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# Positive weak solutions of elliptic Dirichlet problems with singularities in both the dependent and the independent variables

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**Abstract.** We consider singular problems of the form $-\Delta u = k(\cdot, u) - h(\cdot, u)$ in $\Omega$, $u = 0$ on $\partial\Omega$, $u > 0$ in $\Omega$, where $\Omega$ is a bounded $C^{1,1}$ domain in $\mathbb{R}^n$, $n \geq 2$, $h : \Omega \times [0,\infty) \to [0,\infty)$ and $k : \Omega \times (0,\infty) \to [0,\infty)$ are Carathéodory functions such that $h(x,\cdot)$ is nondecreasing, and $k(x,\cdot)$ is nonincreasing and singular at the origin *a.e.* $x \in \Omega$. Additionally, $k(\cdot,s)$ and $h(\cdot,s)$ are allowed to be singular on $\partial\Omega$ for $s > 0$. Under suitable additional hypothesis on $h$ and $k$, we prove that the stated problem has a unique weak solution $u \in H_0^1(\Omega)$, and that $u$ belongs to $C(\overline{\Omega})$. The behavior of the solution near $\partial\Omega$ is also addressed.

**Keywords:** singular elliptic problems, variational problems, sub-supersolutions method, finite energy solutions, positive solutions.

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## 1 Introduction and statement of the main results

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with $C^{1,1}$ boundary, and consider a singular semilinear elliptic problem of the form

$$\begin{cases} -\Delta u = k(\cdot, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $k : \Omega \times (0,\infty) \to [0,\infty)$ is a Carathéodory function (i.e., $k(\cdot,s)$ is measurable for any $s \in (0,\infty)$ and $k(x,\cdot)$ is continuous on $(0,\infty)$ *a.e.* $x \in \Omega$), with $k = k(x,s)$ allowed to be singular at $s = 0$.

Singular problems like (1.1) arise, for instance, in the study of chemical catalysts process, non-Newtonian fluids, the temperature of some electrical conductors whose resistance depends on the temperature, thin films, and micro electro-mechanical systems (see e.g., [4,7,13, 15–17,26,33–35], and the references therein).

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Problem (1.1) was studied, in the case where $k(x,s) = a(x)s^{-\alpha}$, under different sets of assumptions on $a$ and $\alpha$, in [2,8,11,13,16,23,36].

In [14] existence and nonexistence theorems were stated for Lane–Emden–Fowler equations with convection and singular potential.

Recently, Chu, Gao and Gao [6], studied problems of the form $-\operatorname{div}(M(x)\nabla u) = a(x)u^{-\alpha(x)}$ in $\Omega$, $u = 0$ on $\partial\Omega$, where $a$ belongs to a suitable Lebesgue space. Among other results, they found a very weak solution in $H_0^1(\Omega)$ (with test functions in $C_c^1(\Omega)$) when $0 < \alpha < 2$ and $0 < \alpha \in C(\overline{\Omega})$.

Problems of the form (1.1), with $k = k(x,s)$ singular at $s = 0$, and with $k(\cdot,s)$ allowed to exhibit some kind of singularity on $\partial\Omega$, were studied in [1,24,28,30,31,37,38].

Diaz, Hernandez and Rakotoson [12] considered the problem

$$\begin{cases} -\Delta u = ad_\Omega^{-\gamma}u^{-\beta} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{1.2}$$

where $d_\Omega := \operatorname{dist}(\cdot,\partial\Omega)$, $\gamma < 2$, and $a \in L^\infty(\Omega)$ satisfies $\inf_\Omega a > 0$. They studied the existence of solutions $u \in L^1(\Omega, d_\Omega)$ (the $d_\Omega$-weighted Lebesgue space) in the following very weak sense:

$$-\int_\Omega u\Delta\varphi = \int_\Omega ad_\Omega^{-\gamma}\varphi \quad \text{for any } \varphi \in C^2(\overline{\Omega}) \text{ such that } \varphi = 0 \text{ on } \partial\Omega. \tag{1.3}$$

Notice that the space of test functions involved in their notion of solution is strictly smaller than the corresponding space in the present paper (as given in Definition 1.1 below). In Theorem 2 they find, when $\beta + \gamma < 1$, a very weak solution of problem (1.2), and prove that it belongs to $W_0^1(\Omega, \|\cdot\|_{N(r),\infty}) \cap W_{\mathrm{loc}}^{2,q}(\Omega)$ for any $r \in (0,1)$ and $q \in [1,\infty)$, where $W_0^1(\Omega, \|\cdot\|_{N(r),\infty})$ is the space of the functions $w : \Omega \to \mathbb{R}$ such that $w \in W^{1,1}(\Omega)$ and $|\nabla w|$ belongs to the Lorentz space $L^{N(r),\infty}(\Omega)$, with $N(r) := \frac{n}{n-1+r}$.

