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Aleksandr Lyapunov, the man who created the modern theory of stability*

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Abstract. The outstanding Russian mathematician Aleksandr M. Lyapunov passed away one hundred years ago, on November 6, 1918. Honouring his memory, we recall the main events of his life when he was a student, then from the years in Saint Petersburg until 1885, from the Kharkov period, finally from his second period in Saint Petersburg from 1902. We recount the main fields of his scientific activity (stability theory, potential theory, probability theory, shape of planets) concerning stability theory and chaos in details.

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Aleksandr Mikhailovich Lyapunov, an outstanding scholar not only of the Russian mathematics but, we can say without exaggeration, of the history of mathematics all over the world passed away one hundred years ago, on November 6, 1918.

Lyapunov was born in Jaroslav (Russia) on May 25, 1857. His father was astronomer at the observatory of Kazan; later on he worked as a director of the secondary school of Jaroslav. There were seven children in the family, but only three sons reached adult age. Aleksandr was the eldest son. His brothers also were talented: his elder brother, Sergei was a famous composer, the younger, Boris dealt with Slavic linguistics and became a member of the Russian Academy of Sciences. The little Aleksandr completed his first schooling at home as a private student of his father and, after the death of his father, of his uncle. In 1870, the family moved to Nizhnii-Novgorod. There he was a student of the local secondary school, where he graduated with Gold Medal in 1876. Inspired by the creative atmosphere both at home and among his nearest relatives, by that time he had already developed a passionate interest in sciences.

After completing secondary school, the young Lyapunov enrolled at the Faculty of Mathematics of the University of Saint Petersburg. This period was the golden age of the mathematical school at Saint Petersburg thanks to the founder of the school, the brilliant mathematician Pafnutii L. Chebyshev and to his students, among others Aleksandr N. Korkin and Yegor I.

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Zolotarev. Lyapunov graduated two years earlier than usual, obtaining first-class honours, then he worked at the Department of Theoretical Mechanics of the university. His first period in Saint Petersburg lasted until 1885. Later on, in 1902, after he was elected a full member of the Russian Academy of Sciences, he returned here.

The period 1876–1885 in Saint Petersburg was decisive from the point of view of Lyapunov's whole scientific career thanks to the inspiring atmosphere of the Saint Petersburg mathematical school assured not only by the professors and scholars teaching and doing research here but also by the enriching and stimulating society of students learning at the University. The young man finding his feet was enormously influenced by Chebyshev. We know from his reminiscences that he liked Chebyshev's lectures very much. The character of his later scientific activity was determined by these lectures and the personal advices, opinions, and instructions obtained from his master. Like Chebyshev's other followers, he also had the general conception that also mathematical researches have to remain on the ground of realities. Only research topics originated from applications can be valuable, and only those theories can be really useful which originate from investigations of special cases of the real world.

After graduating, in 1882, Lyapunov visited Chebyshev many times and asked his advice about the topic for a master thesis. Chebyshev expressed his strong opinion that it is not worth dealing with easy mathematical problems solvable by known general methods even if the questions and the answers to the questions are formally new. Every young mathematician well up in scientific research must try to solve some serious mathematical problem of recognized difficulty. In this manner he proposed to Lyapunov, a researcher at the Department of Theoretical Mechanics, the following problem. It is known that the equilibrium forms of liquids rotating with small velocities are ellipsoids, but these ellipsoids disappear at certain critical values of the velocity of the rotation. Is it true that the ellipsoid is transformed at a critical velocity into new equilibrium forms which differ by a small amount from the equilibrium ellipsoid at velocities near the critical values? In modern language, this is a bifurcation problem, which is of basic importance in the development of shapes of planets. Chebyshev told Lyapunov: "If you solve this problem, then you become a well-known mathematician at one stroke." Lyapunov did not obtain any instructions how to start dealing with the problem, nevertheless, he started working with great enthusiasm. Later on he learned that Chebyshev posed the same problem for Zolotarev and Sofia Kovalevskaya as well. It is not known if they dealt with the question. After several unsuccessful trials, Lyapunov laid aside the problem (but only for a while, he often returned to it and published very important results about it until the end of his life) and started dealing with a problem he discovered during the unsuccessful trials: are the equilibrium ellipsoids at small velocities stable? Is it possible that we, and all posterity are indebted to this deviation for the birth of modern stability theory? In 1884, he wrote his master thesis on this topic, which he defended at the University of Saint Petersburg in 1885. One of the referees of the thesis (opponents) was Dimitrii K. Bobilev, who encouraged him to publish the results. Chebyshev and Bobilev were justified by the events: the name of the 27-year-old Lyapunov became well-known due to his thesis. Immediately after the defense, a review was published about it in *Bulletin Astronomique*, then in 1904 *Annales de l'Université de Toulouse* published its complete translation into French. In 1885, Lyapunov was appointed as associate professor (Privatdocent) and was invited to head the Department of Mechanics of the University of Kharkov.

