, which implies that the equation is no longer a pointwise identity, problem (1.1) is called the nonlocal problem. This phenomenon provokes some mathematical difficulties, which makes the study of such a class of problem particularly interesting. Its physical motivation about the operator $\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u$, which appears in the Kirchhoff equation. Thus, problem (1.1) is always called Kirchhoff-type problem. The Kirchhoff equation is related to the following stationary analogue of the equation

$$\begin{cases} u_{tt} - \left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

proposed by Kirchhoff [14] in 1883 as an extension of the classical D'Alembert's wave equation for free vibration of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. In problem (1.2), u denotes the displacement, $f(x,u)$ the external force and b the initial tension while a is related to the intrinsic properties of the string (such as Young's modulus). It is worth pointing out that problem (1.2) received much attention only after the work of Lions [23] where a function analysis framework was proposed to the problem. After that, the Kirchhoff-type problem has been extensively investigated, for examples [1,4,9–13,15,17–22,24,25,27–37].

To our best knowledge, the pioneer work on the Kirchhoff-type problem with critical Sobolev exponent is Alves, Corrêa and Figueiredo [1], they considered the following critical Sobolev exponent problem

$$\begin{cases} -\left[M\left(\int_\Omega |\nabla u|^2 dx\right)\right]\Delta u = u^5 + \lambda f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^3$, $M : \mathbb{R}^+ \to \mathbb{R}^+$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ are continuous functions, $F(x,u) = \int_0^u f(x,s)ds$ is superquadratic at the origin and subcritical at infinity. By using the variational method, under appropriate conditions, they obtained that problem (1.3) has a positive solution for all $\lambda > 0$ large enough. After that, the Kirchhoff-type problem with critical exponent has been extensively studied, and some important and interesting results have been obtained, see [4,8–13,15–21,24,27–29,33–37].

Particularly, Lei, Liao and Tang [16] studied the following singular Kirchhoff-type problem with critical exponent

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = u^5 + \lambda u^{-\gamma}, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.4}$$

using the variational method and perturbation method, they obtained two positive solutions for problem (1.4) when $\lambda > 0$ small. After that, Liu et al. generalized [16] to $\mathbb{R}^4$ with the following equation

$$\begin{cases} -\left(a + b \int_\Omega |\nabla u|^2 dx\right) \Delta u = \mu u^3 + \frac{\lambda}{|x|^\beta u^\gamma}, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.5}$$

where $\Omega \subset \mathbb{R}^4$ a bounded smooth domain and $\lambda, \mu > 0$, $0 \leq \beta < 3$, see [24]. When $0 < \gamma < \frac{1}{2}$ and $2(1 + \gamma) < \beta < 3$, by the same methods in [16], they also got two positive solutions for problem (1.4) when $\mu > bS^2$ and $\lambda > 0$ small, where S is the best Sobolev constant in $\mathbb{R}^4$.

Based on [16] and [24], the mountain-pass level value is the most obstacle in proving the existence of the second solution of problem (1.4). This obstacle stems from the local term $b \int_\Omega |\nabla u|^2 dx$, which shows that the difference between the classic elliptic problem(that is, $b = 0$) and the Kirchhoff-type problem. In this paper, we give another way to overcome this obstacle. We add a supperlinear term $g(x, u)$ in problem (1.4), that is problem (1.1). Combining with the perturbation method and variational method, we obtain two positive solutions for problem (1.1).

For all $u \in H_0^1(\Omega)$, we define

$$I(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{6}\int_\Omega (u^+)^6 dx - \int_\Omega G(x, u^+) dx - \frac{\lambda}{1 - \gamma} \int_\Omega (u^+)^{1-\gamma} dx,$$

where $G(x, u) = \int_0^u g(x, s) ds$ and $H_0^1(\Omega)$ is a Sobolev space equipped with the norm $\|u\| = \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}}$. Obviously, the energy functional I does not belong to $C^1(H_0^1(\Omega), \mathbb{R})$. Note that a function u is called a weak solution of problem (1.1) if $u \in H_0^1(\Omega)$ such that

$$\left(a + b\|u\|^2\right) \int_\Omega (\nabla u, \nabla \varphi) dx - \int_\Omega (u^+)^5 \varphi dx - \int_\Omega g(x, u^+) \varphi dx - \lambda \int_\Omega (u^+)^{-\gamma} \varphi dx = 0, \tag{1.6}$$

for all $\varphi \in H_0^1(\Omega)$.

Let S be the best Sobolev constant, namely

$$S := \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega |u|^6 dx\right)^{\frac{1}{3}}}. \tag{1.7}$$

Our main results can be described as follows.

Theorem 1.1. *Assume that $a, b, \lambda > 0$, $0 < \gamma < 1$ and g satisfies (g_1)–(g_4), then there exists $\Lambda > 0$ such that problem (1.1) possesses two positive solutions for all $0 < \lambda < \Lambda$.*

Remark 1.2. To the best of our knowledge, our result is up to date. As we known, [16] is the first paper which considered the singular Kirchhoff-type problem with critical exponent, that is, problem (1.4). However, there exists a small gap in the proof of the second positive solution, that is, the estimation of $B(t_\varepsilon v_\varepsilon)$ in Page 533 of [16]. Indeed, when using the inequality of (3.14) in Page 532 of [16] to estimate $B(t_\varepsilon v_\varepsilon)$, they need check $\frac{\alpha}{t_\varepsilon v_\varepsilon}$ is small enough for $|x| \leq \varepsilon^{\frac{1-\gamma}{16}}$ and ε small. However, it maybe is not true. Obviously, if $|x| \leq \varepsilon^{\frac{1-\gamma}{16}}$ and $\varepsilon \to 0$, for some $C > 0$, one has $\frac{\alpha}{t_\varepsilon v_\varepsilon} \geq C\alpha \frac{(\varepsilon + |x|^2)^{\frac{1}{2}}}{\varepsilon^{\frac{1}{4}}}$, which does not implies that $\frac{\alpha}{t_\varepsilon v_\varepsilon}$ is small. So far there is no way to

correct it. In [24], the authors avoided the similar question by multiplying $|x|^{-\beta}$ in front of the singular term $\frac{\lambda}{u^\gamma}$, see problem (1.5). In here, in order to arrive at the same effect, we add a continuous subcritical function g in the right hand side of equation (1.4).

Comparing with Theorem 1.1 in [28], our problem (1.1) is a singular perturbing problem of that paper. Thanks to this perturbation, we get another solution. Moreover, our condition (g_3) is more general than the following condition (g_3') in [28],

(g_3') There exists a constant $\theta \in (4,6)$ such that $g(x,s)s - \theta G(x,s) \geq 0$ for all $x \in \Omega$ and $s \geq 0$, where $G(x,s) = \int_0^s g(x,t)dt$.

We should point out that [28] generalized a part of Brézis–Nirenberg's result in [7] to the Kirchhoff-type problem.

Our condition (g_4) is first given by [7], which is used to estimate the level of the mountain-pass value. Thanks to (g_4), we obtain the second positive solution of problem (1.1).

In view of the typical power nonlinearities, Theorem 1.1 allows us to ensure the following corollary.

Corollary 1.3. *Assume that $a, b, \lambda > 0$, $0 < \gamma < 1$, $4 < p < 6$ and $g(x,u) = u^{p-1}$, then there exists $\tilde{\Lambda} > 0$ such that problem (1.1) possesses two positive solutions for all $0 < \lambda < \tilde{\Lambda}$.*

Remark 1.4. When $g(x,u) = u^{p-1}(4 < p < 6)$, then clearly g satisfies (g_1)–(g_4). For the proof, we can consider instead with $g(x,u) = (u^+)^{p-1}$.

This paper is organized as following: in Section 2, we consider an auxiliary problem, and in Section 3, we give the proof of Theorem 1.1. For the convenience of writing, we denote $C, C_1, C_2, \ldots$ as various positive constants in the following.

