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Ground state sign-changing solutions for Kirchhoff equations with logarithmic nonlinearity

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Abstract. In this paper, we study Kirchhoff equations with logarithmic nonlinearity:

$$\begin{cases} -(a+b\int_\Omega |\nabla u|^2)\Delta u + V(x)u = |u|^{p-2}u\ln u^2, & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega, \end{cases}$$

where $a,b>0$ are constants, $4<p<2^*$, Ω is a smooth bounded domain of $\mathbb{R}^3$ and $V:\Omega\to\mathbb{R}$. Using constraint variational method, topological degree theory and some new energy estimate inequalities, we prove the existence of ground state solutions and ground state sign-changing solutions with precisely two nodal domains. In particular, some new tricks are used to overcome the difficulties that $|u|^{p-2}u\ln u^2$ is sign-changing and satisfies neither the monotonicity condition nor the Ambrosetti–Rabinowitz condition.

Keywords: logarithmic nonlinearity, ground state solution, sign-changing solution, Kirchhoff equations.

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1 Introduction

In this paper, we investigate the following Kirchhoff equation with logarithmic nonlinearity:

$$\begin{cases} -(a+b\int_\Omega |\nabla u|^2)\Delta u + V(x)u = |u|^{p-2}u\ln u^2, & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $a,b>0$ are constants, $4<p<2^*$, Ω is a smooth bounded domain of $\mathbb{R}^3$ and $V:\Omega\to\mathbb{R}$ satisfies

(V) $V\in\mathcal{C}(\Omega,\mathbb{R})$ and $\inf_{x\in\Omega}V(x)>0$.

In the past years, there have been increasing interests in studying logarithmic nonlinearity due to its relevance in quantum mechanics, quantum optics, nuclear physics, transport and diffusion phenomena, open quantum systems, effective quantum gravity, theory of superfluidity and Bose–Instein consideration (see [32] and the references therein).

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Denote by $H_0^1(\Omega)$ the Sobolev space equipped with the norm and inner product

$$\|u\| = \left(\int_\Omega a|\nabla u|^2 + V(x)u^2 dx\right)^{\frac{1}{2}}, \qquad \langle u, v\rangle = \int_\Omega [a\nabla u \cdot \nabla v + V(x)uv]dx,$$

under the assumption (V). This norm is equivalent to the standard norm of $H_0^1(\Omega)$.

Define the energy functional $I : \ H_0^1(\Omega) \to \mathbb{R}$

$$\begin{aligned} I(u) = \frac{1}{2}\int_\Omega \left[a|\nabla u|^2 + V(x)u^2\right] dx + \frac{b}{4}\left(\int_\Omega |\nabla u|^2 dx\right)^2 \\ + \frac{2}{p^2}\int_\Omega |u|^p dx - \frac{1}{p}\int_\Omega |u|^p \ln u^2 dx. \end{aligned} \tag{1.2}$$

By elementary computation, we have

$$\lim_{t\to 0} \frac{t^{p-1}\ln t^2}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{t^{p-1}\ln t^2}{t^{q-1}} = 0, \tag{1.3}$$

where $q \in (p, 2^*)$. Then for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|t|^{p-1}|\ln t^2| \le \varepsilon|t| + C_\varepsilon|t|^{q-1}, \qquad \forall t \in \mathbb{R}\backslash\{0\}. \tag{1.4}$$

By a similar argument of [23] and (1.4), we have that $I \in \mathcal{C}^1(H_0^1(\Omega), \mathbb{R})$ and

$$\begin{aligned} \langle I'(u), v\rangle = & \int_\Omega \left[a|\nabla u \cdot \nabla v + V(x)uv\right] dx + b\int_\Omega |\nabla u|^2 dx \int_\Omega \nabla u \cdot \nabla v dx \\ & - \int_\Omega |u|^{p-2}uv \ln u^2 dx \end{aligned} \tag{1.5}$$

for all $u, v \in H_0^1(\Omega)$. $u \in H_0^1(\Omega)$ is a weak solution of (1.1) if and only if u is a critical point of I. Moreover, if $u \in H_0^1(\Omega)$ is a solution of (1.1) and $u^\pm \neq 0$, then u is a sign-changing solution of (1.1), where

$$u^+(x) := \max\{u(x), 0\}, \qquad u^-(x) := \min\{u(x), 0\}.$$

From (1.5), one has

$$\begin{aligned} \langle I'(u), u^\pm\rangle = & \int_\Omega \left[a|\nabla u^\pm|^2 + V(x)(u^\pm)^2\right] dx + b\int_\Omega |\nabla u|^2 dx \int_\Omega |\nabla u^\pm|^2 dx \\ & - \int_\Omega |u^\pm|^p \ln(u^\pm)^2 dx. \end{aligned} \tag{1.6}$$

As we know, (1.1) is a special form of the following Kirchhoff type problem:

$$\begin{cases} -(a + b\int_\Omega |\nabla u|^2)\Delta u + V(x)u = f(u), & \text{in } \Omega, \\ u = 0, & \text{on } \Omega, \end{cases} \tag{1.7}$$

where $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$. System (1.7) is related to the stationary analogue of the Kirchhoff equation

$$u_{tt} - \left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x, u) \tag{1.8}$$

proposed by Kirchhoff in [14] as an extension of the classical D'Alembert's wave equations for free vibration of elastic strings. For more mathematical and physical background of the problem (1.7), we refer the readers to the papers [1,2,4,5] and the references therein.