Regarding the case $\beta + \gamma > 1$, in theorem 1 they find a very weak solution of problem (1.2) that belongs to $C(\overline{\Omega}) \cap W_{\mathrm{loc}}^{2,q}(\Omega)$ for any $q \in [1,\infty)$; and in Theorem 5, they prove that, when $\beta + \gamma > 1$ and $\gamma < 2$, the solution that they found belongs to $H_0^1(\Omega)$ if, and only if, $\beta + 2\gamma < 3$. Additionally, in Theorem 4, they prove that, when $\beta + \gamma > 1$, there exist positive constants $c_1$ and $c_2$ such that the found solution $u$ satisfies $c_1 d_\Omega^{\frac{2-\gamma}{1+\beta}} \leq u \leq c_1 d_\Omega^{\frac{2-\gamma}{1+\beta}}$ in $\Omega$. However, it is not obvious that, if $u \in H_0^1(\Omega)$, then $u$ is a weak solution of problem (1.2), i.e., that (1.3) holds for any $\varphi \in H_0^1(\Omega)$.

The existence of classical solutions $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of problem (1.2) was addressed by Mâagli [27] in the case when $a \in C_{\mathrm{loc}}^\sigma(\Omega)$ for some $\sigma \in (0,1)$, and $d_\Omega^\gamma a$ belongs to a suitable class related to the notion of Karamata classes. Our results heavily depend on those found in [27], which are summarized in Remark 2.5 below.

The interested reader can find an updated panoramic view of the area in the research books [19], [32], and in the survey article [18].

In this work we consider problem (1.1) when $k : \Omega \times (0,\infty) \to [0,\infty)$ is a Carathéodory function, $k = k(x,s)$ is allowed to be singular at $s = 0$, in the sense that

$$\lim_{s\to 0^+} k(\cdot,s) = \infty \quad a.e. \text{ in } \Omega,$$

and $k(\cdot,s)$ is allowed to be singular on $\partial\Omega$, in the sense that

$$\lim_{\Omega\ni x\to y} k(x,s) = \infty \quad \text{for any } (y,s) \in \partial\Omega \times (0,\infty).$$

Additionally, we allow the introduction of a second term, and consider the problem

$$\begin{cases} -\Delta u = k(\cdot, u) - h(\cdot, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{1.4}$$

where $h = h(\cdot, s)$ is allowed to be singular on $\partial\Omega$ for any $s > 0$. Under some further assumptions on $k$ and $h$, we prove existence and uniqueness results for weak solutions of problems (1.1) and (1.4).

The notion of weak solution that we use in this work is the usual one, given by the following definition.

**Definition 1.1.** Let $\psi : \Omega \to \mathbb{R}$ be a measurable function such that $\psi\varphi \in L^1(\Omega)$ for all $\varphi$ in $H_0^1(\Omega)$. We say that $u : \Omega \to \mathbb{R}$ is a weak solution of the problem

$$-\Delta u = \psi \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1.5}$$

if $u \in H_0^1(\Omega)$ and $\int_\Omega \langle \nabla u, \nabla \varphi \rangle = \int_\Omega \psi\varphi$ for all $\varphi$ in $H_0^1(\Omega)$.

Similarly, for $u \in H_0^1(\Omega)$, we say that $u$ is a weak supersolution (respectively a weak subsolution) of problem (1.5), and we write

$$\begin{cases} -\Delta u \geq \psi & \text{in } \Omega \text{ (resp. } -\Delta u \leq \psi \text{ in } \Omega\text{ )}, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

to mean $\int_\Omega \langle \nabla u, \nabla \varphi \rangle \geq \int_\Omega \psi\varphi$ (resp. $\int_\Omega \langle \nabla u, \nabla \varphi \rangle \leq \int_\Omega \psi\varphi$) for all nonnegative $\varphi$ in $H_0^1(\Omega)$.

**Definition 1.2.** Let $d_\Omega : \Omega \to \mathbb{R}$ be the distance function $d_\Omega := \operatorname{dist}(\cdot, \partial\Omega)$. For $\beta \geq 0$ and $\gamma \in [0, 2)$ define $\vartheta_{\beta,\gamma} : \Omega \to \mathbb{R}$ by:

- $\vartheta_{\beta,\gamma} := d_\Omega$ if $\beta + \gamma < 1$,
- $\vartheta_{\beta,\gamma} := d_\Omega \ln(\omega_0 d_\Omega^{-1})$ if $\beta + \gamma = 1$, where $\omega_0$ is an arbitrary number, which we fix from now on, such that $\omega_0 > \operatorname{diam}(\Omega)$,
- $\vartheta_{\beta,\gamma} := d_\Omega^{\frac{2-\gamma}{1+\beta}}$ if $\beta + \gamma > 1$.