2   The auxiliary problem

In order to overcome the difficulty of the singular term, for every $\epsilon > 0$, we study the following perturbation problem

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = (u^+)^5 + g(x,u) + \lambda(u^+ + \epsilon)^{-\gamma}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \quad (2.1)$$

where $u^+ = \max\{u,0\}$. The energy functional corresponding to problem (2.1) is

$$I_\epsilon(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{6}\int_\Omega (u^+)^6 dx - \int_\Omega G(x,u^+)dx - \frac{\lambda}{1-\gamma}\int_\Omega \left[(u^+ + \epsilon)^{1-\gamma} - \epsilon^{1-\gamma}\right]dx,$$

Obviously, the energy functional I_ϵ is of class C^1 on $H_0^1(\Omega)$. As well known that there exists a one-to-one correspondence between all solutions of problem (2.1) and the critical points of I_ϵ on $H_0^1(\Omega)$. We mean a function u is called a weak solution of problem (2.1) if $u \in H_0^1(\Omega)$ such that

$$(a + b\|u\|^2)\int_\Omega (\nabla u, \nabla \varphi)dx - \int_\Omega (u^+)^5 \varphi dx - \int_\Omega g(x,u^+)\varphi dx - \lambda \int_\Omega \frac{\varphi}{(u^+ + \epsilon)^\gamma}dx = 0, \quad (2.2)$$

for all $\varphi \in H_0^1(\Omega)$.

First, we prove that I_ϵ satisfies the local $(PS)_c$ condition.

Lemma 2.1. *Suppose that $a, b, \lambda > 0$, $0 < \gamma < 1$ and g satisfies $(g_1) - (g_3)$, then I_ϵ satisfies the $(PS)_c$ condition, where $c < \Theta - D\lambda^{\frac{2}{1+\gamma}}$ with $D = D(\gamma, S, \kappa, \Omega)$ is a positive constant and*

$$\Theta = \frac{abS^3}{4} + \frac{b^3 S^6}{24} + \frac{aS\sqrt{b^2 S^4 + 4aS}}{6} + \frac{b^2 S^4 \sqrt{b^2 S^4 + 4aS}}{24}.$$

Proof. Suppose that $\{u_n\}$ is a $(PS)_c$ sequence for $c \in (0, \Theta - D\lambda^{\frac{2}{1+\gamma}})$, that is,

$$I_\epsilon(u_n) \to c, \qquad I'_\epsilon(u_n) \to 0, \tag{2.3}$$

as $n \to +\infty$. We claim that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. In fact, from (g_1) and (g_2), there exists $C_0 > 0$ such that

$$\left| \frac{1}{5}g(x, s)s - G(x, s) \right| \leq \frac{1}{30}|s|^6 + C_0. \tag{2.4}$$

Note that the subadditivity of $t^{1-\gamma}$, one has

$$(u_n^+ + \epsilon)^{1-\gamma} - \epsilon^{1-\gamma} \leq (u_n^+)^{1-\gamma}. \tag{2.5}$$

Consequently, combining with the Sobolev inequality, it follows from (2.3) and (2.5) that

$$\begin{aligned}
1 + c &+ o(1)\|u_n\| \\
&\geq I_\epsilon(u_n) - \frac{1}{5}\langle I'_\epsilon(u_n), u_n \rangle \\
&= \frac{3a}{10}\|u_n\|^2 + \frac{b}{20}\|u_n\|^4 + \int_\Omega \left[\frac{1}{5}g(x, u_n^+)u_n^+ - G(x, u_n^+) \right] dx \\
&\quad + \frac{1}{30}\int_\Omega (u_n^+)^6 dx - \frac{\lambda}{1-\gamma}\int_\Omega \left[(u_n^+ + \epsilon)^{1-\gamma} - \epsilon^{1-\gamma} \right] dx - \frac{\lambda}{5\epsilon^\gamma}\int_\Omega u_n^- dx \\
&\geq \frac{3a}{10}\|u_n\|^2 + \frac{b}{20}\|u_n\|^4 - \frac{\lambda}{1-\gamma}\int_\Omega (u_n^+)^{1-\gamma} dx - \frac{\lambda}{5\epsilon^\gamma}\int_\Omega u_n^- dx - C_0|\Omega|, \\
&\geq \frac{3a}{10}\|u_n\|^2 + \frac{b}{20}\|u_n\|^4 - C\|u_n\|^{1-\gamma} - C_1\|u_n\| - C_0|\Omega|,
\end{aligned} \tag{2.6}$$

since $0 < \gamma < 1$, which implies that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Going if necessary to a subsequence, still denoted by $\{u_n\}$, there exists $u \in H_0^1(\Omega)$ such that

$$\begin{cases}
u_n \rightharpoonup u, & \text{weakly in } H_0^1(\Omega), \\
u_n \to u, & \text{strongly in } L^s(\Omega), \ 1 \leq s < 6, \\
u_n(x) \to u(x), & \text{a.e. in } \Omega, \\
\text{there exists } k \in L^1(\Omega) \text{ such that for all } n, \ |u_n(x)| \leq k(x) \text{ a.e. in } \Omega.
\end{cases} \tag{2.7}$$

For every $\epsilon > 0$, since

$$\frac{|u|}{(u_n^+ + \epsilon)^\gamma} \leq \frac{|u|}{\epsilon^\gamma},$$

by the dominated convergence theorem and (2.7), one has

$$\lim_{n \to \infty} \int_\Omega (u_n^+ + \epsilon)^{-\gamma} u \, dx = \int_\Omega (u^+ + \epsilon)^{-\gamma} u \, dx. \tag{2.8}$$

Moreover, for every $\epsilon > 0$, by (2.7), one gets

$$
\begin{aligned}
\left| \frac{u_n}{(u_n^+ + \epsilon)^\gamma} \right| &\leq \left| \frac{u_n}{(u_n^+ + \epsilon)^\gamma} \right| \\
&\leq \frac{|u_n|}{\epsilon^\gamma} \\
&\leq \frac{1}{\epsilon^\gamma} k(x).
\end{aligned}
$$

Therefore, it follows from the dominated convergence theorem that

$$
\lim_{n\to\infty} \int_\Omega (u_n^+ + \epsilon)^{-\gamma} u_n dx = \int_\Omega (u^+ + \epsilon)^{-\gamma} u dx. \tag{2.9}
$$

From (2.7), one also has

$$
\int_\Omega |\nabla u_n|^2 dx = \int_\Omega |\nabla w_n|^2 dx + \int_\Omega |\nabla u|^2 dx + o(1), \tag{2.10}
$$

$$
\left(\int_\Omega |\nabla u_n|^2 dx \right)^2 = \|w_n\|^4 + \|u\|^4 + 2\|w_n\|^2 \|u\|^2 + o(1). \tag{2.11}
$$

By (g_2) and (2.7), one has

$$
\int_\Omega g(x, u_n^+) u dx = \int_\Omega g(x, u^+) u dx + o(1), \tag{2.12}
$$

$$
\int_\Omega g(x, u_n^+) u_n dx = \int_\Omega g(x, u^+) u dx + o(1), \tag{2.13}
$$

$$
\int_\Omega G(x, u_n^+) dx = \int_\Omega G(x, u^+) dx + o(1). \tag{2.14}
$$

As usually, letting $w_n = u_n - u$, we need prove that $\|w_n\| \to 0$ as $n \to \infty$. Let $\lim_{n\to\infty} \|w_n\| = l \geq 0$. If $l = 0$, our conclusion is true. Suppose that $l > 0$. By the Brézis–Lieb Lemma (see [6]), one has

$$
\int_\Omega (u_n^+)^6 dx = \int_\Omega (w_n^+)^6 dx + \int_\Omega (u^+)^6 dx + o(1). \tag{2.15}
$$

From (2.3), (2.7), (2.9) and (2.13), one obtains

$$
a\|u_n\|^2 + b\|u_n\|^4 - \int_\Omega (u_n^+)^6 dx - \int_\Omega g(x, u^+) u dx - \lambda \int_\Omega (u^+ + \epsilon)^{-\gamma} u dx = o(1),
$$

consequently, it follows from (2.10), (2.11) and (2.15) that

$$
\begin{aligned}
a\|u\|^2 &+ a\|w_n\|^2 + b\|u\|^4 + b\|w_n\|^4 + 2b\|w_n\|^2\|u\|^2 \\
&- \int_\Omega (w_n^+)^6 dx - \int_\Omega (u^+)^6 dx - \int_\Omega g(x, u^+) u dx - \lambda \int_\Omega (u^+ + \epsilon)^{-\gamma} u dx = o(1). \tag{2.16}
\end{aligned}
$$