Kirchhoff equation (1.8) received increasingly more attention after Lion's [15] proposed an abstract functional analysis framework to it. There are many important results on the existence of positive solutions, multiple solutions and ground state solutions for Kirchhoff equations, see for example, [6–9, 11, 12, 18, 19, 25–27, 29, 30] and the references therein.

Recently, many researchers began to study the sign-changing solutions for (1.7). When $V(x) \equiv 0$, Zhang [31] obtained sign-changing solutions for (1.7) via invariant sets of descent flow under the following (AR) condition

(AR) there exists $v > 4$ such that $vF(x,t) \leq tf(x,t)$ for $|t|$ large, where $F(x,t) = \int_0^t f(x,s)ds$.

Shuai [22] proved the existence of sign-changing solutions for system (1.7) when $f(x,u) = f(u)$ satisfies the Nehari type monotonicity condition:

(F) $\frac{f(t)}{|t|^3}$ is increasing on $(-\infty, 0) \cup (0, +\infty)$

and some other conditions. To obtain a constant sign solution and a sign-changing solution for the following Kirchhoff-type equation:

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^2 dx\right) \Delta u = \lambda f(u), & \text{in } \Omega, \\ u = 0, & \text{on } \Omega, \end{cases} \tag{1.9}$$

Lu [17] also proposed the following monotonicity condition

(F′) there exists $\mu \in (2, 2^*)$ such that $\frac{f(t)}{|t|^{\mu-2}t}$ is nondecreasing in $|t| > 0$.

In particular, letting $a = 1$ and $b = 0$ in (1.1) leads to the following Schrödinger equation:

$$\begin{cases} -\Delta u + V(x)u = |u|^{p-2}u \ln u^2, & x \in \Omega, \\ u \in H_0^1(\Omega). \end{cases} \tag{1.10}$$

System (1.10) has received much attention in mathematical analysis and applications. D'Avenia [3] proved the existence of infinitely many solutions of (1.10) with $p = 2$ in the framework of the non-smooth critical point theory, which is developed by Degiovanni and Zani [10]. When $p = 2$ and V satisfies the following condition

(V′) $V \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$, $\lim_{|x|\to\infty} V(x) = \sup_{x\in\mathbb{R}^N} V(x) := V_\infty \in (-1, \infty)$ and the spectrum $\sigma(-\Delta + V + 1) \subset (0, \infty)$.

Ji [13] obtained a positive ground state solution of (1.10). For more results on the logarithmic Schrödinger equation, we refer the readers to [23, 24] and the references therein.

Motivated by the works mentioned above, in the present paper, we intend to prove the existence of ground state solutions and sign-changing solutions for (1.1). It is worth pointing out that the methods used in [17, 22, 31] rely heavily on the monotonicity conditions (F), (F′) or (AR) condition, so their methods do not work for (1.1) because $f(x,u) = |u|^{p-2}u \ln u^2$ satisfies neither the monotonicity conditions (F), (F′) or (AR) condition. Furthermore, due to the existence of the nonlocal term $\left(\int_\Omega |\nabla u|^2 dx\right) \Delta u$, the method dealing with (1.10) can not be applicable for (1.1). Therefore, a natural question is whether we can still find sign-changing solutions for Kirchhoff equation with logarithmic nonlinearity. The present paper will give an affirmative response and establish the relation between the energy of sign-changing solutions and ground state solutions of (1.1). To the best of our knowledge, there are only a few results of sign-changing solutions for system (1.1).

Now, we state the main result.

Theorem 1.1. *Assume that* (V) *holds. Then problem* (1.1) *has a sign-changing solution* $\tilde{u} \in \mathcal{M}$ *with precisely two nodal domains such that* $I(\tilde{u}) = \inf_{\mathcal{M}} I := m$*, where*

$$\mathcal{M} = \{u \in H_0^1(\Omega),\ u^{\pm} \neq 0, \text{ and } \langle \mathrm{I'(u), u^+} \rangle = \langle \mathrm{I'(u), u^-} \rangle = 0\}.$$

Theorem 1.2. *Assume that* (V) *holds. Then problem* (1.1) *has a ground state solution* $\bar{u} \in \mathcal{N}$ *such that* $I(\bar{u}) = \inf_{\mathcal{N}} I := c$*, where*

$$\mathcal{N} = \{u \in H_0^1(\Omega) \backslash \{0\}, \langle I'(u), u \rangle = 0\}.$$

Moreover, $m \geq 2c$.

