Electronic Journal of Qualitative Theory of Differential Equations
2019, No. 5, 1–16; https://doi.org/10.14232/ejqtde.2019.1.5 www.math.u-szeged.hu/ejqtde/

Infinitely many solutions to quasilinear Schrödinger equations with critical exponent

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Received 2 July 2018, appeared 17 January 2019
Communicated by Dimitri Mugnai

Abstract. This paper is concerned with the following quasilinear Schrödinger equations with critical exponent:

$$-\Delta_p u + V(x)|u|^{p-2}u - \Delta_p(|u|^{2\omega})|u|^{2\omega-2}u = ak(x)|u|^{q-2}u + b|u|^{2\omega p^*-2}u, \qquad x \in \mathbb{R}^N.$$

Here $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator with $1 < p < N$, $p^* = \frac{Np}{N-p}$ is the critical Sobolev exponent. $1 \le 2\omega < q < 2\omega p$, a and b are suitable positive parameters, $V \in C(\mathbb{R}^N, [0,\infty))$, $k \in C(\mathbb{R}^N, \mathbb{R})$. With the help of the concentration-compactness principle and R. Kajikiya's new version of symmetric Mountain Pass Lemma, we obtain infinitely many solutions which tend to zero under mild assumptions on V and k.

Keywords: critical exponent, concentration-compactness principle, symmetric Mountain Pass Theorem.

2010 Mathematics Subject Classification: 35A15, 35J10, 35J62.

1 Introduction and main result

In this paper, we establish the existence of infinitely many solutions which tend to zero for the following quasilinear Schrödinger equations with critical exponent

$$-\Delta_p u + V(x)|u|^{p-2}u - \Delta_p(|u|^{2\omega})|u|^{2\omega-2}u = ak(x)|u|^{q-2}u + b|u|^{2\omega p^*-2}u, \qquad x \in \mathbb{R}^N. \tag{1.1}$$

The energy functional associated with (1.1) is given by

$$\begin{aligned} I(u) = {} & \frac{1}{p}\int_{\mathbb{R}^N}\Big(|\nabla u|^p + V(x)|u|^p\Big)\mathrm{d}x + \frac{(2\omega)^{p-1}}{p}\int_{\mathbb{R}^N}|u|^{p(2\omega-1)}|\nabla u|^p\mathrm{d}x \\ & - \frac{a}{q}\int_{\mathbb{R}^N}k(x)|u|^q\mathrm{d}x - \frac{b}{2\omega p^*}\int_{\mathbb{R}^N}|u|^{2\omega p^*}\mathrm{d}x. \end{aligned} \tag{1.2}$$

Here $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator with $1 < p < N$, $p^* = \frac{Np}{N-p}$ is the critical Sobolev exponent. $1 \le 2\omega < q < 2\omega p$, a and b are positive parameters. $V(x)$ and $k(x)$ are continuous and satisfy the following conditions:

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($\mathbf{V}$) $V \in C(\mathbb{R}^N, [0, \infty))$ satisfies $\inf_{x \in \mathbb{R}^N} V(x) \geq V_0 > 0$, and for each $M > 0$, $\mathrm{meas}\{x \in \mathbb{R}^N : V(x) \leq M\} < +\infty$, where V_0 is a constant and meas denotes the Lebesgue measure in $\mathbb{R}^N$.

($\mathbf{K}$) $0 < k(x) \in L^r(\mathbb{R}^N)$ with $r = \frac{2\omega p^*}{2\omega p^* - q}$.

In recent years, a great attention has been focused on the study of solutions to quasilinear Schrödinger equations. Such equations arise in various branches of mathematical physics. For example, when $p = 2, \omega = 1$, the solutions of (1.1) are related to the existence of solitary wave solutions for quasilinear Schrödinger equations

$$ih\frac{\partial \Psi}{\partial t} = -\Delta \Psi + W(x)\Psi - \tilde{h}(|\Psi|^2)\Psi - \kappa \Delta[\rho(|\Psi|^2)]\rho'(|\Psi|^2)\Psi, \tag{1.3}$$

where $\Psi : \mathbb{R} \times \mathbb{R}^N \to C, W : \mathbb{R}^N \to \mathbb{R}$ is a given potential, κ, h are real constants and $\rho, \tilde{h}$ are real functions. This type of equations appear more naturally in mathematical physics and have been derived as models of several physical phenomena corresponding to various types of $\rho(s)$. In the case $\rho(s) = s$, (1.3) was used for the superfluid film equation in plasma physics by Kurihara in [12] and [13]. In the case $\rho(s) = (1+s)^{1/2}$, (1.3) models the self-channeling of a high-power ultrashort laser in matter (see [4,6]). Considering the case $\rho(s) = s^\alpha, \kappa > 0$ and putting $\Psi(t,x) = \exp(-\frac{iFt}{h})u(x), F \in \mathbb{R}$ is some real constant, it is clear that $\Psi(t,x)$ solves (1.3) if and only if $u(x)$ solves the following elliptic equation:

$$-\Delta u + V(x)u - \alpha\kappa\Delta(|u|^{2\alpha})|u|^{2\alpha-2}u = \tilde{h}(x,u), \qquad x \in \mathbb{R}^N, \tag{1.4}$$

where we have renamed $W(x) - F$ to be $V(x)$.

For the case $\alpha\kappa = 1, \tilde{h}(x,u) = \theta|u|^{p-1}u$, Poppenberg, Schmitt and Wang in [19] studied the equation (1.4) by translating it into an ODE

$$-u'' + V(x)u - (u^2)''u = \theta|u|^{p-1}u, \qquad x \in \mathbb{R}, \tag{1.5}$$

and then a ground state solution $u \in W^{1,2}(\mathbb{R})$ of problem (1.5) was obtained. They also got that the equation (1.5) admits a positive solution $u \in W^{1,2}(\mathbb{R})$ for any arbitrarily large values of θ. Later, Liu, Wang and Wang in [17] established the existence of ground states of soliton-type solutions for (1.4) as in the case $\alpha = 1, \kappa = \frac{1}{2}$ by the variational methods. Using a constrained minimization argument, Liu, Wang and Wang in [16] established the existence of a positive ground state solution for (1.4). As we know, Nehari method is used to get the existence results of ground state solutions in [10] and the problem is transformed to a semilinear one in [2,9] by a change of variables. Recently, the author in [23] studied the equation (1.4) and obtained that it has a positive and a negative weak solution under proper conditions of α, V, g. A natural question is that weather there exist infinitely many solutions for equations like (1.4). The authors in [7,8] investigated the following type quasilinear elliptic equation:

$$-\Delta u + V(x)u - \Delta I(u^2)I'(u^2)u = h(u), \qquad x \in \mathbb{R}^N. \tag{1.6}$$

Let

$$g^2(u) = 1 + \frac{[(I(u^2))']^2}{2}.$$

Problem (1.6) can be reduced to the following quasilinear elliptic equations:

$$-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = h(u), \qquad x \in \mathbb{R}^N. \tag{1.7}$$

By using the Pohozaev identity, the author has the nonexistence result for (1.7).

To the best of our knowledge, the existence of nontrivial radial solutions for (1.4) with $g(x,u) = \mu u^{2(2^*)-1}$ was firstly studied by Moameni in [18], where the Orlicz space as the same as it was used in [17]. However, it seems that there is almost no work on the existence of infinitely many solutions to the quasilinear Schrödinger problem in $\mathbb{R}^N$ involving critical nonlinearities and generalized potential $V(x)$.

Motivated by the above discussions, the main goal of this paper is to study the existence of infinitely many solutions which tend to zero to the problem (1.1). The lack of compactness of the embedding from $W^{1,p}(\mathbb{R}^N)$ into $L^{p^*}(\mathbb{R}^N)$ prevents us from using the variational methods in a standard way. To overcome the lack of compactness caused by the Sobolev embeddings in unbounded domains and the critical exponent, some new estimates for (1.1) are needed to be re-established. We apply Lions' concentration-compactness principle [14, 15] to give a more detailed analysis for the compactness of our problem. Thanks to the new version of symmetric Mountain Pass Lemma in [11], we give the proof of our main result. As far as we know, there are few results on this question, so the research in this paper is meaningful.