It follows from (2.3), (2.8) and (2.13) that

$$
\begin{aligned}
0 &= \lim_{n\to\infty} \langle I_\epsilon'(u_n), u \rangle \\
&= a\|u\|^2 + b\|u\|^4 + bl^2\|u\|^2 - \int_\Omega (u^+)^6 dx - \int_\Omega g(x, u^+) u dx - \lambda \int_\Omega (u^+ + \epsilon)^{-\gamma} u dx. \tag{2.17}
\end{aligned}
$$

Moreover, by (2.3), for any $\varphi \in H_0^1(\Omega)$, one has $\lim_{n\to\infty} \langle I_\epsilon'(u_n), \varphi \rangle = 0$, that is,

$$
(a + bd) \int_\Omega (\nabla u, \nabla \varphi) dx - \int_\Omega (u^+)^5 \varphi dx - \int_\Omega g(x, u^+) \varphi dx - \lambda \int_\Omega (u^+ + \epsilon)^{-\gamma} \varphi dx = 0, \tag{2.18}
$$

where $d = \|u_n\|^2 + o(1)$ is a positive constant. Particularly, choosing $\varphi = u^-$ in (2.18), one has $u^- = 0$. Thus, we have $u \geq 0$ in Ω. On the one hand, from (2.5), (2.17) and (g_3), by the Hölder inequality, Sobolev inequality and Poincaré inequality, we have

$$
\begin{aligned}
I_\epsilon(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{6}\int_\Omega (u^+)^6 dx - \int_\Omega G(x,u)dx \\
&\quad - \frac{\lambda}{1-\gamma}\int_\Omega \left[(u^+ + \epsilon)^{1-\gamma} - \epsilon^{1-\gamma}\right]dx \\
&= \frac{a}{4}\|u\|^2 + \frac{1}{12}\int_\Omega (u^+)^6 dx + \int_\Omega \left[\frac{1}{4}g(x,u^+)u - G(x,u^+)\right]dx \\
&\quad - \frac{\lambda}{1-\gamma}\int_\Omega \left[(u^+ + \epsilon)^{1-\gamma} - (u^+)^{1-\gamma}\right]dx + \frac{\lambda}{4}\int_\Omega (u^+ + \epsilon)^{-\gamma}u\,dx - \frac{bl^2}{4}\|u\|^2 \quad (2.19) \\
&\geq \frac{a}{4}\|u\|^2 - \frac{\lambda_1(a-\kappa)}{4}\int_\Omega u^2 dx - \frac{\lambda}{1-\gamma}\int_\Omega u^{1-\gamma}dx - \frac{bl^2}{4}\|u\|^2 \\
&\geq \frac{\kappa}{4}\|u\|^2 - \frac{\lambda}{1-\gamma}|\Omega|^{\frac{5+\gamma}{6}}S^{-\frac{1-\gamma}{2}}\|u\|^{1-\gamma} - \frac{bl^2}{4}\|u\|^2 \\
&\geq -D\lambda^{\frac{2}{1+\gamma}} - \frac{bl^2}{4}\|u\|^2,
\end{aligned}
$$

where the last inequality is obtained by the Young inequality and

$$
D = \frac{1+\gamma}{1-\gamma}2^{\frac{-2\gamma}{1+\gamma}}(\kappa S)^{-\frac{1-\gamma}{1+\gamma}}|\Omega|^{\frac{5+\gamma}{3(1+\gamma)}}.
$$

On the other hand, it follows from (2.14), (2.16) and (2.17) that

$$
a\|w_n\|^2 + b\|w_n\|^4 + b\|w_n\|^2\|u\|^2 - \int_\Omega (w_n^+)^6 dx = o(1), \tag{2.20}
$$

and

$$
I_\epsilon(u_n) = I_\epsilon(u) + \frac{a}{2}\|w_n\|^2 + \frac{b}{4}\|w_n\|^4 + \frac{b}{2}\|w_n\|^2\|u\|^2 - \frac{1}{6}\int_\Omega (w_n^+)^6 dx + o(1). \tag{2.21}
$$

From (1.7), one has

$$
\int_\Omega (w_n^+)^6 dx \leq \int_\Omega |w_n|^6 dx \leq \frac{\|w_n\|^6}{S^3},
$$

consequently, it follows from (2.20) that

$$
al^2 + bl^4 + bl^2\|u\|^2 \leq \frac{l^6}{S^3},
$$

which implies that

$$
l^2 \geq \frac{bS^3 + \sqrt{b^2S^6 + 4S^3(a+b\|u\|^2)}}{2}. \tag{2.22}
$$

Thus, from (2.20)–(2.22), we obtain

$$
\begin{aligned}
I_\epsilon(u) &= \lim_{n\to\infty}\left[I_\epsilon(u_n) - \frac{a}{2}\|w_n\|^2 - \frac{b}{4}\|w_n\|^4 - \frac{b}{2}\|w_n\|^2\|u\|^2 + \frac{1}{6}\int_\Omega (w_n^+)^6 dx\right]\\
&= c - \left(\frac{a}{3}l^2 + \frac{b}{12}l^4 + \frac{b}{3}l^2\|u\|^2\right)\\
&\leq c - \left[\frac{a\left(bS^3 + \sqrt{b^2 S^6 + 4S^3(a + b\|u\|^2)}\right)}{6}\right.\\
&\quad + \frac{b}{48}\left(bS^3 + \sqrt{b^2 S^6 + 4S^3(a + b\|u\|^2)}\right)^2\\
&\quad + \left.\frac{b\|u\|^2}{6}\left(bS^3 + \sqrt{b^2 S^6 + 4S^3(a + b\|u\|^2)}\right)\right] - \frac{bl^2}{4}\|u\|^2\\
&\leq c - \left(\frac{abS^3}{4} + \frac{b^3 S^6}{24} + \frac{aS\sqrt{b^2 S^4 + 4aS}}{6} + \frac{b^2 S^4 \sqrt{b^2 S^4 + 4aS}}{24}\right) - \frac{bl^2}{4}\|u\|^2\\
&< -D\lambda^{\frac{2}{1+\gamma}} - \frac{bl^2}{4}\|u\|^2,
\end{aligned}
$$

which contradicts (2.19). Hence, $l \equiv 0$, that is, $u_n \to u$ in $H_0^1(\Omega)$ as $n \to \infty$. Therefore, I_ϵ satisfies the $(PS)_c$ condition for all $c < \Theta - D\lambda^{\frac{2}{1+\gamma}}$. This completes the proof of Lemma 2.1. □

As well known, the function

$$
U(x) = \frac{(3\varepsilon^2)^{\frac{1}{4}}}{(\varepsilon^2 + |x|^2)^{\frac{1}{2}}}, \qquad x \in \mathbb{R}^3, \tag{2.23}
$$

is an extremal function for the minimum problem (1.7), that is, it is a positive solution of the following problem

$$
-\Delta u = u^5, \qquad \forall x \in \mathbb{R}^3.
$$

Now, we estimate the level value of functional I_ϵ and obtain the following lemma.

Lemma 2.2. *Assume that* $a, b, \lambda > 0$, $0 < \gamma < 1$ *and g satisfies* (g_1), (g_2) *and* (g_4), *then there exists* $u_0 \in H_0^1(\Omega)$, *such that* $\sup_{t\geq 0} I_\epsilon(tu_0) < \Theta - D\lambda^{\frac{2}{1+\gamma}}$ *for all* $0 < \lambda < \lambda^*$, *where* Θ *and* D *are defined by Lemma* 2.1 *and the positive constant* λ^* *is independent of* u_0 *and* ϵ.