Before describing Lyapunov's Kharkov period, which was extremely rich in results, we have to recall his two personal connections from Saint Petersburg. The first one is Bobilev already mentioned, who was one of his teachers and who had been his mentor since the beginning of his university studies, and with whom he remained in connection until Bobilev's death. Bobilev realized Lyapunov's talent at the first moment and permanently supported him through his scientific carrier. He was the supervisor of Lyapunov's competition essay which was awarded a Gold Medal in 1880. The results of the competition essay were published in two scientific papers. In his reminiscences Lyapunov expressed his gratitude to his former teacher for the support of almost 40 years emphasizing the importance for him of the attention and care, that the busy scholar always found time for him, helped him making the first steps, directed him in labyrinths of the scientific literature, weeded out his first, often naive texts from his young days. This also shows the importance and significance of a good teacher and mentor in the development even of the most outstanding talents. Lyapunov's other important link developed in Saint Petersburg and also lasting for a lifetime was one with Andrei A. Markov. In spite of the fact that Markov was only one year older than Lyapunov, his influence on the development of Lyapunov's scientific activity was decisive. There is no doubt that without this connection Lyapunov could not have reached such significant results also in probability theory.

Lyapunov started working in Kharkov with a great enthusiasm. He took over and reformed the courses in mechanics both at the University and at the Technical College of Kharkov immediately. He wrote new textbooks and lecture notes. We can be convinced of the high quality of these materials also today because in 1982 the Ukrainian Academy of Sciences published them [4]. In addition to his teaching duties he tried to find time also for scientific research, but this was essentially more difficult than it was at Saint Petersburg. He had an enormous working capacity. He worked also at night typically until 4–5 o'clock, but it often happened also that he went to the university to give lectures without sleeping. Perhaps this resulted in the steep rise of his scientific star. He published two fundamental papers in stability theory, which he had started dealing with while at Saint Petersburg. Lyapunov established fundamental results in probability theory, in the topic of central limit theorems. He became interested in this topic back in Saint Petersburg, at Chebyshev's lectures. These results were presented by Markov to the Russian Academy of Sciences. Moreover, what is the most important for us, mathematicians dealing with dynamics: in 1888, he started publishing his general results about stability of motions of mechanical systems of finite degree of freedom. In 1892, he issued the work "The general problem of stability of motions", in which he established the modern abstract mathematical theory of stability (later on we will return to this moment). He submitted this work as doctoral thesis to the University of Moscow and defended it in September of 1892. One of the referees (opponents) was Nikolai Zhukovskii, the "father of flying". Similarly to his master thesis, a complete translation into French was brought out in *Annales de l'Université de Toulouse*. In 1893, he was appointed full professor at the University of Kharkov.