Now we first give the definition of weak solutions for problem (1.1).

Definition 1.1. We say that $u \in W^{1,p}(\mathbb{R}^N) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ is a weak solution of (1.1), if

$$\begin{aligned}\int_{\mathbb{R}^N} &\left(|\nabla u|^{p-2}\nabla u \nabla\varphi + V(x)|u|^{p-2}u\varphi\right)\mathrm{d}x \\ &+ (2\omega)^{p-1}\int_{\mathbb{R}^N}|u|^{p(2\omega-1)}|\nabla u|^{p-2}\nabla u\nabla\varphi \mathrm{d}x + (2\omega)^{p-1}\int_{\mathbb{R}^N}|\nabla u|^p|u|^{p(2\omega-1)-2}u\varphi\mathrm{d}x \\ &- a\int_{\mathbb{R}^N}k(x)|u|^{q-2}u\varphi\mathrm{d}x - b\int_{\mathbb{R}^N}|u|^{2\omega p^*-2}u\varphi\mathrm{d}x = 0\end{aligned}$$

for any $\varphi \in C_0^\infty(\mathbb{R}^N)$.

In the sequel we will omit the term weak when referring to solutions that satisfy the conditions of Definition 1.1. Our main result of this paper is stated as follows.

Theorem 1.2. *Suppose that* $(\mathbf{V})$ *and* $(\mathbf{K})$ *hold,* $1 \le 2\omega < q < 2\omega p$. *Then*

(i) $\forall b > 0, \exists\, a_0 > 0$ *such that if* $0 < a < a_0$, *problem* (1.1) *has a sequence of solutions* $\{u_n\}$ *with* $I(u_n) < 0$, $I(u_n) \to 0$ *and* $\lim_{n\to\infty} u_n = 0$.

(ii) $\forall a > 0, \exists\, b_0 > 0$ *such that if* $0 < b < b_0$, *problem* (1.1) *has a sequence of solutions* $\{u_n\}$ *with* $I(u_n) < 0$, $I(u_n) \to 0$ *and* $\lim_{n\to\infty} u_n = 0$.

Remark 1.3. From Theorem 1.2 it is natural to raise the open problems: What if $2\omega p < q < p^*$? This problem would be investigated by the authors in future works.

The outline of this paper is as follows. Reformulation of the problem and some preliminaries are given in the forthcoming section. In Section 3, behavior of (PS) sequences are established. The proof of Theorem 1.2 is given in Section 4.

We denote that $L^p(\mathbb{R}^N)$ is the usual Lebesgue space with the norm $\|u\|_p^p = \int_{\mathbb{R}^N}|u|^p\mathrm{d}x, 1 \le p < +\infty$. $\|u\|^p = \int_{\mathbb{R}^N}|\nabla u|^p\mathrm{d}x$, $\|u\|_V^p = \int_{\mathbb{R}^N}(|\nabla u|^p + V(x)|u|^p)\mathrm{d}x$. $S = \inf_{u\in W^{1,p}(\mathbb{R}^N)\setminus\{0\}}\frac{\|u\|^p}{\|u\|_{p^*}^p}$ is the best Sobolev constant. Various positive constants are denoted by C and C_i.

2 Reformulation of the problem and preliminaries

The purpose of this section is to establish the variational structure of (1.1) and the main difficulty arises from the function space where the energy functional (1.2) is not well defined in $W^{1,p}(\mathbb{R}^N)$. For example, if $1 < p < N$ and u is defined by

$$u(x) = |x|^{(p-N)/2\omega p} \quad \text{for } x \in B_1 \setminus \{0\},$$

we then have that $u \in W^{1,p}(\mathbb{R}^N)$, but

$$\int_{\mathbb{R}^N} |u|^{p(2\omega-1)} |\nabla u|^p \mathrm{d}x = +\infty.$$

To overcome this difficulty, we employ an argument developed by Liu, Wang and Wang in [17] or Colin and Jeanjean in [5]. We use the change of variables $v = f^{-1}(u)$, where f is defined by

$$f'(t) = \frac{1}{[1 + (2\omega)^{p-1}|f(t)|^{p(2\omega-1)}]^{\frac{1}{p}}},$$

and $f(0) = 0$ on $[0, +\infty)$ and by $f(t) = -f(-t)$ on $(-\infty, 0]$. The following result is due to Adachi and Watanabe in [1] which collects some properties of f.

Lemma 2.1. *The function $f(t)$ enjoys the following properties:*

(1) *f is uniquely defined C^∞ function and invertible.*

(2) *$|f'(t)| \leq 1$, $|f(t)| \leq (2\omega)^{\frac{1}{2\omega p}} |t|^{\frac{1}{2\omega}}$ for all $t \in \mathbb{R}$.*

(3) *$\frac{f(t)}{t} \to 1$ as $t \to 0$.*

(4) *$\frac{f(t)}{t^{\frac{1}{2\omega}}} \to a > 0$ as $t \to +\infty$.*

(5) *$\frac{1}{2\omega} f(t) \leq t f'(t) \leq f(t)$ for all $t \geq 0$.*

(6) *There exists a positive constant C such that*

$$|f(t)| \geq \begin{cases} C|t|, & |t| \leq 1, \\ C|t|^{\frac{1}{2\omega}}, & |t| > 1. \end{cases}$$

After the above change of variables, we can rewrite our energy functional (1.2) in the terms of v:

$$J(v) = \frac{1}{p} \int_{\mathbb{R}^N} \Big(|\nabla v|^p + V(x)|f(v)|^p \Big) \mathrm{d}x - \frac{a}{q} \int_{\mathbb{R}^N} k(x)|f(v)|^q \mathrm{d}x - \frac{b}{2\omega p^*} \int_{\mathbb{R}^N} |f(v)|^{2\omega p^*} \mathrm{d}x.$$

We first give the proof of the following weakly continuous lemma.

Lemma 2.2.

(i) *The functional $\mathcal{F}(v) = \int_{\mathbb{R}^N} k(x)|f(v)|^q \mathrm{d}x$ is well defined and weakly continuous on $W^{1,p}(\mathbb{R}^N)$. Moreover, $\mathcal{F}(v)$ is continuously differentiable, its derivative $\mathcal{F}' : W^{1,p}(\mathbb{R}^N) \to (W^{1,p}(\mathbb{R}^N))^*$ is given by*

$$\langle \mathcal{F}'(v), g \rangle = q \int_{\mathbb{R}^N} k(x)|f(v)|^{q-2} f(v) f'(v) g \mathrm{d}x, \qquad \forall\, g \in W^{1,p}(\mathbb{R}^N).$$

(*ii*) *The functional* $\mathcal{G}(v) = \int_{\mathbb{R}^N} f(v)^{2\omega p^*} \mathrm{d}x$ *is well defined. Moreover,* $\mathcal{G}(v)$ *is continuously differentiable, its derivative* $\mathcal{G}' : W^{1,p}(\mathbb{R}^N) \to (W^{1,p}(\mathbb{R}^N))^*$ *is given by*

$$\langle \mathcal{G}'(v), g\rangle = 2\omega p^* \int_{\mathbb{R}^N} |f(v)|^{2\omega p^*-2} f(v) f'(v) g \mathrm{d}x, \qquad \forall\, g \in W^{1,p}(\mathbb{R}^N).$$