Proof. Define a cut-off function $\eta \in C_0^\infty(\Omega)$ such that $0 \leq \eta \leq 1$, $|\nabla\eta| \leq C_1$. For some $\delta > 0$, we define

$$
\eta(x) = \begin{cases} 1, & |x| \leq \delta, \\ 0, & |x| \geq 2\delta, \end{cases}
$$

and $\{x : |x| \leq 2\delta\} \subset \omega$, where ω is defined by (g_4). Set $u_\varepsilon = \eta(x)U(x)$, where $U(x)$ is defined by (2.23). As well known (see [7, 26]), one has

$$
\|u_\varepsilon\|^2 = \|U\|^2 + O(\varepsilon) = S^{\frac{3}{2}} + O(\varepsilon), \tag{2.24}
$$

$$
|u_\varepsilon|_6^6 = |U|_6^6 + O(\varepsilon^3) = S^{\frac{3}{2}} + O(\varepsilon^3), \tag{2.25}
$$

$$
\begin{cases}
C_2\varepsilon^{\frac{s}{2}} \leq \int_\Omega u_\varepsilon^s dx \leq C_3\varepsilon^{\frac{s}{2}}, & 1 \leq s < 3, \\
C_4\varepsilon^{\frac{s}{2}}|\ln\varepsilon| \leq \int_\Omega u_\varepsilon^s dx \leq C_5\varepsilon^{\frac{s}{2}}|\ln\varepsilon|, & s = 3, \\
C_6\varepsilon^{\frac{6-s}{2}} \leq \int_\Omega u_\varepsilon^s dx \leq C_7\varepsilon^{\frac{6-s}{2}}, & 3 < s < 6.
\end{cases} \tag{2.26}
$$

Moreover, from [35], we also have

$$
\begin{cases}
\|u_\varepsilon\|^4 = S^3 + O(\varepsilon); \\
\|u_\varepsilon\|^6 = S^{\frac{9}{2}} + O(\varepsilon); \\
\|u_\varepsilon\|^8 = S^6 + O(\varepsilon); \\
\|u_\varepsilon\|^{12} = S^9 + O(\varepsilon).
\end{cases}
\tag{2.27}
$$

For all $t \geq 0$, we define $I_\epsilon(tu_\varepsilon)$ by

$$
I_\epsilon(tu_\varepsilon) = \frac{at^2}{2}\|u_\varepsilon\|^2 + \frac{bt^4}{4}\|u_\varepsilon\|^4 - \frac{t^6}{6}\int_\Omega u_\varepsilon^6 dx - \int_\Omega G(x, tu_\varepsilon)dx
$$
$$
- \frac{\lambda}{1-\gamma}\int_\Omega \left[(tu_\varepsilon + \epsilon)^{1-\gamma} - \epsilon^{1-\gamma}\right] dx,
$$

from (g_2), we have

$$
\lim_{t \to +0} I_\epsilon(tu_\varepsilon) = 0, \quad \text{uniformly for all } 0 < \varepsilon < \varepsilon_0,
$$

and

$$
\lim_{t \to +\infty} I_\epsilon(tu_\varepsilon) = -\infty, \quad \text{uniformly for all } 0 < \varepsilon < \varepsilon_0,
$$

where $\varepsilon_0 > 0$ is a small constant. Thus $\sup_{t \geq 0} I_\epsilon(tu_\varepsilon)$ attains for some $t_\varepsilon > 0$. Using the following conclusion of *Step 1* in the proof of Theorem 2.3, one has $I_\epsilon(t_\varepsilon u_\varepsilon) > \rho > 0$. So, by the continuity of I_ϵ, there exist two constants $t_0, T_0 > 0$, which independent of ε, such that $t_0 < t_\varepsilon < T_0$. Set $I_\epsilon(tu_\varepsilon) = I_{\varepsilon,1}(t) - I_{\varepsilon,2}(t) - I_{\varepsilon,3}(t)$, where

$$
I_{\varepsilon,1}(t) = \frac{a}{2}t^2\|u_\varepsilon\|^2 + \frac{b}{4}t^4\|u_\varepsilon\|^4 - \frac{t^6}{6}\int_\Omega u_\varepsilon^6 dx,
$$

and

$$
I_{\varepsilon,2}(t) = \int_\Omega G(x, tu_\varepsilon)dx,
$$
$$
I_{\varepsilon,3}(t) = \frac{\lambda}{1-\gamma}\int_\Omega \left[(tu_\varepsilon + \epsilon)^{1-\gamma} - \epsilon^{1-\gamma}\right] dx.
$$

First, we estimate the value of $I_{\varepsilon,1}$. Since $I'_{\varepsilon,1}(t) = at\|u_\varepsilon\|^2 + bt^3\|u_\varepsilon\|^4 - t^5\int_\Omega u_\varepsilon^6 dx$, let $I'_{\varepsilon,1}(t) = 0$, that is,

$$
a\|u_\varepsilon\|^2 + bt^2\|u_\varepsilon\|^4 - t^4\int_\Omega u_\varepsilon^6 dx = 0,
\tag{2.28}
$$

one obtains

$$
T_\varepsilon^2 = \frac{b\|u_\varepsilon\|^4 + \sqrt{b^2\|u_\varepsilon\|^8 + 4a\|u_\varepsilon\|^2 \int_\Omega u_\varepsilon^6 dx}}{2\int_\Omega u_\varepsilon^6 dx}.
$$

Then $I'_{\varepsilon,1}(t) > 0$ for all $0 < t < T_\varepsilon$ and $I'_{\varepsilon,1}(t) < 0$ for all $t > T_\varepsilon$, so $I_{\varepsilon,1}(t)$ attains its maximum

at T_ε. Thus, it follows from (2.24), (2.25), (2.27) and (2.28) that

$$
\begin{aligned}
I_{\varepsilon,1}(t) &\leq I_{\varepsilon,1}(T_\varepsilon) \\
&= T_\varepsilon^2 \left(\frac{a}{2} \|u_\varepsilon\|^2 + \frac{b}{4} T_\varepsilon^2 \|u_\varepsilon\|^4 - \frac{T_\varepsilon^4}{6} \int_\Omega u_\varepsilon^6 dx \right) \\
&= T_\varepsilon^2 \left(\frac{a}{3} \|u_\varepsilon\|^2 + \frac{b}{12} T_\varepsilon^2 \|u_\varepsilon\|^4 \right) \\
&= \frac{ab\|u_\varepsilon\|^6 + a\|u_\varepsilon\|^2 \sqrt{b^2\|u_\varepsilon\|^8 + 4a\|u_\varepsilon\|^2 \int_\Omega u_\varepsilon^6 dx}}{6 \int_\Omega u_\varepsilon^6 dx} \\
&\quad + \frac{b^3\|u_\varepsilon\|^{12} + 2ab\|u_\varepsilon\|^6 \int_\Omega u_\varepsilon^6 dx}{24 \left(\int_\Omega u_\varepsilon^6 dx \right)^2} \\
&\quad + \frac{b^2\|u_\varepsilon\|^4 \sqrt{b^2\|u_\varepsilon\|^8 + 4a\|u_\varepsilon\|^2 \int_\Omega u_\varepsilon^6 dx}}{24 \left(\int_\Omega u_\varepsilon^6 dx \right)^2} \\
&= \frac{ab\|u_\varepsilon\|^6}{4 \int_\Omega u_\varepsilon^6 dx} + \frac{b^3\|u_\varepsilon\|^{12}}{24(\int_\Omega u_\varepsilon^6 dx)^2} \\
&\quad + \frac{a\|u_\varepsilon\|^2 \sqrt{b^2\|u_\varepsilon\|^8 + 4a\|u_\varepsilon\|^2 \int_\Omega u_\varepsilon^6 dx}}{6 \int_\Omega u_\varepsilon^6 dx} \\
&\quad + \frac{b^2\|u_\varepsilon\|^4 \sqrt{b^2\|u_\varepsilon\|^8 + 4a\|u_\varepsilon\|^2 \int_\Omega u_\varepsilon^6 dx}}{24 \left(\int_\Omega u_\varepsilon^6 dx \right)^2} \\
&= \frac{ab(S^{\frac{9}{2}} + O(\varepsilon))}{4(S^{\frac{3}{2}} + o(\varepsilon))} + \frac{b^3(S^9 + O(\varepsilon))}{24(S^{\frac{3}{2}} + o(\varepsilon))^2} \\
&\quad + \frac{a(S^{\frac{3}{2}} + O(\varepsilon))\sqrt{b^2 S^6 + 4a S^3 + O(\varepsilon)}}{6(S^{\frac{3}{2}} + o(\varepsilon))} \\
&\quad + \frac{b^2(S^6 + O(\varepsilon))\sqrt{b^2 S^6 + 4a S^3 + O(\varepsilon)}}{24(S^{\frac{3}{2}} + o(\varepsilon))^2} \\
&\leq \frac{abS^3}{4} + \frac{b^3 S^6}{24} + \frac{aS\sqrt{b^2 S^4 + 4a S}}{6} + \frac{b^2 S^4 \sqrt{b^2 S^4 + 4a S}}{24} + C_8 \varepsilon \\
&= \Theta + C_8 \varepsilon.
\end{aligned}
\tag{2.29}
$$