During both periods of his life at Saint Petersburg, Lyapunov was an extraordinarily reserved man. He was in contact only with his nearest relatives, he maintained relationship only with very few of his university colleagues as well. His way of life and behaviour were absolutely different in Kharkov. Daytime he was usually in his university work-room, whose door was always open for his colleagues and pupils. They used to have informal fruitful chats, of course mainly about mathematics. His best student was the future academician Vladimir A. Steklov, who described his first meeting with Lyapunov. To understand the story it is necessary to know the political atmosphere at universities in Russia in those years. In 1863, a new activity regulation was introduced for the higher education guaranteeing essentially more rights to students than the earlier one, and several privileges of nobility were ceased. In 1884, a new minister of education was appointed named Ivan Delyanov, who started liquidating the new attainments of 1863; for example, he wanted to ban women from universities. The majority of students were opposed to these reactionary steps and started demonstrations. When the news circulated through Kharkov that a new professor was appointed to the university, the belief was spread that the new professor is also a “reactionary creation of Delyanov". Students decided to make his duties impossible. For the first lecture the new lecturer was accompanied into the auditorium filled to capacity and was introduced to audience by the dean of the faculty. Already it was a surprise that instead of an “old reactionary creation" it was a definitely handsome young man, scarcely older than the students, who entered the hall in company of the dean. When the dean left the hall, the new lecturer immediately started speaking about mechanics in a voice a little bit trembling with excitement. Then came the real surprise. Mechanics was not a favourite subject of students. The new lecturer spoke about the concepts which seemed earlier dry and mysterious, as thrillingly and clearly that the prejudice suddenly dissolved, and since that day Lyapunov held a high position in students' estimation. He achieved great respect with his enthusiasm, thoroughness, deep knowledge and with his purpose to lift audience to a very high level of learning. Due to his efforts, these heights were also approached by those students not particularly interested in mechanics.

Lyapunov actively took part in the public life at the University of Kharkov. He was characterized by one of his professor-colleagues in the following way: “A. M. Lyapunov belonged to the professors who created the real soul of the university, who made the university living and prospering, who personified the ideal of the teacher and the scholar. He was a man without any evil intent, who was permanently living in the pure atmosphere of the science." His activity at the Mathematical Society of Kharkov was also significant. In the period 1899–1902 he was the president of the Society and editor of the Proceedings of the Society. He presented all of his results at meetings of the Society where his students, Steklov and Nikolai N. Saltikov also often gave lectures.

Lyapunov was elected in 1900 a corresponding member, in 1901 a full member of the Russian Academy of Sciences, and he was appointed head of the Department of Applied Mathematics of Saint Petersburg, which position had been unfilled from 1894, following the death of Chebyshev. In 1902, he moved back to Saint Petersburg and the Kharkov period of his life ended. He always affectionately thought of the years spent there; by Steklov's telling he considered these the happiest period of his life.

From 1902, Lyapunov lived in Saint Petersburg except the last year of his life spent in Odessa. In his second period in Saint Petersburg he did not teach any more, all of his time was devoted to scientific research. He returned to the problem which was posed to him by Chebyshev in 1882, to the problem of development of the equilibrium form of heavy liquid rotating with varying velocity. Finally he succeeded in solving the problem completely. This steadfastness of purpose, grim determination, and scientific effort was rightly called by Steklov a real feat. In Saint Petersburg his life was rather aloof. He was not an outgoing man, he did not visit social events except concerts of his brother. His only relaxation was finding opportunity from time to time for admiration of nature. He liked very much planting and taking care of garden and indoor plants and trees, his flat was decorated with ficuses and plants.

He had active correspondence with several mathematicians from abroad, among them Henri Poincaré and Émile Picard. In 1908, he was invited to the Fourth International Congress of Mathematicians in Rome, where he met many of his pen-friends. Unfortunately, he never succeeded in meeting personally Poincaré, with whom he had the strongest scientific connection.

From 1909, he took part in the great venture of the Russian Academy of Sciences in collecting and publishing all works of Leonhard Euler. He was the editor of the 18th and 19th volumes of the mathematical works.

He obtained several acknowledgements for his scientific and pedagogical services. He was honorary doctor of universities of Saint Petersburg, Kharkov, and Kazan, external member of the Accademia dei Lincei of Rome, corresponding member of the Academy of Paris, honorary member of the Mathematical Society of Kharkov, and member of several other scientific societies.

In June 1917, he moved together with his wife to Odessa, where his younger brother lived. In the second year of his stay in Kharkov Lyapunov had married his second cousin, who had shared being a private student of his father at Jaroslav. At the beginning of the 1900s, his wife came down with pulmonary tuberculosis and her state relapsed more and more. Doctors suggested change of air, they traveled to Odessa for this reason. Lyapunov launched a special course for 1918/19 at the University of Odessa about “Shapes of Celestial Bodies". Unfortunately, his wife died on October 31, 1918. On the same day Lyapunov put a bullet through his own head with a revolver. He asked to be buried into the same grave with his wife. He passed away on November 3, 1918.