Proof. Firstly, by (3) and (4) in Lemma 2.1, it is clear that $\mathcal{F}(v)$ and $\mathcal{G}(v)$ are well defined on $W^{1,p}(\mathbb{R}^N)$. Next, we prove that $\mathcal{F}(v), \mathcal{G}(v) \in C^1(\mathbb{R}^N)$. It suffices to show that both $\mathcal{F}(v)$ and $\mathcal{G}(v)$ have continuous Gateaux derivatives on $W^{1,p}(\mathbb{R}^N)$. We only prove that $\mathcal{F}(v)$ has continuous Gateaux derivatives on $W^{1,p}(\mathbb{R}^N)$ since the case of the proof for $\mathcal{G}(v)$ is simpler. Our proof is the same as the proof of Lemma 3.10 in [22] , for the convenience of the readers, we present the process. Let $v, g \in W^{1,p}(\mathbb{R}^N)$. Given $0 < |t| < 1$, by the mean value theorem, there exists $\lambda \in (0,1)$ such that

$$\begin{aligned}
\frac{||f(v+tg)|^q - |f(v)|^q|}{|t|} &= q|f(v+t\lambda g)|^{q-1}|f'(v+t\lambda g)||g| \\
&= q|f(v+t\lambda g)|^q \frac{|f'(v+t\lambda g)|}{|f(v+t\lambda g)|}|g| \\
&\le C|v+t\lambda g|^{\frac{q}{2\omega}}|v+t\lambda g|^{-1}|g| \\
&= C|v+t\lambda g|^{\frac{q-2\omega}{2\omega}}|g| \\
&\le C(|v|^{\frac{q-2\omega}{2\omega}}|g| + |g|^{\frac{q}{2\omega}}),
\end{aligned}$$

where the conclusions of Lemma 2.1 (2) and (5) are used. By the Hölder inequality and assumption of $(\mathbf{K})$, we have

$$\int_{\mathbb{R}^N} k(x)(|v|^{\frac{q-2\omega}{2\omega}}|g| + |g|^{\frac{q}{2\omega}})\mathrm{d}x \le \|k(x)\|_r \|g\|_{p^*}(\|v\|^{\frac{q-2\omega}{2\omega}} + \|g\|^{\frac{q-2\omega}{2\omega}}).$$

It follows from the Lebesgue Dominated Convergence Theorem that $\mathcal{F}(v)$ is Gateaux differentiable and

$$\langle \mathcal{F}'(v), g\rangle = q\int_{\mathbb{R}^N} k(x)|f(v)|^{q-2}f(v)f'(v)g\mathrm{d}x.$$

Now, we give the proof of continuity of Gateaux derivative. Assume that $v_n \to v$ in $W^{1,p}(\mathbb{R}^N)$, then $f^2(v_n) \to f^2(v)$ in $W^{1,p}(\mathbb{R}^N)$. By the continuity of the embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$, we get that $f^2(v_n) \to f^2(v)$ in $L^{p^*}(\mathbb{R}^N)$. Define $\mathcal{K}(v) = k(x)|f(v)|^{q-2}f(v)f'(v)$. Then $\mathcal{K} \in (L^{p^*}(\mathbb{R}^N), C(L^{p^*}(\mathbb{R}^N))')$. It follows that $\mathcal{K}(v_n) \to \mathcal{K}(v)$ in $(L^{p^*}(\mathbb{R}^N))'$. Using the Hölder and Sobolev inequalities, we have

$$\langle \mathcal{F}'(v_n) - \mathcal{F}'(v), g\rangle \le \|\mathcal{K}(v_n) - \mathcal{K}(v)\|_{(p^*)'}\|g\|_{p^*} \le C\|\mathcal{K}(v_n) - \mathcal{K}(v)\|_{(p^*)'}\|g\|.$$

Hence $\|\mathcal{F}'(v_n) - \mathcal{F}'(v)\| \to 0$ and $\mathcal{F} \in C^1$. □

From the above analysis we can get that $J(v)$ is well defined on $W^{1,p}(\mathbb{R}^N)$ under the assumptions of $(\mathbf{V})$ and $(\mathbf{K})$. The standard arguments applied in [20,22] show that $J(v)$ belongs to $C^1(W^{1,p}(\mathbb{R}^N), \mathbb{R})$. As in [5], we note that if v is a nontrivial critical point of J, v then is a nontrivial solution of the problem

$$-\Delta_p v + V(x)|f(v)|^{p-2}f(v)f'(v) = ak(x)|f(v)|^{q-2}f(v)f'(v) + b|f(v)|^{2\omega p^*-2}f(v)f'(v). \quad (2.1)$$

Therefore, let $u = f(v)$ and since $(f^{-1})'(t) = [1 + (2\omega)^{p-1}|f(t)|^{p(2\omega-1)}]^{\frac{1}{p}}$, we conclude that u is a nontrivial solution of the problem (1.1).

Now we can restate Theorem 1.2 as follows.

Theorem 2.3. *Suppose that* $(\mathbf{V})$ *and* $(\mathbf{K})$ *are held,* $\omega > 1/2, 2\omega < q < 2\omega p$. *Then*

(i) $\forall b > 0, \exists\, \widetilde{a_0} > 0$ *such that if* $0 < a < \widetilde{a_0}$, *problem* (2.1) *has a sequence of solutions* $\{v_n\}$ *with* $J(v_n) < 0, J(v_n) \to 0$ *and* $\lim_{n\to\infty} v_n = 0$.

(ii) $\forall a > 0, \exists\, \widetilde{b_0} > 0$ *such that if* $0 < b < \widetilde{b_0}$, *problem* (2.1) *has a sequence of solutions* $\{v_n\}$ *with* $J(v_n) < 0, J(v_n) \to 0$ *and* $\lim_{n\to\infty} v_n = 0$.

3 Properties of $(PS)_c$ sequences

In this section, we perform a careful analysis of the behavior of minimizing sequences with the aid of Lions' concentration–compactness principle [14,15], which allows us to recover the compactness below some critical threshold.

Let E be a real Banach space and $J : E \to \mathbb{R}$ be a function of class C^1. We say that $\{v_n\} \subset E$ is a $(PS)_c$ sequence if $J(v_n) \to c$ and $J'(v_n) \to 0$. J is said to satisfy the Palais–Smale condition at level c ($(PS)_c$ for short) if any $(PS)_c$ sequence contains a convergent subsequence.

Lemma 3.1. *Assume* $(\mathbf{V})$ *and* $(\mathbf{K})$, $\{v_n\} \subset W^{1,p}(\mathbb{R}^N)$ *be a* $(PS)_c$ *sequence for* J *at level* $c < 0$ *and* $2\omega < q < 2\omega p$. *Then*

(i) *there exists* $C > 0$ *such that, for all* $n \in \mathbb{N}$, $\|v_n\|_V \le C$;

(ii) $\forall b > 0, \exists\, a_* > 0$ *such that if* $0 < a < a_*$, *then* J *satisfies* $(PS)_c$;

(iii) $\forall a > 0, \exists\, b_* > 0$ *such that if* $0 < b < b_*$, *then* J *satisfies* $(PS)_c$.

Proof. At first, we prove that $\{v_n\}$ is bounded in $W^{1,p}(\mathbb{R}^N)$. Let $\{v_n\}$ be a $(PS)_c$ sequence in $W^{1,p}(\mathbb{R}^N)$ such that for all $\phi \in C_0^\infty(\mathbb{R}^N)$, we have that

$$\begin{aligned} c + o_n(\|v_n\|) = J(v_n) = {} & \frac{1}{p}\int_{\mathbb{R}^N}\Big(|\nabla v_n|^p + V(x)|f(v_n)|^p\Big)\mathrm{d}x \\ & - \frac{a}{q}\int_{\mathbb{R}^N} k(x)|f(v_n)|^q \mathrm{d}x - \frac{b}{2\omega p^*}\int_{\mathbb{R}^N}|f(v_n)|^{2\omega p^*}\mathrm{d}x, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} o_n(\|v_n\|) = \langle J'(v_n), \varphi\rangle = {} & \int_{\mathbb{R}^N}\Big(|\nabla v_n|^{p-2}\nabla v_n \nabla\varphi + V(x)|f(v_n)|^{p-2}f(v_n)f'(v_n)\varphi\Big)\mathrm{d}x \\ & - a\int_{\mathbb{R}^N} k(x)|f(v_n)|^{q-2}f(v_n)f'(v_n)\varphi \mathrm{d}x \\ & - b\int_{\mathbb{R}^N}|f(v_n)|^{2\omega p^*-2}f(v_n)f'(v_n)\varphi \mathrm{d}x. \end{aligned} \tag{3.2}$$