Second, we estimate the value of $I_{\varepsilon,2}$. We claim that

$$
\lim_{\varepsilon \to 0^+} \frac{\int_\Omega G(x, t_\varepsilon u_\varepsilon) dx}{\varepsilon} = +\infty.
\tag{2.30}
$$

Let $m(t) = \inf_{x \in w} g(x, t)$, from (g_1) and (g_4), we have

$$
g(x, t) \geq m(t) \geq 0, \qquad \lim_{t \to +\infty} \frac{m(t)}{t^3} = +\infty,
$$

for almost $x \in \omega$ and $t > 0$. Consequently, for any $\mu > 0$, there exists $A > 0$ such that

$M(t) \geq \mu t^4$ for all $t \geq A$, where $M(t) = \int_0^t m(s)ds$. Thus, one has

$$
\begin{aligned}
\frac{\int_\Omega G(x, t_\varepsilon u_\varepsilon)dx}{\varepsilon} &\geq \varepsilon^{-1} \int_{|x|<\delta} G(x, t_\varepsilon u_\varepsilon)dx \\
&\geq \varepsilon^{-1} \int_{|x|<\delta} M(t_\varepsilon u_\varepsilon)dx \\
&= \varepsilon^{-1} \int_0^\delta M\left[\frac{t_\varepsilon 3^{\frac{1}{4}} \varepsilon^{\frac{1}{2}}}{(\varepsilon^2 + r^2)^{\frac{1}{2}}} \right] r^2 dr \\
&= \varepsilon^2 \int_0^{\delta\varepsilon^{-1}} M\left[\frac{t_\varepsilon 3^{\frac{1}{4}} \varepsilon^{-\frac{1}{2}}}{(1 + r^2)^{\frac{1}{2}}} \right] r^2 dr \\
&= \varepsilon^2 \int_0^{\varepsilon^{-1}} M\left[\frac{t_\varepsilon 3^{\frac{1}{4}} \varepsilon^{-\frac{1}{2}}}{(1 + r^2)^{\frac{1}{2}}} \right] r^2 dr - \varepsilon^2 \int_{\delta\varepsilon^{-1}}^{\varepsilon^{-1}} M\left[\frac{t_\varepsilon 3^{\frac{1}{4}} \varepsilon^{-\frac{1}{2}}}{(1 + r^2)^{\frac{1}{2}}} \right] r^2 dr.
\end{aligned}
$$ (2.31)

Since $m(t) > 0$ for all $t > 0$, we obtain $M(t)$ is increasing for all $t > 0$. From (g_2), one has $M(t) \leq Ct^2$ for all $t > 0$ small enough. Consequently, one gets

$$
\left| \varepsilon^2 \int_{\delta\varepsilon^{-1}}^{\varepsilon^{-1}} M\left[\frac{t_\varepsilon 3^{\frac{1}{4}} \varepsilon^{-\frac{1}{2}}}{(1 + r^2)^{\frac{1}{2}}} \right] r^2 dr \right| \leq C\varepsilon^{-1} M(t_\varepsilon 3^{\frac{1}{4}} \varepsilon^{\frac{1}{2}}) \leq C\varepsilon^{-1} M(T_0 3^{\frac{1}{4}} \varepsilon^{\frac{1}{2}}) \leq C,
$$ (2.32)

for all $\varepsilon > 0$ small enough. Fixing A, there exists $B > 0$ such that $\frac{t_\varepsilon 3^{\frac{1}{4}} \varepsilon^{-\frac{1}{2}}}{(1+r^2)^{\frac{1}{2}}} \geq A$ for all $1 < r < B\varepsilon^{-\frac{1}{2}}$. Therefore, one obtains

$$
\begin{aligned}
\liminf_{\varepsilon \to 0^+} \varepsilon^2 \int_0^{\varepsilon^{-1}} M\left[\frac{t_\varepsilon 3^{\frac{1}{4}} \varepsilon^{-\frac{1}{2}}}{(1 + r^2)^{\frac{1}{2}}} \right] r^2 dr &\geq \liminf_{\varepsilon \to 0^+} \varepsilon^2 \int_1^{B\varepsilon^{-\frac{1}{2}}} M\left[\frac{t_\varepsilon 3^{\frac{1}{4}} \varepsilon^{-\frac{1}{2}}}{(1 + r^2)^{\frac{1}{2}}} \right] r^2 dr \\
&\geq \liminf_{\varepsilon \to 0^+} C\mu\varepsilon^2 \int_1^{B\varepsilon^{-\frac{1}{2}}} \frac{\varepsilon^{-2} r^2}{(1 + r^2)^2} dr \\
&= \int_1^{+\infty} \frac{\varepsilon^{-2} r^2}{(1 + r^2)^2} dr \\
&= +\infty.
\end{aligned}
$$ (2.33)

According to (2.31)–(2.33), (2.30) is obtained. Finally, we estimate the value of $I_{\varepsilon,3}$. From (2.26), since $0 < t_0 < t_\varepsilon < T_0$, one has

$$
\begin{aligned}
I_{\varepsilon,3}(t_\varepsilon) &= \frac{\lambda}{1 - \gamma} \int_\Omega \left[(t_\varepsilon u_\varepsilon + \epsilon)^{1-\gamma} - \epsilon^{1-\gamma} \right] dx \\
&\geq 0.
\end{aligned}
$$ (2.34)

Thus, from (2.29), (2.30) and (2.34), there exists a large enough positive constant $C_9 > C_8$ such that

$$
\begin{aligned}
I_\varepsilon(tu_\varepsilon) &= I_{\varepsilon,1}(t) - I_{\varepsilon,2}(t) - I_{\varepsilon,3}(t) \\
&\leq I_{\varepsilon,1}(t_\varepsilon) - I_{\varepsilon,2}(t_\varepsilon) - I_{\varepsilon,3}(t_\varepsilon) \\
&< \Theta + C_8 \varepsilon - C_9 \varepsilon \\
&< \Theta - (C_9 - C_8)\varepsilon \\
&\leq \Theta - D\lambda^{\frac{2}{1+\gamma}},
\end{aligned}
$$

provided that $\varepsilon > 0$ small enough and $0 < \lambda < \left(\frac{(C_9-C_8)\varepsilon}{D}\right)^{\frac{1+\gamma}{2}}$. Thus there exists $\lambda^* = \left(\frac{(C_9-C_8)\varepsilon}{D}\right)^{\frac{1+\gamma}{2}} > 0$, choosing $u_0 = u_\varepsilon$, such that $I_\epsilon(tu_0) < \Theta - D\lambda^{\frac{2}{1+\gamma}}$ for all $0 < \lambda < \lambda^*$. This completes the proof of Lemma 2.2. $\qquad\square$

Therefore, we can obtain the following conclusion for problem (2.1).

Theorem 2.3. *Assume that $a, b, \lambda > 0$, $0 < \gamma < 1$ and g satisfies (g_1)–(g_4), then there exists $\Lambda > 0$ such that problem (2.1) possesses two positive solutions for all $0 < \lambda < \Lambda$ and every $\epsilon > 0$. Moreover, one of the solutions is a positive ground state solution.*

Proof. We divide three steps to prove Theorem 2.3.

Step 1. We prove that there exists a positive local minimizer solution of problem (2.1).

First, we claim that there exist $\lambda_* > 0$ and $R, \rho > 0$ such that $I_\epsilon(u)|_{u \in S_R} \geq \rho$ and $\inf_{u \in B_R} I_\epsilon(u) < 0$ for $\lambda \in (0, \lambda_*)$, where $S_R = \{u \in H_0^1(\Omega) : \|u\| = R\}$, $B_R = \{u \in H_0^1(\Omega) : \|u\| \leq R\}$. In fact, by (g_1) and (g_2), we infer that

$$|G(x,s)| \leq \frac{a\lambda_1}{4}|s|^2 + C_{10}|s|^6,$$

for all $x \in \overline{\Omega}$ and $s \in \mathbb{R}$. Consequently, by the Hölder inequality and (1.7) and (2.5), we have

$$
\begin{aligned}
I_\epsilon(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{6}\int_\Omega (u^+)^6 dx - \int_\Omega G(x, u^+) dx \\
&\quad - \frac{\lambda}{1-\gamma}\int_\Omega \left[(u^+ + \epsilon)^{1-\gamma} - \epsilon^{1-\gamma}\right] dx \\
&\geq \frac{a}{4}\|u\|^2 + \frac{b}{4}\|u\|^4 - C_{11}\|u\|^6 - \frac{\lambda|\Omega|^{\frac{5+\gamma}{6}}}{(1-\gamma)S^{\frac{1-\gamma}{2}}}\|u\|^{1-\gamma} \\
&\geq \frac{a}{4}\|u\|^2 - C_{11}\|u\|^6 - \frac{\lambda|\Omega|^{\frac{5+\gamma}{6}}}{(1-\gamma)S^{\frac{1-\gamma}{2}}}\|u\|^{1-\gamma} \\
&= \frac{\|u\|^{1-\gamma}}{4}\left(a\|u\|^{1+\gamma} - C_{12}\|u\|^{5+\gamma} - B\lambda\right)
\end{aligned}
\tag{2.35}
$$

where $C_{12} = 4C_{11}$ and $B = \frac{4|\Omega|^{\frac{5+\gamma}{6}}}{(1-\gamma)S^{\frac{1-\gamma}{2}}}$. Let

$$h(t) = at^{1+\gamma} - C_{12}t^{5+\gamma} - B\lambda, \quad \forall t \in [0, +\infty).$$