To obtain this result, we must overcome the following difficulties:

1) The fact that $\frac{|u|^{p-2}u \ln u^2}{u^3}$ is not increasing prevent us from using the Nehari manifold method in [17,22].

2) It is more complicated to show the boundedness of minimizing sequences of $c = \inf_{\mathcal{N}} I$ and $m = \inf_{\mathcal{M}} I$.

3) Compared with the case that $a = 1$ and $b = 0$, the presence of the nonlocal term $\left(\int_\Omega |\nabla u|^2 dx\right) \Delta u$ brings us some new troubles. More specifically, the functional:

$$\chi: \ H_0^1(\Omega) \to \mathbb{R} : u \mapsto \int_\Omega |\nabla u|^2 dx \int_\Omega \nabla u \cdot \nabla v$$

is not weakly continuous for any $v \in H_0^1(\Omega)$, which cause great obstacles when proving that the limit of $(PS)_c$ sequence is indeed a nontrivial solution of (1.1).

Next, we give some notations. We denote the ball centered at x with the radius r by $B(x,r)$ and the norm of $L^i(\Omega)$ is denoted by $|\cdot|_i$ for $1 \leq i < \infty$. We shall denote by $C_i, i = 1, 2, \dots$ for various positive constants.

2 Preliminary lemmas

Firstly, we establish an energy estimate inequality related to $I(u)$, $I(su^+ + tu^-)$, $\langle I'(u), u^+ \rangle$ and $\langle I'(u), u^- \rangle$ to overcome the difficulty that the logarithmic nonlinearity $|u|^{p-2}u \ln u^2$ does not satisfy (F).

Lemma 2.1. *For all* $u \in H_0^1(\Omega)$ *and* $s, t \geq 0$*, there holds*

$$\begin{aligned} I(u) \geq\ & I(su^+ + tu^-) + \frac{1-s^p}{p} \langle I'(u), u^+ \rangle + \frac{1-t^p}{p} \langle I'(u), u^- \rangle + \left(\frac{1-s^2}{2} - \frac{1-s^p}{p} \right) \|u^+\|^2 \\ & + \left(\frac{1-t^2}{2} - \frac{1-t^p}{p} \right) \|u^-\|^2 + b \left[\left(\frac{1-s^4}{4} - \frac{1-s^p}{p} \right) |\nabla u^+|_2^4 + \left(\frac{1-t^4}{4} - \frac{1-t^p}{p} \right) |\nabla u^-|_2^4 \right] \\ & + b \frac{s^p + t^p - 2s^2 t^2}{4} |\nabla u^+|_2^2 |\nabla u^-|_2^2. \end{aligned} \tag{2.1}$$

Proof. It is obvious that (2.1) holds for $u = 0$, then we consider the case $u \neq 0$. Through a preliminary calculation, we have

$$2(1 - \tau^p) + p\tau^p \ln \tau^2 > 0, \qquad \forall\, \tau \in (0,1) \cup (1, +\infty). \tag{2.2}$$

Set

$$\Omega_u^+ = \{x \in \Omega : u(x) \geq 0\}, \qquad \Omega_u^- = \{x \in \Omega : u(x) < 0\}.$$

For $u \in H_0^1(\Omega)\backslash\{0\}$, one has

$$\begin{aligned}
&\int_\Omega |su^+ + tu^-|^p \ln(su^+ + tu^-)^2 dx \\
&\quad = \int_{\Omega^+} |su^+ + tu^-|^p \ln(su^+ + tu^-)^2 dx + \int_{\Omega^-} |su^+ + tu^-|^p \ln(su^+ + tu^-)^2 dx \\
&\quad = \int_{\Omega^+} |su^+|^p \ln(su^+)^2 dx + \int_{\Omega^-} |tu^-|^p \ln(tu^-)^2 dx \\
&\quad = \int_\Omega \left[|su^+|^p \ln(su^+)^2 + |tu^-|^p \ln(tu^-)^2\right] dx \\
&\quad = \int_\Omega \left[|su^+|^p \left(\ln(u^+)^2 + \ln s^2\right) + |tu^-|^p \left(\ln(u^-)^2 + \ln t^2\right)\right] dx, \qquad \forall\, s,t \geq 0.
\end{aligned} \tag{2.3}$$