We assume, from now on, $n \geq 2$. Let us state our main results.

**Theorem 1.3.** *Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with $C^{1,1}$ boundary, and let $k : \Omega \times (0, \infty) \to \mathbb{R}$ satisfy the following conditions:*

*k1) $k$ is a nonnegative Carathéodory function;*

*k2) $s \to k(\cdot, s)$ is nonincreasing on $(0, \infty)$ a.e. $x \in \Omega$;*

*k3) there exist $\beta \geq 0$, $\gamma \geq 0$, and $B_2 > 0$ such that, for any $s > 0$, $k(\cdot, s) \leq B_2 d_\Omega^{-\gamma} s^{-\beta}$ a.e. in $\Omega$;*

*k4) there exist $\delta > 0$ and $B_1 > 0$ such that, for any $s \in (0, \delta)$, $k(\cdot, s) \geq B_1 d_\Omega^{-\gamma} s^{-\beta}$ a.e. in $\Omega$.*

*Let $h : \Omega \times [0, \infty) \to \mathbb{R}$ satisfy the following conditions:*

*h1) $h$ is a Carathéodory function;*

*h2)* $h(x,\cdot)$ *is nondecreasing on* $[0,\infty)$, *and* $h(\cdot,0)=0$ *a.e. in* $\Omega$;

*h3)* $h(\cdot,s)\leq B_3 d_\Omega^{-\eta} s^p$ *a.e. in* $\Omega$ *for all* $s\in[0,\infty)$, *with* $B_3>0$, $p>1$, $0<\eta<\gamma+p+\beta$ *if* $\beta+\gamma\leq 1$, *and* $0<\eta<\gamma+(p+\beta)\frac{2-\gamma}{1+\beta}$ *if* $\beta+\gamma>1$.

*Then:*

*i) If* $\beta+2\gamma<3$, *then problem* (1.4) *has a unique weak solution* $u\in H_0^1(\Omega)$. *Moreover,* $u\in W_{\mathrm{loc}}^{2,q}(\Omega)\cap C(\overline{\Omega})$ *for any* $q\in[1,\infty)$, *and* $u$ *satisfies* $c\vartheta_{\beta,\gamma}\leq u\leq c'\vartheta_{\beta,\gamma}$ *in* $\Omega$, *for some positive constants* $c$ *and* $c'$.

*ii) If problem* (1.4) *has a weak solution* $u\in H_0^1(\Omega)\cap C(\overline{\Omega})$, *then* $\beta+2\gamma<3$.

Note that, in particular, Theorem 1.3 says that $-\Delta u=d_\Omega^{-\gamma}$ in $\Omega$, $u=0$ on $\partial\Omega$, $u>0$ in $\Omega$, has a weak solution if, and only if, $\gamma<\frac{3}{2}$.

The next theorem states that, when $h$ is identically zero, the assertion *i)* of Theorem 1.3 remains valid if the condition *k4)* is replaced by the following milder condition:

*k5)* there exist $\delta>0$ and a measurable set $E\subset\Omega$ such that $|E|>0$ and $\inf_{E\times(0,\delta)}k>0$, where inf stands for the essential infimum, and $|E|$ denotes the Lebesgue measure of $E$.

**Theorem 1.4.** *Let* $\Omega$ *be a bounded domain in* $\mathbb{R}^n$ *with* $C^{1,1}$ *boundary, and let* $k:\Omega\times(0,\infty)\to\mathbb{R}$ *satisfy the conditions k1)–k3) of Theorem* 1.3. *Assume that* $\beta+\gamma<\frac{3}{2}$, *and that the condition k5) holds. Then problem* (1.1) *has a unique weak solution* $u\in H_0^1(\Omega)$, *and* $u\in W_{\mathrm{loc}}^{2,q}(\Omega)\cap C(\overline{\Omega})$ *for any* $q\in[1,\infty)$, *and* $cd_\Omega\leq u\leq c'\vartheta_{\beta,\gamma}$ *in* $\Omega$, *for some positive constants* $c$ *and* $c'$.

Concerning the case when $h$ is nonidentically zero, our next result shows that the assertion *i)* of theorem 1.3 holds under a weaker condition than *k4)*, at the expense of strengthening *h3)*.

**Theorem 1.5.** *Let* $\Omega$ *be a bounded domain in* $\mathbb{R}^n$ *with* $C^{1,1}$ *boundary, and let* $k:\Omega\times(0,\infty)\to\mathbb{R}$ *satisfy the conditions k1)–k3) of Theorem* 1.3, *with* $\beta$ *and* $\gamma$ *satisfying* $\beta>0$, $\gamma\in[0,2)$, *and* $\beta+\gamma<\frac{3}{2}$. *Assume also the following condition:*

*k6) there exist* $\delta>0$ *and* $B_1>0$ *such that, for any* $s\in(0,\delta)$, $k(\cdot,s)\geq B_1 s^{-\beta}$ *a.e. in* $\Omega$.