Choose $\varphi = \varphi_n = [1 + (2\omega)^{p-1}|f(v_n)|^{p(2\omega-1)}]^{\frac{1}{p}} f(v_n)$, we have $\varphi_n \in W^{1,p}(\mathbb{R}^N)$ and then $|\varphi_n| < 2\omega|v_n|$. Since

$$\nabla\varphi_n = \left[1 + \frac{(2\omega-1)(2\omega)^{p-1}|f(v_n)|^{p(\omega-1)}}{1 + (2\omega)^{p-1}|f(v_n)|^{p(\omega-1)}}\right]\nabla v_n \le (2\omega)\nabla v_n, \qquad f'(v_n)\varphi_n = f(v_n),$$

we get $\|\varphi_n\| \le C\|v_n\|$. It follows from (3.2) that

$$\begin{aligned} o_n(\|v_n\|) = \langle J'(v_n), \varphi_n\rangle \le {} & \int_{\mathbb{R}^N}(2\omega|\nabla v_n|^p + V(x)|f(v_n)|^p)\mathrm{d}x \\ & - a\int_{\mathbb{R}^N} k(x)|f(v_n)|^q \mathrm{d}x - b\int_{\mathbb{R}^N}|f(v_n)|^{2\omega p^*}\mathrm{d}x. \end{aligned} \tag{3.3}$$

Since

$$
\begin{aligned}
|\nabla f^{2\omega}(v_n)|^p &= |2\omega f^{2\omega-1}(v_n)f'(v_n)\nabla v_n|^p \\
&= (2\omega)^p \frac{|f^{2\omega-1}(v_n)|^p}{1+(2\omega)^{p-1}|f(v_n)|^{p(2\omega-1)}}|\nabla v_n|^p \\
&= \left[2\omega \frac{(2\omega)^{p-1}|f(v_n)|^{p(2\omega-1)}}{1+(2\omega)^{p-1}|f(v_n)|^{p(2\omega-1)}}\right]|\nabla v_n|^p \\
&\leq 2\omega|\nabla v_n|^p,
\end{aligned} \tag{3.4}
$$

we get

$$
\begin{aligned}
0 > c + o_n(\|v_n\|) &= J(v_n) - \frac{1}{2\omega p^*}\langle J'(v_n), \varphi_n\rangle \\
&\geq \frac{1}{N}\int_{\mathbb{R}^N}|\nabla v_n|^p \mathrm{d}x + \left(\frac{1}{p} - \frac{1}{2\omega p^*}\right)\int_{\mathbb{R}^N} V(x)|f(v_n)|^p \mathrm{d}x - \frac{a}{qr}\int_{\mathbb{R}^N} k(x)|f(v_n)|^q \mathrm{d}x \\
&\geq \frac{1}{N}\int_{\mathbb{R}^N}(|\nabla v_n|^p + V(x)|f(v_n)|^p)\mathrm{d}x - \frac{a}{qr}\|k(x)\|_r \left(\int_{\mathbb{R}^N}|f^{2\omega}(v_n)|^{p^*}\mathrm{d}x\right)^{\frac{q}{2\omega p^*}} \\
&\geq \frac{1}{N}\int_{\mathbb{R}^N}(|\nabla v_n|^p + V(x)|f(v_n)|^p)\mathrm{d}x - \frac{a}{qr}C_1\left(\int_{\mathbb{R}^N}|\nabla f^{2\omega}(v_n)|^p \mathrm{d}x\right)^{\frac{q}{2\omega p}} \\
&\geq \frac{1}{N}\int_{\mathbb{R}^N}(|\nabla v_n|^p + V(x)|f(v_n)|^p)\mathrm{d}x - C_2\left[\int_{\mathbb{R}^N}|\nabla v_n|^p \mathrm{d}x\right]^{\frac{q}{2\omega p}} \\
&\geq \frac{1}{N}\int_{\mathbb{R}^N}(|\nabla v_n|^p + V(x)|f(v_n)|^p)\mathrm{d}x - C_2\left[\int_{\mathbb{R}^N}(|\nabla v_n|^p + V(x)|f(v_n)|^p)\mathrm{d}x\right]^{\frac{q}{2\omega p}},
\end{aligned} \tag{3.5}
$$

which implies that for n large enough, there exists $C > 0$ such that

$$
\int_{\mathbb{R}^N}(|\nabla v_n|^p + V(x)|f(v_n)|^p)\mathrm{d}x \leq C. \tag{3.6}
$$

In the following, we need to show $\{v_n\}$ is bounded in $W^{1,p}(\mathbb{R}^N)$. From (3.6), we need to prove that $\int_{\mathbb{R}^N} V(x)|v_n|^p \mathrm{d}x$ is bounded. By $(\mathbf{V})$,

$$
\int_{\{x:|v_n|>1\}} V(x)|v_n|^p \mathrm{d}x \leq M\int_{\{x:|v_n|>1\}}|v_n|^{p^*}\mathrm{d}x \leq MS^{-\frac{p^*}{p}}\left(\int_{\{x:|v_n|>1\}}|\nabla v_n|^p \mathrm{d}x\right)^{\frac{p^*}{p}},
$$

and using Lemma 2.1 (6),

$$
\int_{\{x:|v_n|\leq 1\}} V(x)|v_n|^p \mathrm{d}x \leq \frac{1}{C^2}\int_{\{x:|v_n|\leq 1\}} V(x)|f(v_n)|^p \mathrm{d}x \leq \frac{1}{C^2}\int_{\mathbb{R}^N} V(x)|f(v_n)|^p \mathrm{d}x.
$$

These estimates imply that $\{v_n\}$ is bounded in $W^{1,p}(\mathbb{R}^N)$. Then $\{f(v_n)\}$ is also bounded in $W^{1,p}(\mathbb{R}^N)$. Therefore we can assume that

$$
\begin{aligned}
&v_n \rightharpoonup v && \text{weakly in } W^{1,p}(\mathbb{R}^N), \\
&v_n \to v && \text{a.e. in } \mathbb{R}^N, \\
&v_n \to v && \text{strongly in } L^t_{\mathrm{loc}}(\mathbb{R}^N) \text{ for all } t \in [1, p^*).
\end{aligned}
$$

Since $f \in C^\infty$, we have

$$
\begin{aligned}
&f^{2\omega}(v_n) \rightharpoonup f^{2\omega}(v) && \text{weakly in } W^{1,p}(\mathbb{R}^N), \\
&f^{2\omega}(v_n) \to f^{2\omega}(v) && \text{a.e. in } \mathbb{R}^N.
\end{aligned}
$$

In view of the concentration–compactness principle [14, 15], there exist a subsequence, still denoted by $\{f(v_n)\}$, $\mu, \nu \in M(\mathbb{R}^N \cup \{\infty\})$ which are the positive finite Radon measures on $\mathbb{R}^N \cup \{\infty\}$, an at most countable set $\mathcal{J}$, a set of different points $\{x_j\} \subset \mathbb{R}^N$, and real numbers μ_j, ν_j such that the following convergence hold in the sense of measures

$$|\nabla f^{2\omega}(v_n)|^p \rightharpoonup \mathrm{d}\mu \geq |\nabla f^{2\omega}(v)|^p + \sum_{j\in\mathcal{J}} \mu_j \delta_{x_j},$$
$$|f^{2\omega}(v_n)|^{p^*} \rightharpoonup \mathrm{d}\nu = |f^{2\omega}(v)|^{p^*} + \sum_{j\in\mathcal{J}} \nu_j \delta_{x_j}.$$

From the above two equations and the Sobolev inequalities, it follows easily that

$$\mu_j \geq S\nu_j^{\frac{p}{p^*}} \quad \text{for all } j \in \mathcal{J}. \tag{3.7}$$

Concentration at infinity of the sequence $\{u_n\}$ is described by the following quantities:

$$\mu_\infty := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\{x:|x|>R\}} |\nabla f^{2\omega}(v_n)|^p \mathrm{d}x,$$
$$\nu_\infty := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\{x:|x|>R\}} |f^{2\omega}(v_n)|^{p^*} \mathrm{d}x.$$

We claim that

$$\mathcal{J} \text{ is finite and, for } j \in \mathcal{J}, \text{ either } \nu_j = 0 \text{ or } \nu_j \geq (b^{-1}S)^{N/2}.$$