It follows from (1.2), (1.6), (2.2) and (2.3) that

$$\begin{aligned}
&I(u) - I(su^+ + tu^-) \\
&= \frac{1}{2}\left(\|u\|^2 - \|su^+ + tu^-\|^2\right) + \frac{b}{4}(|\nabla u|_2^4 - |\nabla(su^+ + tu^-)|_2^4) + \frac{2}{p^2}\int_\Omega \left[|u|^p - |su^+ + tu^-|^p\right] dx \\
&\quad - \frac{1}{p}\int_\Omega \left[|u|^p \ln u^2 - |su^+ + tu^-|^p \ln(su^+ + tu^-)^2\right] dx \\
&= \frac{1-s^2}{2}\|u^+\|^2 + \frac{1-t^2}{2}\|u^-\|^2 + \frac{b(1-s^4)}{4}|\nabla u^+|_2^4 + \frac{b(1-t^4)}{4}|\nabla u^-|_2^4 + \frac{b(1-s^2t^2)}{2}|\nabla u^+|_2^2|\nabla u^-|_2^2 \\
&\quad + \frac{2}{p^2}\int_\Omega \left[|u^+|^p - |su^+|^p + |u^-|^p - |tu^-|^p\right] dx \\
&\quad - \frac{1}{p}\int_\Omega \left[|u^+|^p \ln(u^+)^2 - |su^+|^p \ln(u^+)^2 - |su^+|^p \ln s^2\right] dx \\
&\quad - \frac{1}{p}\int_\Omega \left[|u^-|^p \ln(u^-)^2 - |tu^-|^p \ln(u^-)^2 - |tu^-|^p \ln t^2\right] dx \\
&= \frac{1-s^p}{p}\langle I'(u), u^+\rangle + \frac{1-t^p}{p}\langle I'(u), u^-\rangle + \left(\frac{1-s^2}{2} - \frac{1-s^p}{p}\right)\|u^+\|^2 + \left(\frac{1-t^2}{2} - \frac{1-t^p}{p}\right)\|u^-\|^2 \\
&\quad + b\left[\left(\frac{1-s^4}{4} - \frac{1-s^p}{p}\right)|\nabla u^+|_2^4 + \left(\frac{1-t^4}{4} - \frac{1-t^p}{p}\right)|\nabla u^-|_2^4 + \frac{s^p + t^p - 2s^2t^2}{4}|\nabla u^+|_2^2|\nabla u^-|_2^2\right] \\
&\quad + \frac{2(1-s^p) + ps^p \ln s^2}{p^2}\int_\Omega |u^+|^p dx + \frac{2(1-t^p) + pt^p \ln t^2}{p^2}\int_\Omega |u^-|^p dx \\
&\geq \frac{1-s^p}{p}\langle I'(u), u^+\rangle + \frac{1-t^p}{p}\langle I'(u), u^-\rangle + \left(\frac{1-s^2}{2} - \frac{1-s^p}{p}\right)\|u^+\|^2 + \left(\frac{1-t^2}{2} - \frac{1-t^p}{p}\right)\|u^-\|^2 \\
&\quad + b\left[\left(\frac{1-s^4}{4} - \frac{1-s^p}{p}\right)|\nabla u^+|_2^4 + \left(\frac{1-t^4}{4} - \frac{1-t^p}{p}\right)|\nabla u^-|_2^4 + \frac{s^p + t^p - 2s^2t^2}{4}|\nabla u^+|_2^2|\nabla u^-|_2^2\right].
\end{aligned} \tag{2.4}$$

Hence, (2.1) holds for all $u \in H_0^1(\Omega)$ and $s,t \geq 0$. □

Let $s = t$ in (2.1), we can obtain the following corollary.

Corollary 2.2. *For all $u \in H_0^1(\Omega)$ and $t \geq 0$, there holds*

$$I(u) \geq I(tu) + \frac{1-t^p}{p}\langle I'(u), u\rangle + \left(\frac{1-t^2}{2} - \frac{1-t^p}{p}\right)\|u\|^2 + b\left(\frac{1-t^4}{4} - \frac{1-t^p}{p}\right)|\nabla u|_2^4. \tag{2.5}$$

In view of Lemma 2.1 and Corollary 2.2, we have the following corollaries.

Corollary 2.3. *For any $u \in \mathcal{M}$, there holds $I(u) = \max_{s,t\geq 0} I(su^+ + tu^-)$.*

Corollary 2.4. *For any $u \in \mathcal{N}$, there holds $I(u) = \max_{t\geq 0} I(tu)$.*

Lemma 2.5. *For any $u \in H_0^1(\Omega)\backslash\{0\}$, there exists an unique $t_u > 0$ such that $t_u u \in \mathcal{N}$.*

Proof. First, we prove the existence of t_u. Let $u \in \mathcal{N}$ be fixed and define a function $g(t) = \langle I'(tu), tu\rangle$ on $[0, +\infty)$. Then,

$$g(t) = \langle I'(tu), tu\rangle = t^2\|u\|^2 + bt^4|\nabla u|_2^4 - \int_\Omega |tu|^p \ln(tu)^2, \qquad \forall\, t > 0. \tag{2.6}$$

It follows from (1.4) and (2.6) that $\lim_{t\to 0^+} g(t) = 0$, $g(t) > 0$ for $t > 0$ small and $g(t) < 0$ for t large. Since $g(t)$ is continuous, there exits $t_u > 0$ such that $g(t_u) = 0$.