Then we have $h'(t) = t^\gamma[a(1+\gamma) - C_{12}(5+\gamma)t^4]$. Let $h'(t) = 0$, one has

$$t_{\max} = \left[\frac{a(1+\gamma)}{C_{12}(5+\gamma)}\right]^{\frac{1}{4}}, \qquad \max_{t \geq 0} h(t) = h(t_{\max}) = \frac{4a}{5+\gamma}\left[\frac{a(1+\gamma)}{C_{12}(5+\gamma)}\right]^{\frac{1+\gamma}{4}} - B\lambda.$$

Therefore, choosing $\lambda_* = \frac{4a}{B(5+\gamma)}\left[\frac{a(1+\gamma)}{C_{12}(5+\gamma)}\right]^{\frac{1+\gamma}{4}}$ and $R = t_{\max}$, according to (2.35), there exists $\rho > 0$ such that $I_\epsilon(u)|_{u \in S_R} \geq \rho$ for all $0 < \lambda < \lambda_*$. By (g_2), for $u \in H_0^1(\Omega)$ with $u^+ > 0$ it holds

$$
\begin{aligned}
\lim_{t \to 0^+} \frac{I_\epsilon(tu)}{t} &= -\frac{\lambda}{1-\gamma}\lim_{t \to 0^+}\frac{1}{t}\int_\Omega [(tu^+ + \epsilon)^{1-\gamma} - \epsilon^{1-\gamma}] dx \\
&= -\frac{\lambda}{1-\gamma}\lim_{t \to 0^+}\int_\Omega \frac{(1-\gamma)\xi^{-\gamma}tu^+}{t} dx \qquad (\epsilon < \xi < tu^+ + \epsilon) \\
&= -\lambda\int_\Omega \frac{u^+}{\epsilon^\gamma} dx \qquad (\text{as } t \to 0^+, \xi \to \epsilon) \\
&< 0.
\end{aligned}
$$

Thus there exists $u \in H_0^1(\Omega)$ with $\|u\|$ small enough such that $I_\epsilon(u) < 0$. Thus, we have $\inf_{u \in B_R} I_\epsilon(u) < 0$. Therefore, our claim is true.

Denote $m_\epsilon = \inf_{u \in B_R} I_\epsilon(u)$, there exists a minimizing sequence $\{u_n\} \subset H_0^1(\Omega)$ such that $\lim_{n \to \infty} I_\epsilon(u_n) = m_\epsilon$. Applying Corollary 4.1 in [26], there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$ such that $I_\epsilon'(u_n) \to 0$ as $n \to \infty$. Choosing $\lambda_{**} = \left(\frac{\Theta}{D}\right)^{\frac{1+\gamma}{2}}$ such that $\Theta - D\lambda^{\frac{2}{1+\gamma}} > 0$ for any $0 < \lambda < \lambda_{**}$. Then, taking $\lambda^{**} = \min\{\lambda_*, \lambda_{**}\}$, for any $0 < \lambda < \lambda^{**}$, by Lemma 2.1, one has there exists $u_\epsilon \in H_0^1(\Omega)$ such that $I_\epsilon(u_\epsilon) = \lim_{n \to \infty} I_\epsilon(u_n) = m_\epsilon < 0$. Thus u_ϵ is nonzero solution of problem (2.1). Let $u_\epsilon^- = \max\{-u_\epsilon, 0\}$, by $\langle I_\epsilon'(u_\epsilon), u_\epsilon^- \rangle = 0$, one has $u_\epsilon^- = 0$. Thus, $u_\epsilon \geq 0$. By the strong maximum principle, one has $u_\epsilon > 0$ in Ω. Therefore, u_ϵ is a positive local minimizer solution of problem (2.1) for all $0 < \lambda < \lambda^{**}$.

Step 2. We prove that there exists a positive mountain-pass type solution of problem (2.1).

By (g_1) and (g_2), there exists $C_\epsilon > 0$ such that

$$|G(x, s)| \leq \varepsilon |s|^6 + C_\varepsilon.$$

Consequently, for $u \in H_0^1(\Omega)\backslash\{0\}$, one has

$$\lim_{t \to +\infty} \frac{|\int_\Omega G(x, tu)dx|}{t^6} \leq \lim_{t \to +\infty} \frac{\varepsilon t^6 \int_\Omega u^6 dx + C_\varepsilon |\Omega|}{t^6} = \varepsilon \int_\Omega u^6 dx.$$

By the arbitrary of ε, one gets

$$\lim_{t \to +\infty} \frac{|\int_\Omega G(x, tu)dx|}{t^6} = 0.$$

Thus, we have

$$\lim_{t \to +\infty} \frac{I_\epsilon(tu)}{t^6} = -\int_\Omega u^6 dx.$$

Consequently, there exists $\tilde{u} \in H_0^1(\Omega)$ such that $\|\tilde{u}\| > R$ and $I_\epsilon(\tilde{u}) < 0$. Let $0 < \lambda < \lambda_*$. According to *Step 1*, the functional I_ϵ satisfies the geometry of the mountain-pass lemma. Let c be defined by

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\epsilon(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0, \gamma(1) = \tilde{u}\}$. Obviously, one has $0 < \Theta - D\lambda^{\frac{2}{1+\gamma}}$. According to Lemma 2.1 and Lemma 2.2, there exists $\{u_n\} \subset H_0^1(\Omega)$ such that

$$I_\epsilon(u_n) \to c > \rho \quad \text{and} \quad I_\epsilon'(u_n) \to 0,$$

then $\{u_n\}$ has a convergent subsequence (still denoted by $\{u_n\}$) in $H_0^1(\Omega)$. We assume $u_n \to v_\epsilon$ in $H_0^1(\Omega)$ as $n \to \infty$. Then applying the mountain-pass lemma (see [3] Theorem 2.1), one gets $\lim_{n \to \infty} I_\epsilon(u_n) = I_\epsilon(v_\epsilon) = c > \rho > 0$ and $I_\epsilon'(v_\epsilon) = 0$. Thus, v_ϵ is a nonzero solution of problem (2.1). Similar to u_ϵ in *Step 1*, by the strong maximum principle, one has $v_\epsilon > 0$ in Ω. Therefore, v_ϵ is a positive solution of problem (2.1) with $I_\epsilon(v_\epsilon) > \rho > 0$ for all $0 < \lambda < \lambda^*$. Therefore, choosing $\Lambda = \min\{\lambda^*, \lambda^{**}\}$, u_ϵ and v_ϵ are two positive solutions of problem (2.1).

Step 3. We prove that there exists a positive ground state solution of problem (2.1).