In fact, for $\varepsilon > 0$, letting x_j be a singular point of the measures μ_j and ν_j, $\phi_j(x)$ be a smooth cut–off function centered at x_j such that $0 \leq \phi_j(x) \leq 1$, $\phi_j(x) \equiv 0$ on $|x - x_j| \geq 2$, $\phi_j(x) \equiv 1$ on $|x - x_j| \leq 1$, and $|\nabla\phi_j(x)| \leq 2$ for all $x \in \mathbb{R}^N$. Letting $\phi_j^\varepsilon(x) = \phi_j(\frac{x}{\varepsilon}), \psi_n = [1 + (2\omega)^{p-1}|f(v_n)|^{p(2\omega-1)}]^{\frac{1}{p}} f(v_n)$, then we get that $\{\psi_n\}$ is bounded in $W^{1,p}(\mathbb{R}^N)$. Testing $J'(v_n)$ with $\psi_n\phi_j^\varepsilon$, we obtain $\lim_{n\to\infty}\langle J'(v_n), \psi_n\phi_j^\varepsilon(x)\rangle = 0$, that is

$$\begin{aligned}
&- \lim_{n\to\infty} \int_{\mathbb{R}^N} [1 + (2\omega)^{p-1}|f(v_n)|^{p(2\omega-1)}]^{\frac{1}{p}} f(v_n)|\nabla v_n|^{p-2}\nabla v_n \nabla\phi_j^\varepsilon \mathrm{d}x \\
&= \lim_{n\to\infty} \bigg[\int_{\mathbb{R}^N} \frac{1 + 2\omega(2\omega)^{p-1}|f(v_n)|^{p(2\omega-1)}}{1 + (2\omega)^{p-1}|f(v_n)|^{p(2\omega-1)}} |\nabla v_n|^p \phi_j^\varepsilon \mathrm{d}x + \int_{\mathbb{R}^N} V(x)|f(v_n)|^p \phi_j^\varepsilon \mathrm{d}x \\
&\qquad - a\int_{\mathbb{R}^N} k(x)|f(v_n)|^q \phi_j^\varepsilon \mathrm{d}x - b\int_{\mathbb{R}^N} |f(v_n)|^{2\omega p^*}\phi_j^\varepsilon \mathrm{d}x \bigg].
\end{aligned} \tag{3.8}$$

In the following we estimate each term in (3.8). By Lemma 2.1 (5) and the expression of f', we have

$$\frac{|f(v_n)|}{f'(v_n)} \leq 2\omega|v_n| \Rightarrow [1 + (2\omega)^{p-1}|f(v_n)|^{p(2\omega-1)}]^{\frac{1}{p}}|f(v_n)| \leq 2\omega|v_n|.$$

Thus

$$\begin{aligned}
0 &\leq \lim_{\varepsilon\to0}\lim_{n\to\infty} \left| \int_{\mathbb{R}^N} [1 + (2\omega)^{p-1}|f(v_n)|^{p(2\omega-1)}]^{\frac{1}{p}} f(v_n)|\nabla v_n|^{p-2}\nabla v_n\nabla\phi_j^\varepsilon \mathrm{d}x \right| \\
&\leq \lim_{\varepsilon\to0}\lim_{n\to\infty} \int_{\mathbb{R}^N} \left|2\omega v_n|\nabla v_n|^{p-2}\nabla v_n\nabla\phi_j^\varepsilon\right| \mathrm{d}x \\
&\leq \lim_{\varepsilon\to0}\lim_{n\to\infty} 2\omega\Big(\int_{\mathbb{R}^N} |\nabla v_n|^p \mathrm{d}x\Big)^{\frac{p-1}{p}} \Big(\int_{\mathbb{R}^N} |v_n\nabla\phi_j^\varepsilon|^p \mathrm{d}x\Big)^{\frac{1}{p}} \\
&\leq \lim_{\varepsilon\to0}\lim_{n\to\infty} C\Big(\int_{\mathbb{R}^N} |v_n\nabla\phi_j^\varepsilon|^p \mathrm{d}x\Big)^{\frac{1}{p}} \\
&\leq C\lim_{\varepsilon\to0}\Big(\int_{B(x_j,2\varepsilon)} |v_n|^{p\cdot\frac{p^*}{p}} \mathrm{d}x\Big)^{\frac{1}{p^*}} \Big(\int_{B(x_j,2\varepsilon)} |\nabla\phi_j^\varepsilon|^{p\cdot\frac{N}{p}} \mathrm{d}x\Big)^{\frac{1}{N}} = 0.
\end{aligned} \tag{3.9}$$

Also we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla f^{2\omega}(v_n)|^p\phi_j^\varepsilon \mathrm{d}x=\int_{\mathbb{R}^N}\phi_j^\varepsilon \mathrm{d}x\geq\int_{\mathbb{R}^N}|\nabla f^{2\omega}(v)|^p\phi_j^\varepsilon \mathrm{d}x+\mu_j, \tag{3.10}$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|f(v_n)|^{2\omega p^*}\phi_j^\varepsilon \mathrm{d}x=\int_{\mathbb{R}^N}\phi_j^\varepsilon \mathrm{d}\nu\geq\int_{\mathbb{R}^N}|f(v)|^{2\omega p^*}\phi_j^\varepsilon \mathrm{d}x+\nu_j. \tag{3.11}$$

By the weak continuity of $\mathcal{F}(v)$, we get

$$\lim_{\varepsilon\to0}\lim_{n\to\infty}\int_{\mathbb{R}^N}k(x)|f(v_n)|^q\phi_j^\varepsilon \mathrm{d}x=0. \tag{3.12}$$

From (3.9)–(3.12), by the weak continuity of $\mathcal{F}$, we have

$$\begin{aligned}0=&\lim_{\varepsilon\to0}\lim_{n\to\infty}\Bigg[\int_{\mathbb{R}^N}\frac{1+2\omega(2\omega)^{p-1}|f(v_n)|^{p(2\omega-1)}}{1+(2\omega)^{p-1}|f(v_n)|^{p(2\omega-1)}}|\nabla v_n|^p\phi_j^\varepsilon \mathrm{d}x+\int_{\mathbb{R}^N}V(x)|f(v_n)|^p\phi_j^\varepsilon \mathrm{d}x\\&-a\int_{\mathbb{R}^N}k(x)|f(v_n)|^q\phi_j^\varepsilon \mathrm{d}x-b\int_{\mathbb{R}^N}|f(v_n)|^{2\omega p^*}\phi_j^\varepsilon \mathrm{d}x\Bigg]\\\geq&\lim_{\varepsilon\to0}\lim_{n\to\infty}\left[\int_{\mathbb{R}^N}|\nabla f^{2\omega}(v_n)|^p\phi_j^\varepsilon \mathrm{d}x-a\int_{\mathbb{R}^N}k(x)|f(v_n)|^q\phi_j^\varepsilon \mathrm{d}x-b\int_{\mathbb{R}^N}|f(v_n)|^{2\omega p^*}\phi_j^\varepsilon \mathrm{d}x\right]\\=&\,\mu_j-b\nu_j.\end{aligned} \tag{3.13}$$

Combining with (3.7), we obtain

$$\text{either}\quad \text{(i) } \nu_j=0\quad\text{or}\quad \text{(ii) } \nu_j\geq(b^{-1}S)^{\frac{N}{p}},$$

which implies that $\mathcal{J}$ is finite. The claim is thereby proved.