Next, we prove the uniqueness of t_u. Arguing by contradiction, we suppose that there exists $u \in H_0^1(\Omega)\backslash\{0\}$ and two positive constants $t_1 \neq t_2$ such that $g(t_1) = g(t_2)$. Since function $f(x) = \frac{1-a^x}{x}$ is monotonically decreasing on $(0, +\infty)$ for $a > 0$ and $a \neq 1$, by (2.5), one has

$$\begin{aligned} I(t_1u) &\geq I(t_2u) + \frac{t_1^p - t_2^p}{t_1^p}\langle I'(t_1u), t_1u\rangle + t_1^2\left[\frac{1-(\frac{t_2}{t_1})^2}{2} - \frac{1-(\frac{t_2}{t_1})^p}{p}\right]\|u\|^2 \\ &\quad + bt^4\left[\frac{1-(\frac{t_2}{t_1})^4}{4} - \frac{1-(\frac{t_2}{t_1})^p}{p}\right]|\nabla u|_2^4 \\ &> I(t_2u), \end{aligned}$$

and

$$\begin{aligned} I(t_2u) &\geq I(t_1u) + \frac{t_2^p - t_1^p}{t_2^p}\langle I'(t_2u), t_2u\rangle + t_2^2\left[\frac{1-(\frac{t_1}{t_2})^2}{2} - \frac{1-(\frac{t_1}{t_2})^p}{p}\right]\|u\|^2 \\ &\quad + bt^4\left[\frac{1-(\frac{t_1}{t_2})^4}{4} - \frac{1-(\frac{t_1}{t_2})^p}{p}\right]|\nabla u|_2^4 \\ &> I(t_1u). \end{aligned}$$

This contradiction shows that $t_u > 0$ is unique for any $u \in H_0^1\backslash\{0\}$. □

Lemma 2.6. *For any $u \in H_0^1(\Omega)$ with $u^\pm \neq 0$, there exists an unique pair (s_u, t_u) of positive numbers such that $s_uu^+ + t_uu^- \in \mathcal{M}$.*

Proof. For any $u \in H_0^1(\Omega)$ with $u^\pm \neq 0$, Let

$$g_1(s,t) = s^2\|u^+\|^2 + bs^4|\nabla u^+|_2^4 + bs^2t^2|\nabla u^+|_2^2|\nabla u^-|_2^2 - \int_\Omega |su^+|^p \ln(su^+)^2 dx, \tag{2.7}$$

and

$$g_2(s,t) = t^2\|u^-\|^2 + bt^4|\nabla u^-|_2^4 + bs^2t^2|\nabla u^+|_2^2|\nabla u^-|_2^2 - \int_\Omega |tu^-|^p \ln(tu^-)^2 dx. \tag{2.8}$$

Using (1.4), it's easy to verify that $g_1(s,s) > 0$ and $g_2(s,s) > 0$ for $s > 0$ small and $g_2(t,t) < 0$ and $g_2(t,t) < 0$ for $t > 0$ large enough. Thus, there exist $0 < r < R$ such that

$$g_1(r,r) > 0, \quad g_2(r,r) > 0; \quad g_1(R,R) < 0, \quad g_2(R,R) < 0. \tag{2.9}$$

From (2.7), (2.8), (2.9), we have

$$g_1(r,t) > 0, \quad g_1(R,t) < 0 \qquad \forall\, t \in [r,R]; \tag{2.10}$$

and

$$g_2(s,r) > 0, \quad g_2(s,R) < 0 \qquad \forall\, s \in [r,R]. \tag{2.11}$$

In view of Miranda's Theorem [20], there exists some point (s_u,t_u) with $r < s_u, t_u < R$ such that $g_1(s_u,t_u) = g_2(s_u,t_u) = 0$, which implies $s_u u^+ + t_u u^- \in \mathcal{M}$. Using (2.1), as a similar argument of Lemma 2.5, we can obtain the uniqueness of (s_u,t_u). □

From Corollaries 2.3, 2.4, Lemmas 2.5 and 2.6, we can obtain the following lemma.

Lemma 2.7. *The following minimax characterizations hold*

$$\inf_{u\in\mathcal{N}} I(u) =: c = \inf_{u\in H_0^1(\Omega), u\neq 0} \max_{t\geq 0} I(tu);$$

and

$$\inf_{u\in\mathcal{M}} I(u) =: m = \inf_{u\in H_0^1(\Omega), u^\pm\neq 0} \max_{s,t\geq 0} I(su^+ + tu^-).$$

Lemma 2.8. *$c > 0$ and $m > 0$ are achieved.*

Proof. For every $u \in \mathcal{N}$, we have $\langle I'(u), u\rangle = 0$. Then by (1.4), (1.5) and the Sobolev embedding theorem, we get

$$\|u\|^2 \leq \|u\|^2 + b|\nabla u|_2^4 = \int_\Omega |u|^p \ln u^2 dx \leq \frac{1}{2}\|u\|^2 + C_1\|u\|^q, \tag{2.12}$$

which implies that there exists a constant $\alpha > 0$ such that $\|u\| \geq \alpha$.