Lyapunov achieved epoch-marking scientific results mainly in four areas of mathematics and mechanics:

1. Stability theory
2. Potential theory
3. Probability theory
4. Shapes of celestial bodies

Stability theory can be divided into two significantly different periods: one before Lyapunov and one starting with Lyapunov’s activity. The milestone is Lyapunov’s doctoral thesis from 1892 mentioned earlier. Before Lyapunov, stability theory as a part of mechanics meant the collection of investigations for certain stability properties of different mechanical systems without even having an exact definition of stability, i.e., there existed no real “theory" of stability. The most general and the most celebrated result was Joseph-Louis Lagrange’s theorem saying that the equilibrium state of a conservative mechanical system is stable if the potential energy has a minimum at the corresponding equilibrium position. Lagrange could prove this theorem only in a special case, for the general case Peter Gustav Lejeune Dirichlet gave a marvelous and simple geometric proof based upon the conservation of the total mechanical energy. Lyapunov’s discovery was nothing else but the realization of the fact that the proof works not only for mechanical systems but also for *arbitrary* systems governed by differential equations (nowadays we call them dynamical systems), provided that we can find an “energy-like" auxiliary function to the system. First of all he defined the stability of a constant solution (equilibrium position) of a system of differential equations saying that this solution is stable if solutions starting from a sufficiently small neighbourhood of the equilibrium position remain arbitrarily near to the equilibrium position for the entire future. In other words, somewhat more generally speaking, if we perturb the initial values of a solution in a sufficiently small amount then the solution starting from this point (perturbed motion) remains arbitrarily near to the original solution (unperturbed motion). Stability problems are difficult mainly for the reason that solutions of differential equations have to be compared on infinite intervals, so numerical methods cannot be applied.

Lyapunov established two methods of stability investigations. These are called the first and the second or direct method of Lyapunov even today. The first method follows the earlier practice of stability investigations, but puts it on a totally new basis. The stability properties of solutions of linear systems of differential equations with constant coefficients were well-known even before Lyapunov. The general solution of such a system is the sum of finite exponential functions, so every stability property is determined by the exponents of these functions. Investigating nonlinear systems one took the first term of the Taylor series of the right-hand side of the system, which is linear, and they dropped the remaining terms saying that they are small in comparison with the linear term. The constant solution of the original nonlinear system was said to be stable if the same constant solution, as a solution of the reduced linear system was stable, and it was said to be unstable otherwise. This method was called stability investigation on the basis of the first approximation. Lyapunov showed in his dissertation that this method having been accepted for hundreds of years can yield false results in certain critical cases. He kept the role of exponential functions, even introducing the concept of characteristic number, he extended this role to linear systems of differential equations with varying coefficients. A real number μ is a characteristic number of a solution if among the functions $\{e^{ct}\}_{c\in\mathbb{R}}$ the function $e^{-\mu t}$ is the "nearest" one to the solution. He showed that solutions of an arbitrary system of linear differential equations with varying coefficients can have only finitely many different characteristic numbers. In addition, he proved that in certain regular cases, the zero solution of the original nonlinear system is stable if all characteristic numbers of the reduced linear system are positive, respectively the zero solution is unstable if the reduced linear system has at least one negative characteristic number. He also showed by examples that if zero is a characteristic number of the reduced linear system (critical case), then the nonlinear terms on the right-hand side can be chosen so that the zero solution of the complete nonlinear system is stable, but they can also be chosen so that it is unstable. This means that the method of stability investigations on the basis of first approximation can produce false results in critical cases.