Let $\mathcal{N} = \{u \in H_0^1(\Omega) : I_\epsilon'(u) = 0\}$ and $m_0 = \inf\limits_{u \in \mathcal{N}} I_\epsilon(u)$. For all $u \in \mathcal{N}$, by the Sobolev inequality, it follows from (2.4) and (2.5) that

$$
\begin{aligned}
I_\epsilon(u) &= I_\epsilon(u) - \frac{1}{5}\langle I_\epsilon'(u), u \rangle \\
&= \frac{3a}{10}\|u\|^2 + \frac{b}{20}\|u\|^4 + \int_\Omega \left[\frac{1}{5}g(x, u^+)u^+ - G(x, u^+)\right] dx \\
&\quad + \frac{1}{30}\int_\Omega (u^+)^6 dx - \frac{\lambda}{1-\gamma}\int_\Omega \left[(u^+ + \epsilon)^{1-\gamma} - \epsilon^{1-\gamma}\right] dx - \frac{\lambda}{5\epsilon^\gamma}\int_\Omega u^- dx \quad (2.36) \\
&\geq \frac{3a}{10}\|u\|^2 + \frac{b}{20}\|u\|^4 - \frac{\lambda}{1-\gamma}\int_\Omega |u|^{1-\gamma} dx - \frac{\lambda}{5\epsilon^\gamma}\int_\Omega |u| dx - C_0|\Omega| \\
&\geq \frac{3a}{10}\|u\|^2 + \frac{b}{20}\|u\|^4 - C_{13}\|u\|^{1-\gamma} - C_{14}\|u\| - C_0|\Omega|,
\end{aligned}
$$

since $0 < \gamma < 1$, which implies that m_0 is well defined. According to *Step 1* and *Step 2*, one has $u_\epsilon, v_\epsilon \in \mathcal{N}$. Thus $m_0 = \inf_{u \in \mathcal{N}} I_\epsilon(u) \leq I_\epsilon(u_\epsilon) < 0$. From (2.36), we can easy obtain that $m_0 > -\infty$. Therefore, for the minimization problem m_0, we can get a $(PS)_{m_0}$ sequence. By Lemma 2.1 and Lemma 2.2, there exists $u \in H_0^1(\Omega)$ such that $I_\epsilon(u) = m_0$ and $I_\epsilon'(u) = 0$. Similar to u_ϵ in *Step 1*, by the strong maximum principle, one obtains that u is a positive ground state solution of problem (2.1). This completes the proof of Theorem 2.3. $\qquad\square$

3 The proof of Theorem 1.1

According to Section 2, we know that Θ, Λ, D are independent of ϵ. Therefore, there exist two sequences of positive solutions $\{u_{\epsilon_n}\}$ and $\{v_{\epsilon_n}\}$ of the auxiliary problem (2.1) with $I_{\epsilon_n}(u_{\epsilon_n}) < 0$ and $I_{\epsilon_n}(v_{\epsilon_n}) > 0$, where $\epsilon_n \to 0^+$ as $n \to +\infty$. Now, we will prove the limits of the two sequences of positive solutions are two different positive solutions of problem (1.1). Now, we give the proof of Theorem 1.1.

Proof of Theorem 1.1. First, we prove that $\{u_{\epsilon_n}\}$ and $\{v_{\epsilon_n}\}$ have a convergent subsequence in $H_0^1(\Omega)$, respectively. Without loss of generality, we only need prove that $\{u_{\epsilon_n}\}$ has a convergent subsequence in $H_0^1(\Omega)$. Similarly, we can also obtain $\{v_{\epsilon_n}\}$ has a convergent subsequence. Since u_{ϵ_n} is the positive solution of problem (2.1), one has

$$
-(a + b\|u_{\epsilon_n}\|^2)\Delta u_{\epsilon_n} = u_{\epsilon_n}^5 + g(x, u_{\epsilon_n}) + \lambda(u_{\epsilon_n} + \epsilon_n)^{1-\gamma} \geq \min\left\{1, \frac{\lambda}{2^\gamma}\right\},
$$

which implies that

$$
-\Delta u_{\epsilon_n} \geq \frac{1}{a + b\|u_{\epsilon_n}\|^2}\min\left\{1, \frac{\lambda}{2^\gamma}\right\}.
$$

Let e be a positive weak solution of the following problem

$$
\begin{cases}
-\Delta u = 1, & x \in \Omega, \\
u = 0, & x \in \partial\Omega,
\end{cases}
$$

and for every $\Omega_0 \subset\subset \Omega$, there exists $e_0 > 0$ such that $e|_{\Omega_0} \geq e_0$. Therefore, by the comparison principle, we get

$$
u_{\epsilon_n} \geq \frac{\min\{1, \frac{\lambda}{2^\gamma}\}}{a + b\|u_{\epsilon_n}\|^2}e.
$$

In particular, from $e|_{\Omega_0} \geq e_0 > 0$, one has

$$u_{\epsilon_n}|_{\Omega_0} \geq \frac{\min\left\{1, \frac{\lambda}{2\gamma}\right\}}{a + b\|u_{\epsilon_n}\|^2} e_0 > 0. \tag{3.1}$$

Similar to (2.6), we can easy obtain that

$$
\begin{aligned}
\Theta - D\lambda^{\frac{2}{1+\gamma}} &\geq I_{\epsilon_n}(u_{\epsilon_n}) - \frac{1}{5}\langle I'_{\epsilon_n}(u_{\epsilon_n}), u_{\epsilon_n}\rangle \\
&\geq \frac{3a}{10}\|u_{\epsilon_n}\|^2 + \frac{b}{20}\|u_{\epsilon_n}\|^4 - \frac{\lambda}{1-\gamma}\int_{\Omega} u_{\epsilon_n}^{1-\gamma}dx - C_0|\Omega| \\
&\geq \frac{3a}{10}\|u_{\epsilon_n}\|^2 + \frac{b}{20}\|u_{\epsilon_n}\|^4 - C\|u_{\epsilon_n}\|^{1-\gamma} - C_0|\Omega|,
\end{aligned}
$$

which implies that $\{u_{\epsilon_n}\}$ is bounded in $H_0^1(\Omega)$. Up to a subsequence, combining with (3.1), there exists $u_* \in H_0^1(\Omega)$ with $u_* > 0$ such that

$$
\begin{cases}
u_{\epsilon_n} \rightharpoonup u_* \text{ weakly in } H_0^1(\Omega), \\
u_{\epsilon_n} \to u_* \text{ strongly in } L^q(\Omega)(1 \leq q < 6), \\
u_{\epsilon_n}(x) \to u_*(x) \quad \text{a.e. in } \Omega.
\end{cases}
\tag{3.2}
$$

Now, we shall prove that $u_{\epsilon_n} \to u_*$ in $H_0^1(\Omega)$ as $\epsilon_n \to 0$. By (g_2) and (3.2), one has

$$\int_{\Omega} g(x, u_{\epsilon_n})u_* dx = \int_{\Omega} g(x, u_*)u_* dx + o(1), \tag{3.3}$$

$$\int_{\Omega} g(x, u_{\epsilon_n})u_{\epsilon_n} dx = \int_{\Omega} g(x, u_*)u_* dx + o(1), \tag{3.4}$$

$$\int_{\Omega} G(x, u_{\epsilon_n})dx = \int_{\Omega} G(x, u_*)dx + o(1). \tag{3.5}$$

As usually, letting $w_{\epsilon_n} = u_{\epsilon_n} - u_*$, we need prove that $\|w_{\epsilon_n}\| \to 0$ as $n \to \infty$. Let $\lim_{n\to\infty}\|w_{\epsilon_n}\| = l \geq 0$. For every $\epsilon_n > 0$, since

$$\frac{u_{\epsilon_n}}{(u_{\epsilon_n} + \epsilon_n)^{\gamma}} \leq u_{\epsilon_n}^{1-\gamma},$$

and similar to (2.3) in [22], one has

$$\int_{\Omega} u_{\epsilon_n}^{1-\gamma}dx = \int_{\Omega} u_*^{1-\gamma}dx + o(1),$$

consequently, by the dominated convergence theorem, one has

$$\lim_{n\to\infty}\int_{\Omega}(u_{\epsilon_n} + \epsilon_n)^{-\gamma}u_{\epsilon_n}dx = \int_{\Omega} u_*^{1-\gamma}dx. \tag{3.6}$$

Since u_{ϵ_n} is a positive solution of problem (2.1) with $\epsilon = \epsilon_n$, it follows from (2.2) that

$$(a + b\|u_{\epsilon_n}\|^2)\int_{\Omega}(\nabla u_{\epsilon_n}, \nabla\varphi)dx - \int_{\Omega} u_{\epsilon_n}^5\varphi dx - \int_{\Omega} g(x, u_{\epsilon_n})\varphi dx - \lambda\int_{\Omega}\frac{\varphi}{(u_{\epsilon_n} + \epsilon_n)^{\gamma}}dx = 0, \tag{3.7}$$

for all $\varphi \in H_0^1(\Omega)$. For any $\phi \in H_0^1(\Omega) \cap C_0(\Omega)$, where $C_0(\Omega)$ is the subset of $C(\Omega)$ consisting of functions with compact support in Ω, by the dominate convergence theorem, it follows from (3.1) with $\Omega_0 = \text{supp }\phi$ that

$$\lim_{n\to\infty}\int_{\Omega}\frac{\phi}{(u_{\epsilon_n} + \epsilon_n)^{\gamma}}dx = \int_{\Omega} u_*^{-\gamma}\phi dx.$$