Let $\{u_n\} \subset \mathcal{M}$ be such that $I(u_n) \to m$. By (1.2) and (1.5), one has

$$\begin{aligned} m + o(1) &= I(u_n) - \frac{1}{p}\langle I'(u_n), u_n\rangle \\ &= \left(\frac{1}{2}-\frac{1}{p}\right)\|u_n\|^2 + b\left(\frac{1}{4}-\frac{1}{p}\right)|\nabla u_n|_2^4 + \frac{2}{p^2}\int_\Omega |u_n|^p dx \geq \left(\frac{1}{2}-\frac{1}{p}\right)\|u_n\|^2. \end{aligned} \tag{2.13}$$

This shows that $\{\|u_n\|\}$ is bounded. Thus, passing to a subsequence, we may assume that $u_n^\pm \rightharpoonup \tilde{u}^\pm$ in $H_0^1(\Omega)$ and $u_n^\pm \to \tilde{u}^\pm$ in $L^s(\Omega)$ for $2 \leq s < 2^*$. Since $\{u_n\} \subset \mathcal{M}$, we have $\langle I'(u_n), u_n^\pm\rangle = 0$. Similar as (2.12), there exists a constant $\beta > 0$ such that $\|u_n^\pm\| \geq \beta$. Using (1.4), (1.6) and the boundedness of $\{u_n\}$, we have

$$\beta^2 \leq \|u_n^\pm\|^2 \leq \|u_n^\pm\|^2 + b|\nabla u_n|_2^2|\nabla u_n^\pm|_2^2 = \int_\Omega |u_n^\pm|^p \ln(u_n^\pm)^2 dx \leq \frac{\beta^2}{2} + C_2\int_\Omega |u_n^\pm|^q dx.$$

Thus,

$$\int_\Omega |u_n^\pm|^q dx \geq \frac{\beta^2}{2C_2}.$$

By the compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^s(\Omega)$ for $2 \leq s < 2^*$, we get

$$\int_\Omega |\tilde{u}^\pm|^q dx \geq \frac{\beta^2}{2C_2},$$

which implies $\tilde{u}^{\pm} \neq 0$. By (1.4), (1.5), [28, A.2], the Lebesgue dominated convergence theorem and the weak semicontinuity of norm, we have

$$\|u^{\pm}\|^2 + b|\nabla u|_2^2|\nabla u^{\pm}|_2^2 \leq \varliminf_{n\to\infty} \left(\|u_n^{\pm}\|^2 + b|\nabla u_n|_2^2|\nabla u_n^{\pm}|^2\right) = \varliminf_{n\to\infty} \int_\Omega |u_n^{\pm}|^p \ln(u_n^{\pm})^2 dx \quad (2.14)$$

$$= \int_\Omega |u^{\pm}|^p \ln(u^{\pm})^2 dx, \quad (2.15)$$

which, together with (1.6), implies

$$\langle I'(\tilde{u}), \tilde{u}^+\rangle \leq 0 \quad \text{and} \quad \langle I'(\tilde{u}), \tilde{u}^-\rangle \leq 0. \quad (2.16)$$

In view of Lemma 2.6, there exist two constants $\tilde{s}, \tilde{t} > 0$ such that

$$\tilde{s}\tilde{u}^+ + \tilde{t}\tilde{u}^- \in \mathcal{M} \quad \text{and} \quad I(\tilde{s}\tilde{u}^+ + \tilde{t}\tilde{u}^-) \geq m. \quad (2.17)$$

Thus, it follows from (1.2), (1.5), (2.1), (2.16), (2.17) and the weak semicontinuity of norm that

$$\begin{aligned}
m &= \lim_{n\to\infty} \left[I(u_n) - \frac{1}{p}\langle I'(u_n), u_n\rangle\right] \\
&= \lim_{n\to\infty} \left[\left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 + b\left(\frac{1}{4} - \frac{1}{p}\right)|\nabla u_n|_2^4 + \frac{2}{p^2}\int_\Omega |u_n|^p dx\right] \\
&\geq \left(\frac{1}{2} - \frac{1}{p}\right)\|\tilde{u}\|^2 + b\left(\frac{1}{4} - \frac{1}{p}\right)|\nabla \tilde{u}|_2^4 + \frac{2}{p^2}\int_\Omega |\tilde{u}|^p dx \\
&= I(\tilde{u}) - \frac{1}{p}\langle I'(\tilde{u}), \tilde{u}\rangle \\
&\geq I(\tilde{s}\tilde{u}^+ + \tilde{t}\tilde{u}^-) + \frac{1-\tilde{s}^p}{p}\langle I'(\tilde{u}), \tilde{u}^+\rangle + \frac{1-\tilde{t}^p}{p}\langle I'(\tilde{u}), \tilde{u}^-\rangle - \frac{1}{p}\langle I'(\tilde{u}), \tilde{u}\rangle \\
&\geq m - \frac{\tilde{s}^p}{p}\langle I'(\tilde{u}), \tilde{u}^+\rangle - \frac{\tilde{t}^p}{p}\langle I'(\tilde{u}), \tilde{u}^-\rangle \geq m.
\end{aligned}$$