The opposites of characteristic numbers are called Lyapunov exponents, which play an important role in the mathematical theory of chaos. This theory is concerned with dynamical models possessing unpredictable motions seeming absolutely random in spite of the fact that the models are absolutely deterministic in the sense that all laws of the described systems or phenomena are known. Systems have deterministic mathematical models in the form of exact differential equations not containing any random component, every motion of which can be repeated as many times as you like with the same courses provided you start the system from exactly the same initial state, but motions are extremely sensitive to small changes of the initial values of state variables. Edward Lorenz called this phenomenon "the butterfly effect" drawing the parallel that if somewhere in Brazil a butterfly starts fluttering its wings, it may result in a tsunami in Texas. This phenomenon is made probable by certain Lyapunov exponents. Let $\mathbf{\Phi}(t;\mathbf{u})$ denote the general solution satisfying the initial condition $\mathbf{\Phi}(0;\mathbf{u}) = \mathbf{u}$ with the initial vector $\mathbf{u}$. If we start the system from the initial point $\mathbf{u} + \Delta\mathbf{u}$ and expand the difference of the two solutions into Taylor series, then we get the approximation $\mathbf{\Phi}(t;\mathbf{u}+\Delta\mathbf{u}) - \mathbf{\Phi}(t;\mathbf{u}) \approx D_{\mathbf{u}}\mathbf{\Phi}(t;\mathbf{u})\Delta\mathbf{u}$. This difference can be large for small $\Delta\mathbf{u}$ if the norm of the matrix derived $D_{\mathbf{u}}\mathbf{\Phi}(t;\mathbf{u})$ is large. However, it is known that this derivative matrix satisfies the so-called variational system of equations, which is a system of linear differential equations with varying coefficients. If this system has at least one positive Lyapunov exponent in the above sense then the norm of the derivative matrix may grow exponentially in time, which means that the difference of the two solutions grows exponentially in time for certain $\Delta\mathbf{u}$. If the phase space is bounded then it follows from this fact that the solution has no predictable future since the phase point of the solution does not converge to an equilibrium position or periodic orbit, but it roams between these until the end of time. This is chaos.

In fact, the second or direct method is the extension of Dirichlet's brilliant proof to an arbitrary system of differential equations. It is called "direct" method because its application does not demand the knowledge of the solutions as does the application of the Lagrange–Dirichlet theorem, it is enough to know the energy. In other words, one can deduce properties of solutions directly from the equations without the knowledge of the solutions themselves. This is a very important advantage of the method because we know the solutions very rarely; typically the solutions are not known, even they cannot be expressed by formulae; these are the non-integrable systems. (For example, this has been proved for the system of equations of motion of three bodies.) In Lyapunov's theorems on stability, only the existence of a scalar valued auxiliary function V (Lyapunov function) is required, which is non-increasing along solutions. This explains why we called this function "energy-like": the mechanical energy of a mechanical system is either constant or is dissipated, i.e., it is non-increasing. For example, by the first theorem of the direct method, if one can find such a function to the system, then the equilibrium solution of the system is stable.

Lyapunov introduced a stronger kind of stability important in practice. From experience, a mechanical system under the action of damping "asymptotically" returns to the equilibrium position. This means that if the system starts from a position near the equilibrium with small initial velocity, then the deviation from the equilibrium and the velocity tend to zero as time goes to infinity. Lyapunov called this property asymptotic stability. Recalling the "naive" version of Lyapunov's definition of stability, we can express this property saying that the effect of the perturbations in the initial values dies away. For example, such a behaviour can be observed in a pendulum, provided that we cannot get rid of damping. Like the mechanical energy, the function V is positive definite, therefore, instead of convergence to zero of the state variables it is enough to guarantee that V tends to zero along solutions, i.e., the composite function with outer function V and with a motion as an inner function tends to zero. The brilliance of the method is concealed in the fact that one can compute the derivative of the composite function and check its decrease, what is more, measure its decrease even if one does not know the solution itself. This is due to the fact that the derivative of the inner function (which is the motion) can be substituted by the right-hand side of the equation which is in our hands. By the second theorem of the direct method, if the derivative of V along solutions is negative definite, then the zero solution is asymptotically stable. This theorem can be used to give exact conditions under which the old method of "stability investigations on the basis of first approximation" works.

It had been conjectured already for a long time before Lyapunov that the converse of the Lagrange–Dirichlet theorem also holds true, i.e., if the potential energy of a conservative mechanical system has no strict minimum at the equilibrium position, then the equilibrium is unstable. Among others, for the investigation of this problem Lyapunov also formulated a theorem guaranteeing instability for the equilibrium; this is the third basic theorem of his method. Using this theorem he proved the converse of the Lagrange–Dirichlet theorem in the case when the potential energy was analytic and the lack of the strict minimum could be deduced from the term of the lowest degree in the series expansion of the potential energy.