To analyze the concentration at ∞, we follow closely the argument used in [21]. By choosing a suitable cut-off function $\varphi\in C_0^\infty(\mathbb{R}^N,[0,1])$ such that $\varphi(x)\equiv0$ on $|x|\leq1$ and $\varphi(x)\equiv1$ on $|x|\geq2$. Setting $\varphi_R(x)=\varphi(\frac{x}{R})$, then $\{\varphi_R\psi_n\}$ is bounded in $W^{1,p}(\mathbb{R}^N)$, and $\lim_{n\to\infty}\langle J'(v_n),\varphi_R\psi_n\rangle=0$, that is

$$\begin{aligned}&-\lim_{n\to\infty}\int_{\mathbb{R}^N}[1+(2\omega)^{p-1}|f(v_n)|^{p(2\omega-1)}]^{\frac1p}f(v_n)|\nabla v_n|^{p-2}\nabla v_n\nabla\varphi_R \mathrm{d}x\\&\quad=\lim_{n\to\infty}\Bigg[\int_{\mathbb{R}^N}\frac{1+2\omega(2\omega)^{p-1}|f(v_n)|^{p(2\omega-1)}}{1+(2\omega)^{p-1}|f(v_n)|^{p(2\omega-1)}}|\nabla v_n|^p\varphi_R \mathrm{d}x+\int_{\mathbb{R}^N}V(x)|f(v_n)|^p\varphi_R \mathrm{d}x\\&\qquad-a\int_{\mathbb{R}^N}k(x)|f(v_n)|^q\varphi_R \mathrm{d}x-b\int_{\mathbb{R}^N}|f(v_n)|^{2\omega p^*}\varphi_R \mathrm{d}x\Bigg].\end{aligned} \tag{3.14}$$

Similar to the process of (3.9), we can get

$$\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N}[1+(2\omega)^{p-1}|f(v_n)|^{p(2\omega-1)}]^{\frac1p}f(v_n)|\nabla v_n|^{p-2}\nabla v_n\nabla\varphi_R \mathrm{d}x=0. \tag{3.15}$$

Using the weak continuity of $\mathcal{F}$, we have

$$\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N}k(x)|f(v_n)|^q\varphi_R \mathrm{d}x=0.$$

Therefore,

$$
\begin{aligned}
0 = & \lim_{R\to\infty}\lim_{n\to\infty}\Bigg[\int_{\mathbb{R}^N}\frac{1+2\omega(2\omega)^{p-1}|f(v_n)|^{p(2\omega-1)}}{1+(2\omega)^{p-1}|f(v_n)|^{p(2\omega-1)}}|\nabla v_n|^p\varphi_R\mathrm{d}x+\int_{\mathbb{R}^N}V(x)|f(v_n)|^p\varphi_R\mathrm{d}x \\
& \qquad - a\int_{\mathbb{R}^N}k(x)|f(v_n)|^q\varphi_R\mathrm{d}x - b\int_{\mathbb{R}^N}|f(v_n)|^{2\omega p^*}\varphi_R\mathrm{d}x\Bigg] \\
\geq & \lim_{R\to\infty}\lim_{n\to\infty}\Bigg[\int_{\mathbb{R}^N}|\nabla f^{2\omega}v_n|^p\varphi_R\mathrm{d}x - a\int_{\mathbb{R}^N}k(x)|f(v_n)|^q\varphi_R\mathrm{d}x - b\int_{\mathbb{R}^N}|f(v_n)|^{2\omega p^*}\varphi_R\mathrm{d}x\Bigg] \\
= & \mu_\infty - b\nu_\infty. \qquad (3.16)
\end{aligned}
$$

By

$$
\mu_\infty \geq S\nu_\infty^{\frac{p}{p^*}}, \qquad (3.17)
$$

we get

$$
\text{either}\quad \text{(iii) } \nu_\infty = 0 \quad \text{or} \quad \text{(iv) } \nu_\infty \geq (b^{-1}S)^{\frac{N}{p}}.
$$

Next, we claim that (ii) and (iv) cannot occur if a and b are chosen properly. In fact, by (3.4) and (**V**), we have

$$
\int_{\mathbb{R}^N}(|\nabla v|^p + V(x)|f(v)|^p)\mathrm{d}x \geq \int_{\mathbb{R}^N}|\nabla v|^p\mathrm{d}x \geq \frac{1}{2\omega}\int_{\mathbb{R}^N}|\nabla f^{2\omega}(v)|^p\mathrm{d}x.
$$

Then if (iv) holds, from the weak lower semicontinuity of the norm and the weak continuity of $\mathcal{F}$, we have,

$$
\begin{aligned}
0 > c = & \lim_{n\to\infty}\left[J(v_n) - \frac{1}{2\omega p^*}\langle J'(v_n),\varphi_n\rangle\right] \\
\geq & \lim_{n\to\infty}\left[\left(\frac{1}{p}-\frac{1}{p^*}\right)\int_{\mathbb{R}^N}(|\nabla v_n|^p + V(x)|f(v_n)|^p)\mathrm{d}x - \frac{a}{qr}\|k(x)\|_r\|f^{2\omega}(v_n)\|_{p^*}^{\frac{q}{2\omega}}\right] \\
\geq & \frac{1}{N}\int_{\mathbb{R}^N}(|\nabla v|^p + V(x)|f(v)|^p)\mathrm{d}x - \frac{a}{qr}\|k(x)\|_r\|f^{2\omega}(v)\|_{p^*}^{\frac{q}{2\omega}} \\
\geq & \frac{1}{N}\cdot\frac{1}{2\omega}\int_{\mathbb{R}^N}|\nabla f^{2\omega}(v)|^p\mathrm{d}x - \frac{a}{qr}\|k(x)\|_r\|f^{2\omega}(v)\|_{p^*}^{\frac{q}{2\omega}} \\
\geq & \frac{1}{2\omega N}S\|f^{2\omega}(v)\|_{p^*}^p - \frac{a}{qr}\|k(x)\|_r\|f^{2\omega}(v)\|_{p^*}^{\frac{q}{2\omega}}.
\end{aligned}
$$

This inequality implies that

$$
\|f^{2\omega}(v)\|_{p^*} \leq Ca^{\frac{2\omega}{2\omega p-q}}.
$$

Therefore from (3.17) and (iv),

$$
\begin{aligned}
0 > c = & \lim_{n\to\infty}\left[J(v_n) - \frac{1}{2\omega p^*}\langle J'(v_n),\varphi_n\rangle\right] \\
\geq & \lim_{n\to\infty}\left[\left(\frac{1}{p}-\frac{1}{p^*}\right)\int_{\mathbb{R}^N}(|\nabla v_n|^p + V(x)|f(v_n)|^p)\mathrm{d}x - \frac{a}{qr}\|k(x)\|_r\|f^{2\omega}(v_n)\|_{p^*}^{\frac{q}{2\omega}}\right] \\
\geq & \lim_{R\to\infty}\lim_{n\to\infty}\left[\frac{1}{N}\int_{\mathbb{R}^N}(|\nabla v_n|^p + V(x)|f(v_n)|^p)\phi_R\mathrm{d}x - \frac{a}{qr}\|k(x)\|_r\|f^{2\omega}(v_n)\|_{p^*}^{\frac{q}{2\omega}}\right] \\
\geq & \frac{1}{N}\mu_\infty S^{\frac{N}{p}} - Ca^{\frac{2\omega}{2\omega p-q}} \\
\geq & \frac{1}{N}b^{\frac{p-N}{p}}S^{\frac{N}{p}} - Ca^{\frac{2\omega}{2\omega p-q}}.
\end{aligned}
$$

However, if $a > 0$ is given, we can choose small b_* so that for every $0 < b < b_*$, the last term on the right-hand side above is greater than zero, which is a contradiction. Similarly, if $b > 0$ is given, we can take small a_* so that for every $0 < a < a_*$, the last term on the right-hand side above is greater than zero. Similarly, we can prove that (ii) cannot occur for each j. Hence

$$\|f^{2\omega}(v_n)\|_{p^*} \to \|f^{2\omega}(v)\|_{p^*} \quad \text{as } n \to \infty,$$

and

$$\int_{\mathbb{R}^N} k(x)(|f(v_n)|^q - |f(v)|^q)\mathrm{d}x \leq \|k(x)\|_r \||f(v_n)|^q - |f(v)|^q\|_{\frac{2\omega p^*}{q}}.$$