This shows

$$\langle I'(\tilde{u}), \tilde{u}^{\pm}\rangle = 0, \qquad I(\tilde{u}) = m,$$

i.e. $\tilde{u} \in \mathcal{M}$ and $I(\tilde{u}) = m$. Since $\tilde{u}^{\pm} \neq 0$, then it follows from (2.1) that

$$m = I(\tilde{u}) \geq \frac{1}{p}\langle \Phi'(\tilde{u}), \tilde{u}^+\rangle + \frac{1}{p}\langle \Phi'(\tilde{u}), \tilde{u}^-\rangle + \left(\frac{1}{2} - \frac{1}{p}\right)\|\tilde{u}^+\|^2 + \left(\frac{1}{2} - \frac{1}{p}\right)\|\tilde{u}^-\|^2 > 0.$$

By a similar argument as above, we have that $c > 0$ is achieved. □

Lemma 2.9. *The minimizers of* $\inf_{\mathcal{N}} I$ *and* $\inf_{\mathcal{M}} I$ *are critical points of* I.

Proof. Assume that $\bar{u} = \bar{u}^+ + \bar{u}^- \in \mathcal{M}$, $I(\bar{u}) = m$ and $I'(\bar{u}) \neq 0$. Then there exist $\delta > 0$ and $\varrho > 0$ such that

$$\|I'(u)\| \geq \varrho, \quad \text{for all } \|u - \bar{u}\| \leq 3\delta \text{ and } u \in H_0^1(\Omega).$$

Let $D = (1/2, 3/2) \times (1/2, 3/2)$. By Lemma 2.1, one has

$$\chi := \max_{(s,t)\in\partial D} I(s\bar{u}^+ + t\bar{u}^-) < m. \quad (2.18)$$

For $\varepsilon := \min\{(m-\chi)/3, \varrho\delta/8\}$ and $S := B(\bar{u}, \delta)$, [28, Lemma 2.3] yields a deformation $\eta \in \mathcal{C}([0,1] \times H_0^1(\Omega), H_0^1(\Omega))$ such that

(i) $\eta(1,u) = u$ *if* $I(u) < m - 2\varepsilon$ *or* $I(u) > m + 2\varepsilon$;

(ii) $\eta(1, I^{m+\varepsilon} \cap B(\bar{u}, \delta)) \subset I^{m-\varepsilon}$;

(iii) $I(\eta(1,u)) \leq I(u)$, $\forall\, u \in H_0^1(\Omega)$.

By Lemma 2.1 and (iii), we have

$$\begin{aligned} I(\eta(1, s\bar{u}^+ + t\bar{u}^-)) &\leq I(s\bar{u}^+ + t\bar{u}^-) < I(\bar{u}) \\ &= m, \qquad \forall\, s,t > 0,\ |s-1|^2 + |t-1|^2 \geq \delta^2/\|\bar{u}\|^2. \end{aligned} \tag{2.19}$$

By Corollary 2.3, we can obtain that $I(s\bar{u}^+ + t\bar{u}^-) \leq I(\bar{u}) = m$ for $s,t > 0$, then it follows from (ii) that

$$I(\eta(1, s\bar{u}^+ + t\bar{u}^-)) \leq m - \varepsilon, \qquad \forall\, s,t \geq 0,\ |s-1|^2 + |t-1|^2 < \delta^2/\|\bar{u}\|^2. \tag{2.20}$$

Thus, it follows from (2.19) and (2.20) that

$$\max_{(s,t)\in\overline{D}} I(\eta(1, s\bar{u}^+ + t\bar{u}^-)) < m. \tag{2.21}$$

Define $h(s,t) = s\bar{u}^+ + t\bar{u}^-$. We now prove that $\eta(1, h(D)) \cap \mathcal{M} \neq \emptyset$, contradicting to the definition of m. Let $\beta(s,t) := \eta(1, h(s,t))$,

$$\Psi_1(s,t) := \big(\langle I'(h(s,t)), \bar{u}^+\rangle, \langle I'(h(s,t)), \bar{u}^-\rangle\big)$$

and

$$\Psi_2(s,t) := \left(\frac{1}{s}\langle I'(\beta(s,t)), (\beta(s,t))^+\rangle, \frac{1}{t}\langle I'(\beta(s,t)), (\beta(s,t))^-\rangle\right).$$

Since $\bar{u} \in \mathcal{M}$, by Lemma (2.6), $(s,t) = (1,1)$ is the unique pair of positive numbers such that $s\bar{u}^+ + t\bar{u}^- \in \mathcal{M}$. Furthermore, that $s\bar{u}^+ + t\bar{u}^- \in \mathcal{M}$ is equivalent to that (s,t) is a solution of the following equation

$$\Psi_1(s,t) = (0,0). \tag{2.22}$$

Therefore, (2.22) has an unique solution $(s,t) = (1,1)$ in D. By virtue of the degree theory, we can derive that $\deg(\Psi_1, D, (0,0)) = 1$. It follows from (2.18) and (i) that $\beta = h$ on ∂D. Consequently, we get

$$\deg(\Psi_2, D, (0,0)) = \deg(\Psi_1, D, (0,0)) = 1,$$

which implies that $\Psi_2(s_0, t_0) = 0$ for some $(s_0, t_0) \in D$, that is $\eta(1, h(s_0, t_0)) = \beta(s_0, t_0) \in \mathcal{M}$. This contradiction shows that $I'(\bar{u}) = 0$.