It is clear for everybody that the direct method is “only" a program for stability investigation. The first step of the program: find a Lyapunov function to the system investigated. The program can say nothing about how to find such a function. For mechanical systems Newton found this function, it is called total mechanical energy. (To be more precise, Newton found one such function, because since that time it turned out that for investigation of certain stability problems in mechanics, several Lyapunov functions different from the energy can be useful, even functions having no mechanical interpretation.) Without doubt, the most difficult step of the method is to find the suitable Lyapunov function. There is no algorithm for this. According to Yevgenii Barbashin, to find a good Lyapunov function is a mixture of fortune and art. Nevertheless, he wrote a monograph on how to make this art in certain particular cases [1].

More than one hundred years have passed since the discovery of the direct method. Knowing the results of this period, we can undoubtedly say that the direct method is the most general and most effective tool of stability investigations in stability theory and its applications among others to technical sciences, natural sciences, especially to biology and economics.

Finally, we briefly review some results achieved by Lyapunov in the other three topics of his activity.

Lyapunov published his results in potential theory in Journal de Mathématiques in 1898. These results were concerned with properties of the potential of single and double layers, with the Dirichlet problem and some basic formulae of the theory of harmonic functions. As is known, Carl Gottfried Neumann suggested a method which was suitable not only for proving the existence of the solution of the Dirichlet problem but also for solving boundary-value problems for the Laplace equation, but the convergence of some infinite series was proved in the method only for the case of convex surfaces. Supposing that Neumann’s method can be applied to the given surface, Lyapunov deduced the basic formulae and the complete theory of harmonic functions independently of the fact that the surface was convex or not.

The central limit theorems about the distribution of sums of independent random variables are very important in probability theory. Chebyshev proved such a theorem under the condition that the random variables can take values increasing to infinity with very small probability, i.e., all moments of the variables are finite. Lyapunov succeeded in removing this serious condition in the theorem. He proved that the theorem is also true if only one moment of every variable is finite. There was another big novelty of Lyapunov’s approach: he used characteristic functions for the proof establishing a very fruitful and efficient method widely applicable in probability theory.

Investigations for the evolution of shapes of celestial bodies originated with Isaac Newton. In his “Principia" he showed that a *homogeneous* body of liquid state rotating with small velocity under the action of gravitational and centrifugal forces has the form of a compressed rotational ellipsoid. Further developing this theory, in his monograph from 1743, Alexis Claude Clairaut studied equilibrium forms of liquids of *non-constant density* rotating with small velocity. He started out from the basic assumption that parts of constant density of the liquid have forms of rotational ellipsoids. Pierre Simon de Laplace went further dropping the condition that the level surfaces are rotational ellipsoids. He divided the inhomogeneous liquid into homogeneous layers and proved that in the first approximation these layers form rotational ellipsoids. Laplace’s investigations were not well-founded mathematically. Lyapunov completed the exact mathematical foundation of the theory. He gave conditions on the density of the fluid under which he could prove the existence of the equilibrium forms. At first, independently of Laplace’s debatable approach, he exactly deduced the nonlinear integral equations modeling the liquid, elaborated a method for the solution of these equations, and compared the results with the theory of Clairaut. In the second technical stage of the work demanding enormous endurance and discipline he checked the convergence of infinite series on which the method was based. It is important to emphasize that, contrary to the earlier investigators, he did not suppose the analyticity of the density function of the liquid. He assumed only the property that the density function decreases going toward the boundary surface, and he allowed that this function may have infinitely many discontinuities. He could solve the Clairaut equation under these conditions using the theory of Stieltjes integral. By his own account this work took 15 years of the second period of his life in Saint Petersburg.

In 1917, when he and his wife moved to Odessa, the University of Odessa asked him to give a course on a subject chosen by him. Lyapunov announced the course titled "Shapes of celestial bodies", in which he intended to summarize the results of his work of the previous 15 years. In spite of the troubled political situation in Russia, his more and more decreasing eyesight, the worsening of his wife's illness he gave seven two-hour lectures from the course and only his tragic death prevented him from fulfilling his duties. This fact characterizes his human probity, his vocation as of a teacher and scholar.

Aleksandr Mikhailovich Lyapunov, the scientific man of celestial bodies was a star of the Russian mathematics running a great arc orbit. The influence of his results both in pure and applied mathematics and in mechanics is inestimable. His legacy, his exemplary scientific and teacher's oeuvre has since then always induced a perpetual respect and appreciation of mathematicians all over the world.

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