Thus, from the weak lower semicontinuity of the norm and $\mathcal{F} \in C^\infty$ we have

$$\begin{aligned}
o(\|v_n\|) &= \langle J'(v_n), \varphi_n \rangle \\
&= \int_{\mathbb{R}^N} (|\nabla v_n|^p + V(x)|f(v_n)|^p)\mathrm{d}x + \int_{\mathbb{R}^N} \left[\frac{(2\omega-1)(2\omega)^{p-1}|f(v_n)|^{p(2\omega-1)}}{1+(2\omega)^{p-1}|f(v_n)|^{p(2\omega-1)}} \right] |\nabla v_n|^p \mathrm{d}x \\
&\quad - a \int_{\mathbb{R}^N} k(x)|f(v_n)|^q \mathrm{d}x - b \int_{\mathbb{R}^N} |f(v_n)|^{2\omega p^*} \mathrm{d}x \\
&= \int_{\mathbb{R}^N} (|\nabla v_n - \nabla v|^p + |\nabla v|^p + V(x)|f(v)|^p)\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} \left[\frac{(2\omega-1)(2\omega)^{p-1}|f(v)|^{p(2\omega-1)}}{1+(2\omega)^{p-1}|f(v)|^{p(2\omega-1)}} \right] |\nabla v|^p \mathrm{d}x \\
&\quad - a \int_{\mathbb{R}^N} k(x)|f(v)|^q \mathrm{d}x - b \int_{\mathbb{R}^N} |f(v)|^{2\omega p^*} \mathrm{d}x + o(\|v_n\|) \\
&= \int_{\mathbb{R}^N} |\nabla v_n - \nabla v|^p \mathrm{d}x + o(\|v_n\|)
\end{aligned}$$

since $J'(v) = 0$. Thus we prove that $\{v_n\}$ strongly converges to v in $W^{1,p}(\mathbb{R}^N)$. □

4 Proofs of the main results

In this section, we use the minimax procedure (see [20]) to prove the existence of infinitely many solutions. Let X be a Banach space and Σ be the class of subsets of $X \setminus \{0\}$ which are closed and symmetric with respect to the origin. For $A \in \Sigma$, we define the genus $\gamma(A)$ by

$$\gamma(A) = \min\{n \in \mathbb{N} : \exists \phi \in C(A, \mathbb{R}^n \setminus \{0\}),\ \phi(z) = -\phi(-z)\}.$$

If there is no mapping as above for any $n \in \mathbb{N}$, then $\gamma(A) = +\infty$. Let Σ_n denote the family of closed symmetric subsets A of X such that $0 \notin A$ and $\gamma(A) \geq n$. We list some properties of the genus (see [11,20]).

Proposition 4.1. *Let A and B be closed symmetric subsets of X which do not contain the origin. Then the following hold:*

(i) If there exists an odd continuous mapping from A to B, then $\gamma(A) \leq \gamma(B)$;

(ii) If there is an odd homeomorphism from A to B, then $\gamma(A) = \gamma(B)$;

(iii) If $\gamma(B) < \infty$, then $\gamma(\overline{A \setminus B}) \geq \gamma(A) - \gamma(B)$;

(iv) n-dimensional sphere S_n has a genus of $n+1$ by the Borsuk–Ulam Theorem;

(v) If A is compact, then $\gamma(A) < +\infty$ and there exists $\delta > 0$ such that $N_\delta(A) \subset \Sigma$ and $\gamma(N_\delta(A)) = \gamma(A)$, where $N_\delta(A) = \{x \in X : \|x - A\| \le \delta\}$.

Thanks to the work of Kajikiya in [11], we take the following version of the symmetric mountain-pass lemma.

Proposition 4.2. *Let E be an infinite-dimensional space and $J \in C^1(E, \mathbb{R})$ and suppose the following conditions hold:*

(A_1) *$J(u)$ is even, bounded from below, $J(0) = 0$ and $J(u)$ satisfies the local Palais–Smale condition (PS for short).*

(A_2) *For each $k \in \mathbb{N}$, there exists an $A_k \in \Sigma_k$ such that $\sup_{u \in A_k} J(u) < 0$.*

Then either (i) or (ii) below holds.

(i) There exists a sequence $\{u_k\}$ such that $J'(u_k) = 0, J(u_k) < 0$ and $\{u_k\}$ converges to zero.

(ii) There exist two sequences $\{u_k\}$ and $\{v_k\}$ such that $J'(u_k) = 0$, $J(u_k) = 0$, $u_k \neq 0$, $\lim_{k\to\infty} u_k = 0; J'(v_k) = 0, J(v_k) < 0$, $\lim_{k\to\infty} J(v_k) = 0$, and $\{v_k\}$ converges to a non-zero limit.

Remark 4.3. From Proposition 4.2 we have a sequence $\{u_k\}$ of critical points such that $J(u_k) \le 0, u_k \neq 0$ and $\lim_{k\to\infty} u_k = 0$.

In order to get infinitely many solutions we need some lemmas. Let $J(v)$ be the functional defined as before, $1 < 2\omega < q < 2\omega p$, and $a > 0, b > 0$. Then, by (3.4),

$$\begin{aligned}
J(v) &= \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla v|^p + V(x)|f(v)|^p)\mathrm{d}x - \frac{a}{q}\int_{\mathbb{R}^N} k(x)|f(v)|^q \mathrm{d}x - \frac{b}{2\omega p^*}\int_{\mathbb{R}^N}|f(v)|^{2\omega p^*}\mathrm{d}x \\
&\ge \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla v|^p + V(x)|f(v)|^p)\mathrm{d}x - \frac{a}{q}\|k(x)\|_r \|f^{2\omega}(v)\|_{p^*}^{\frac{q}{2\omega}} - \frac{b}{2\omega p^*}\int_{\mathbb{R}^N}|f(v)|^{2\omega p^*}\mathrm{d}x \\
&\ge \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla v|^p + V(x)|f(v)|^p)\mathrm{d}x - \frac{a}{q}C_1\Big(\int_{\mathbb{R}^N}|\nabla f^{2\omega}(v)|^p \mathrm{d}x\Big)^{\frac{q}{2\omega p}} \\
&\quad - \frac{b}{2\omega p^*}C_2\Big(\int_{\mathbb{R}^N}|\nabla f^{2\omega}(v)|^p\mathrm{d}x\Big)^{\frac{p^*}{p}} \\
&\ge \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla v|^p + V(x)|f(v)|^p)\mathrm{d}x - \frac{aC_3}{q}\Big(\int_{\mathbb{R}^N}|\nabla v|^p\mathrm{d}x\Big)^{\frac{q}{2\omega p}} - \frac{bC_4}{2\omega p^*}\Big(\int_{\mathbb{R}^N}|\nabla v|^p\mathrm{d}x\Big)^{\frac{p^*}{p}} \\
&\ge C_5\|v\|^p - aC_6\|v\|^{\frac{q}{2\omega}} - bC_7\|v\|^{p^*}.
\end{aligned}$$

Define

$$g(t) = C_5 t^p - aC_6 t^{\frac{q}{2\omega}} - bC_7 t^{p^*}.$$

Since $1 < 2\omega < q < 2\omega p$, it is easy to see that, for the given $b > 0$, we can choose small $a^* > 0$ such that if $0 < a < a^*$, there exists $0 < t_0 < t_1$ such that $g(t) < 0$ for $0 < t < t_0$; $g(t) > 0$ for $t_0 < t < t_1$; $g(t) < 0$ for $t > t_1$.

Similarly, for the given $a > 0$, we can choose small $b^* > 0$ such that if $0 < b < b^*$, there exists $0 < t_0 < t_1$ such that $g(t) < 0$ for $0 < t < t_0$; $g(t) > 0$ for $t_0 < t < t_1$; $g(t) < 0$ for $t > t_1$.

Clearly, $g(t_0) = 0 = g(t_1)$. Following the same idea as in [3], we consider the truncated functional

$$\tilde{J}(v) = \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla v|^p + V(x)|f(v)|^p)\mathrm{d}x - \frac{a}{q}\int_{\mathbb{R}^N} k(x)|f(v)|^q\mathrm{d}x - \frac{b}{2\omega p^*}\psi(v)\int_{\mathbb{R}^N}|f(v)|^{2\omega p^*}\mathrm{d}x,$$

where $\psi(v) = \tau(\|v\|)$ and $\tau : \mathbb{R}^+ \to [0,1]$ is a non-increasing C^∞ function such that $\tau(t) = 1$ if $t \le t_0$ and $\tau(t) = 0$ if $t \ge t_1$. Obviously, $\tilde{J}(v)$ is even. Thus, following Lemma 3.1, we obtain the following lemma.