In a similar way as above, we can prove that any minimizer of $\inf_{\mathcal{N}} I$ is a critical point of I. □

3 Proof of Theorem 1.1

In view of Lemmas 2.8 and 2.9, there exist $\tilde{u} \in \mathcal{M}$ such that

$$I(\tilde{u}) = m, \qquad I'(\tilde{u}) = 0. \tag{3.1}$$

Now, we show that $\tilde{u}$ has exactly two nodal domains. Set $\tilde{u} = u_1 + u_2 + u_3$, where

$$u_1 \geq 0, \quad u_2 \leq 0, \quad \Omega_1 \cap \Omega_2 = \emptyset, \quad u_1|_{\mathbb{R}^N\setminus\Omega_1} = u_2|_{\mathbb{R}^N\setminus\Omega_2} = u_3|_{\Omega_1\cup\Omega_2} = 0, \tag{3.2}$$

$$\Omega_1 := \{x \in \Omega : u_1(x) > 0\}, \qquad \Omega_2 := \{x \in \Omega : u_2(x) < 0\},$$

and Ω_1, Ω_2 are connected open subset of Ω. Setting $v = u_1 + u_2$, we have that $v^+ = u_1$ and $v^- = u_2$, i.e. $v^\pm \neq 0$. Note that $I'(\tilde{u}) = 0$, by a preliminary calculation, we can obtain

$$\langle I'(\tilde{u}), v^+ \rangle = -b|\nabla v^+|_2^2 |\nabla u_3|_2^2, \tag{3.3}$$

and

$$\langle I'(\tilde{u}), v^- \rangle = -b|\nabla v^-|_2^2 |\nabla u_3|_2^2. \tag{3.4}$$

It follows from (1.2), (1.5), (2.1), (3.1), (3.2), (3.3) and (3.4) that

$$\begin{aligned}
m = I(\tilde{u}) &= I(\tilde{u}) - \frac{1}{p}\langle I'(\tilde{u}), \tilde{u}\rangle \\
&= I(v) + I(u_3) + \frac{b}{2}|\nabla u_3|_2^2 |\nabla u|_2^2 - \frac{1}{p}\left[\langle I'(v), v\rangle + \langle I'(u_3), u_3\rangle + 2b|\nabla u_3|_2^2|\nabla v|_2^2\right] \\
&\geq \sup_{s,t\geq 0}\left[I(sv^+ + tv^-) + \frac{1-s^p}{p}\langle I'(v), v^+\rangle + \frac{1-t^p}{p}\langle I'(v), v^-\rangle\right] \\
&\quad + I(u_3) - \frac{1}{p}\langle I'(v), v\rangle - \frac{1}{p}\langle I'(u_3), u_3\rangle \\
&\geq \sup_{s,t\geq 0}\left[I(sv^+ + tv^-) + \frac{bs^p}{p}|\nabla v^+|_2^2|\nabla u_3|_2^2 + \frac{bt^p}{p}|\nabla v^-|_2^2|\nabla u_3|_2^3\right] \\
&\quad + (\frac{1}{2} - \frac{1}{p})\|u_3\|^2 + b\left(\frac{1}{4} - \frac{1}{p}\right)|\nabla u_3|_2^4 + \frac{2}{p^2}\int_\Omega |u_3|^p dx \\
&\geq \max_{s,t\geq 0} I(sv^+ + tv^-) + \left(\frac{1}{2} - \frac{1}{p}\right)\|u_3\|^2 \\
&\geq m + \left(\frac{1}{2} - \frac{1}{p}\right)\|u_3\|^2.
\end{aligned}$$

which implies $u_3 = 0$. Therefore, $\tilde{u}$ has exactly two nodal domains.

4 Proof of Theorem 1.2

In view of Lemmas 2.8 and 2.9, there exist $\bar{u} \in \mathcal{M}$ such that

$$I(\bar{u}) = m, \qquad I'(\bar{u}) = 0. \tag{4.1}$$

Furthermore, it follows from (1.2), (3.1), Corollary 2.3 and Lemma 2.7 that

$$\begin{aligned}
m = I(\tilde{u}) &= \sup_{s,t\geq 0} I(s\tilde{u}^+ + t\tilde{u}^-) \\
&= \sup_{s,t\geq 0}\left[I(s\tilde{u}^+) + I(t\tilde{u}^-) + \frac{bs^2t^2}{2}|\nabla \tilde{u}^+|_2^2|\nabla \tilde{u}|_2^2\right] \\
&\geq \sup_{s\geq 0} I(s\tilde{u}^+) + \sup_{t\geq 0} I(t\tilde{u}^-) \geq 2c > 0.
\end{aligned}$$

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