Consequently, taking the test function $\varphi = \phi \in H_0^1(\Omega) \cap C_0(\Omega)$ in (3.7), and let $n \to \infty$, we can obtain

$$(a + bl^2 + b\|u_*\|^2)\int_\Omega (\nabla u_*, \nabla \phi)dx = \int_\Omega u_*^5 \phi dx + \int_\Omega g(x, u_*)\phi dx + \lambda \int_\Omega u_*^{-\gamma}\phi dx. \qquad (3.8)$$

Similar to prove (4.3) in [16] holds for any $\phi \in H_0^1(\Omega)$, we can prove that (3.8) holds for any $\phi \in H_0^1(\Omega)$ by the same way. Choosing $\phi = u_*$ in (3.8), one has

$$a\|u_*\|^2 + b\|u_*\|^4 + bl^2\|u_*\|^2 - \int_\Omega u_*^6 dx - \int_\Omega g(x, u_*)u_* dx - \lambda \int_\Omega u_*^{1-\gamma}dx = 0. \qquad (3.9)$$

Choosing $\varphi = u_{\epsilon_n}$ in (3.7), let $n \to \infty$, by the Brézis–Lieb Lemma, it follows from (3.2), (3.4) and (3.6) that

$$a\|u_*\|^2 + a\|w_{\epsilon_n}\|^2 + b\|u_*\|^4 + b\|w_{\epsilon_n}\|^4 + 2b\|w_{\epsilon_n}\|^2\|u\|^2$$
$$- \int_\Omega w_{\epsilon_n}^6 dx - \int_\Omega u_*^6 dx - \int_\Omega g(x, u_*)u_* dx - \lambda \int_\Omega u_*^{1-\gamma}dx = o(1). \qquad (3.10)$$

On the one hand, from (2.5), (3.9) and (g_3), by the Hölder inequality, Sobolev inequality and Poincaré inequality, we have

$$\begin{aligned}
I(u_*) &= \frac{a}{2}\|u_*\|^2 + \frac{b}{4}\|u_*\|^4 - \frac{1}{6}\int_\Omega u_*^6 dx - \int_\Omega G(x, u_*)dx - \frac{\lambda}{1-\gamma}\int_\Omega u_*^{1-\gamma}dx \\
&= \frac{a}{4}\|u_*\|^2 + \frac{1}{12}\int_\Omega u_*^6 dx + \int_\Omega \left[\frac{1}{4}g(x, u_*)u - G(x, u_*)\right]dx \\
&\quad - \frac{\lambda}{1-\gamma}\int_\Omega u_*^{1-\gamma}dx - \frac{bl^2}{4}\|u_*\|^2 \\
&\geq \frac{a}{4}\|u_*\|^2 - \frac{\lambda_1(a - \kappa)}{4}\int_\Omega u_*^2 dx - \frac{\lambda}{1-\gamma}\int_\Omega u_*^{1-\gamma}dx - \frac{bl^2}{4}\|u_*\|^2 \\
&\geq \frac{\kappa}{4}\|u_*\|^2 - \frac{\lambda}{1-\gamma}|\Omega|^{\frac{5+\gamma}{6}}S^{-\frac{1-\gamma}{2}}\|u_*\|^{1-\gamma} - \frac{bl^2}{4}\|u_*\|^2 \\
&\geq -D\lambda^{\frac{2}{1+\gamma}} - \frac{bl^2}{4}\|u_*\|^2,
\end{aligned} \qquad (3.11)$$

where the last inequality is obtained by the Young inequality. On the other hand, it follows from (3.5), (3.9) and (3.10) that

$$a\|w_{\epsilon_n}\|^2 + b\|w_{\epsilon_n}\|^4 + b\|w_{\epsilon_n}\|^2\|u_*\|^2 - \int_\Omega w_{\epsilon_n}^6 dx = o(1), \qquad (3.12)$$

and

$$I_{\epsilon_n}(u_{\epsilon_n}) = I(u_*) + \frac{a}{2}\|w_{\epsilon_n}\|^2 + \frac{b}{4}\|w_{\epsilon_n}\|^4 + \frac{b}{2}\|w_{\epsilon_n}\|^2\|u_*\|^2 - \frac{1}{6}\int_\Omega w_{\epsilon_n}^6 dx + o(1). \qquad (3.13)$$

From (1.7), one has

$$\int_\Omega w_{\epsilon_n}^6 dx \leq \frac{\|w_{\epsilon_n}\|^6}{S^3},$$

consequently, it follows from (3.12) that

$$al^2 + bl^4 + bl^2\|u_*\|^2 \leq \frac{l^6}{S^3},$$

which implies that

$$l^2 \geq \frac{bS^3 + \sqrt{b^2S^6 + 4S^3(a + b\|u_*\|^2)}}{2}. \tag{3.14}$$

Since $I_{\epsilon_n}(u_{\epsilon_n}) < \Theta - D\lambda^{\frac{2}{1+\gamma}}$, from (3.12)–(3.14), we obtain

$$\begin{aligned}
I(u_*) &< \lim_{n\to\infty}\left[\Theta - D\lambda^{\frac{2}{1+\gamma}} - \frac{a}{2}\|w_{\epsilon_n}\|^2 - \frac{b}{4}\|w_{\epsilon_n}\|^4 - \frac{b}{2}\|w_{\epsilon_n}\|^2\|u_*\|^2 + \frac{1}{6}\int_\Omega w_{\epsilon_n}^6 dx\right] \\
&= \Theta - D\lambda^{\frac{2}{1+\gamma}} - \left(\frac{a}{3}l^2 + \frac{b}{12}l^4 + \frac{b}{3}l^2\|u_*\|^2\right) \\
&\leq \Theta - D\lambda^{\frac{2}{1+\gamma}} - \left[\frac{a\left(bS^3 + \sqrt{b^2S^6 + 4S^3(a+b\|u_*\|^2)}\right)}{6}\right. \\
&\quad + \frac{b}{48}\left(bS^3 + \sqrt{b^2S^6 + 4S^3(a+b\|u_*\|^2)}\right)^2 \\
&\quad \left. + \frac{b\|u_*\|^2}{6}\left(bS^3 + \sqrt{b^2S^6 + 4S^3(a+b\|u_*\|^2)}\right)\right] - \frac{bl^2}{4}\|u_*\|^2 \\
&\leq \Theta - D\lambda^{\frac{2}{1+\gamma}} - \left(\frac{abS^3}{4} + \frac{b^3S^6}{24} + \frac{aS\sqrt{b^2S^4 + 4aS}}{6} + \frac{b^2S^4\sqrt{b^2S^4 + 4aS}}{24}\right) - \frac{bl^2}{4}\|u_*\|^2 \\
&< -D\lambda^{\frac{2}{1+\gamma}} - \frac{bl^2}{4}\|u_*\|^2,
\end{aligned}$$

which contradicts (3.11). Hence, $l \equiv 0$, that is, $u_{\epsilon_n} \to u_*$ in $H_0^1(\Omega)$ as $n \to \infty$. Moreover, since (3.8) holds any $\phi \in H_0^1(\Omega)$, we get that u_* is a positive solution of problem (1.1). Similarly, for $\{v_{\epsilon_n}\}$, up to a subsequence, there exists $v_* \in H_0^1(\Omega)$ with $v_* > 0$ such that $v_{\epsilon_n} \to v_*$ in $H_0^1(\Omega)$ and v_* is a positive solution of problem (1.1).

Second, we prove that u_* and v_* are two different positive solutions of problem (1.1). According to Theorem 2.3, one has $I(u_*) = \lim_{n\to\infty} I_{\epsilon_n}(u_{\epsilon_n}) = \lim_{n\to\infty} m_{\epsilon_n} \leq 0$ and $I(v_*) = \lim_{n\to\infty} I_{\epsilon_n}(v_{\epsilon_n}) > \rho > 0$. Therefore, u_* and v_* are two different positive solutions of problem (1.1). This completes the proof of Theorem 1.1. $\square$

Acknowledgements

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