Lemma 4.4. *Let $c < 0$ and $2\omega < q < 2\omega p$. Then*

(1) *$\tilde{J} \in C^1$ and $\tilde{J}$ is bounded below.*

(2) *If $\tilde{J}(v) < 0$, then $\|v\| < t_0$ and $\tilde{J}(v) = J(v)$.*

(3) *$\forall b > 0, \exists\, \widetilde{a^*} = \min\{a_*, a^*\} > 0$ such that if $0 < a < \widetilde{a^*}$, then $\tilde{J}$ satisfies $(PS)_c$.*

(4) *$\forall a > 0, \exists\, \widetilde{b^*} = \min\{b_*, b^*\} > 0$ such that if $0 < b < \widetilde{b^*}$, then $\tilde{J}$ satisfies $(PS)_c$.*

Proof. The aforementioned (1) and (2) are immediate. To prove (3) and (4), observe that all (PS) sequences for $\tilde{J}$ with $c < 0$ must be bounded. Similar to the proof of Lemma 3.1, there exists a strong convergent subsequence in $W^{1,p}(\mathbb{R}^N)$. □

Remark 4.5. Denote $K_c = \{v \in W^{1,p}(\mathbb{R}^N); \tilde{J}'(v) = 0, \tilde{J}(v) = c\}$ If a, b are as in (3) or (4) above, it then follows from $(PS)_c$ that $K_c (c < 0)$ is compact.

Lemma 4.6. *Assume that $(\boldsymbol{K})$ is held, then for the given $n \in \mathbb{N}$, there exists $\epsilon_n < 0$ such that*

$$\gamma(\tilde{J}^{\epsilon_n}) := \gamma(\{v \in W^{1,p}(\mathbb{R}^N) : \tilde{J}(v) \le \epsilon_n\}) \ge n.$$

Proof. Let X_n be a n-dimensional subspace of $W^{1,p}(\mathbb{R}^N)$. For any $v \in X_n, v \ne 0$, write $v = r_n w$ with $w \in X_n, \|w\| = 1$ and then $r_n = \|v\|$. From the assumption $(\mathbf{K})$, it is easy to see that, for every $w \in X_n$ with $\|w\| = 1$, there exists $d_n > 0$ such that $\int_{\mathbb{R}^N} k(x)|w|^{\frac{q}{2\omega}} \mathrm{d}x \ge d_n$. Thus for $0 < r_n < t_0$ and Lemma 2.1 (2),(5), we have

$$\begin{aligned}
\tilde{J}(v) = J(v) &= \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla v|^p + V(x)|f(v)|^p)\mathrm{d}x - \frac{a}{q}\int_{\mathbb{R}^N} k(x)|f(v)|^q \mathrm{d}x - \frac{b}{2\omega p^*}\int_{\mathbb{R}^N} |f(v)|^{2\omega p^*}\mathrm{d}x \\
&\le \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla v|^p + V(x)(C|v|^{\frac{p}{2\omega}} + C))\mathrm{d}x \\
&\quad - \frac{a}{q}\int_{\mathbb{R}^N} k(x)(C|v|^{\frac{q}{2\omega}} + C)\mathrm{d}x - \frac{b}{2\omega p^*}\int_{\mathbb{R}^N} (|v|^{p^*} + C)\mathrm{d}x \\
&\le \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla v|^p + V(x)C|v|^{\frac{p}{2\omega}})\mathrm{d}x - \frac{Ca}{q}\int_{\mathbb{R}^N} k(x)|v|^{\frac{q}{2\omega}}\mathrm{d}x - \frac{Cb}{2\omega p^*}\int_{\mathbb{R}^N} |v|^{p^*}\mathrm{d}x - C \\
&\le C_1 r_n^p + C_2 r_n^{\frac{p}{2\omega}} - a d_n r_n^{\frac{q}{2\omega}} - b C_3 r_n^{p^*} - C_4 \\
&= \epsilon_n.
\end{aligned}$$

Therefore we can choose small $r_n \in (0, t_0)$ such that $\tilde{J}(v) \le \epsilon_n < 0$. Let

$$S_{r_n} = \{v \in X_n : \|v\| = r_n\}. \tag{4.1}$$

Then $S_{r_n} \cap X_n \subset \tilde{J}^{\epsilon_n}$. Hence by Proposition 4.1,

$$\gamma(\tilde{J}^{\epsilon_n}) \ge \gamma(S_{r_n} \cap X_n) = n.$$ □

According to Lemma 4.4 we denote $\Sigma_n = \{A \in \Sigma : \gamma(A) \ge n\}$ and let

$$c_n = \inf_{A \in \Sigma_n} \sup_{v \in A} \tilde{J}(v). \tag{4.2}$$

Then $-\infty < c_n \le \epsilon_n < 0$ since $\tilde{J}^{\epsilon_n} \in \Sigma_n$ and $\tilde{J}$ is bounded from below.

Lemma 4.7. *Let a, b be as in (3) or (4) of Lemma 4.4. Then all c_n (given by (4.2)) are critical values of $\tilde{J}$ and $c_n \to 0$.*

Proof. It is clear that $c_n \leq c_{n+1}$. By (4.2) we have $c_n < 0$. Hence $c_n \to \bar{c} \leq 0$. Moreover, since that all c_n are critical values of $\tilde{J}$ [20], we claim that $\bar{c} = 0$. If $\bar{c} < 0$, then by Remark 4.5, $K_{\bar{c}} = \{v \in W^{1,p}(\mathbb{R}^N); \tilde{J}'(v) = 0, \tilde{J}(v) = \bar{c}\}$ is compact and $K_{\bar{c}} \in \Sigma$, then $\gamma(K_{\bar{c}}) = n_0 < +\infty$ and there exists $\delta > 0$ such that $\gamma(K_{\bar{c}}) = \gamma(N_\delta(K_{\bar{c}})) = n_0$, here $N_\delta(K_{\bar{c}}) = \{x \in X; \|x - K_{\bar{c}}\| \leq \delta\}$. By the deformation lemma [22] there exist $\epsilon > 0$ $(\bar{c} + \epsilon < 0)$ and an odd homeomorphism η such that

$$\eta(\tilde{J}^{\bar{c}+\epsilon} \setminus N_\delta(K_{\bar{c}})) \subset \tilde{J}^{\bar{c}-\epsilon}.$$

Since c_n is increasing and converges to $\bar{c}$, there exists $n \in \mathbb{N}$ such that $c_n > \bar{c} - \epsilon$ and $c_{n+n_0} \leq \bar{c}$. Choose $A \in \Sigma_{n+n_0}$ such that $\sup_{v \in A} \tilde{J}(v) < \bar{c} + \epsilon$. By the properties of γ, we have

$$\gamma(\overline{A \setminus N_\delta(K_{\bar{c}})}) \geq \gamma(A) - \gamma(N_\delta(K_{\bar{c}}))) \geq n, \qquad \gamma(\overline{\eta(A \setminus N_\delta(K_{\bar{c}}))}) \geq n.$$

Therefore $\overline{\eta(A \setminus N_\delta(K_{\bar{c}}))} \in \Sigma_n$. Consequently $\sup_{v \in \overline{\eta(A \setminus N_\delta(K_{\bar{c}}))}} \tilde{J}(v) \geq c_n > \bar{c} - \epsilon$, a contradiction, hence $c_n \to 0$. □

Proof of Theorem 2.3. By Lemma 4.4 (2), $\tilde{J}(v) = J(v)$ if $\tilde{J} < 0$. This and Lemma 4.7 give the result. □

Proof of Theorem 1.2. This follows from Theorem 2.3 since $u_m = f(v_m) \neq u_n = f(v_n)$ if $v_m \neq v_n$ and $f \in C^\infty$. □

Acknowledgements

The authors express their gratitude to the referee for his/her careful reading and helpful suggestions. L. Wang was supported by the NSFC (No. 11561024, No. 11701178). J. Wang was supported by the NSFC (No. 11501186).

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