*Let* $h:\Omega\times[0,\infty)\to\mathbb{R}$ *satisfy the conditions h1) and h2) of Theorem* 1.3, *and the following*

*h4)* $h(\cdot,s)\leq B_3 d_\Omega^{-\eta} s^p$ *a.e. in* $\Omega$ *for all* $s\in[0,\infty)$, *with* $B_3>0$, $p>1$, $0<\eta<p-1$ *if* $\beta+\gamma\leq 1$ *and* $0<\eta<(p-1)\frac{2-\gamma}{1+\beta}$ *if* $1<\beta+\gamma<\frac{3}{2}$.

*Then problem* (1.4) *has a unique weak solution* $u$. *Moreover,* $u\in W_{\mathrm{loc}}^{2,q}(\Omega)\cap C(\overline{\Omega})$ *for any* $q\in[1,\infty)$, *and there exist positive constants* $c$ *and* $c'$ *such that* $c'\vartheta_{\beta,0}\leq u\leq c\vartheta_{\beta,\gamma}$ *in* $\Omega$.

**Remark 1.6.** Let us stress that the strength of the singularity, which is the theme in the background of the present work, needs to be limited if one expects weak solutions in $H_0^1(\Omega)$. Indeed, Lazer and McKenna [23] considered the problem $-\Delta u=au^{-\alpha}$ in $\Omega$, $u=0$ on $\partial\Omega$, $u>0$ in $\Omega$, under the assumptions $a\in C^\gamma(\overline{\Omega})$, $\min_{\overline{\Omega}}a>0$, $\alpha>0$, and $\Omega$ a bounded regular domain. They proved that there exists a unique solution $u\in C^2(\Omega)\cap C(\overline{\Omega})$; and that $u\in H_0^1(\Omega)$ if, and only if, $\alpha<3$. A clear-cut simple condition like that is elusive when the right hand side of the equation is not in the form $au^{-\alpha}$; in [21] we addressed such a more general situation, but still did not consider the case when a spatial singularity is added. This latter situation is considered in the present work.

The paper is organized as follows: in Section 2, we collect some preliminary results. Lemma 2.4 is an adaptation of Lemma 3.2 in [22] and states that, under suitable conditions, a solution in the sense of distributions of an elliptic problem, is also a weak solution in $H_0^1(\Omega)$. Remark 2.5 recalls a result, due to Mâagli [27], about existence, uniqueness, and behavior near the boundary, of positive classical solutions of problems of the form $-\Delta u = ad_\Omega^{-\gamma}u^{-\beta}$ in $\Omega$, $u = 0$ on $\partial\Omega$ (for a suitable class of Hölder continuous functions $a$); and Remark 2.7 recalls a sub-supersolution theorem for singular problems due to Loc and Schmitt [25]. In Section 3 we prove Theorems 1.3, 1.4 and 1.5, by combining the results of [27], [25], and Lemma 2.4, jointly with some additional auxiliary results.

## 2 Preliminaries

For $w \in L^1_{\text{loc}}(\Omega)$, we write, as usual, $w \in \left(H_0^1(\Omega)\right)'$ to mean that $w\varphi \in L^1(\Omega)$ for any $\varphi \in H_0^1(\Omega)$, and that the map $\varphi \to \int_\Omega w\varphi$ is continuous on $H_0^1(\Omega)$.

**Remark 2.1.** Let us recall the Hardy inequality (as stated, e.g., in [29, Theorem 1.10.15], see also [3, p. 313]): There exists a positive constant $c$ such that $\|\frac{\varphi}{d_\Omega}\|_{L^2(\Omega)} \le c\|\nabla\varphi\|_{L^2(\Omega)}$ for all $\varphi \in H_0^1(\Omega)$.

**Remark 2.2.** If $\psi \in L^1_{loc}(\Omega)$ and $d_\Omega\psi \in L^2(\Omega)$, then $\psi\varphi \in L^1(\Omega)$ for any $\varphi \in H_0^1(\Omega)$. Moreover, $\psi \in \left(H_0^1(\Omega)\right)'$ and $\|\psi\|_{(H_0^1(\Omega))'} \le c\|d_\Omega\psi\|_2$ with $c$ independent of $\psi$. Indeed, for $\varphi \in H_0^1(\Omega)$, from the Hölder and the Hardy inequalities, $\int_\Omega|\psi\varphi| \le \|d_\Omega\psi\|_2\|d_\Omega^{-1}\varphi\|_2 \le c\|d_\Omega\psi\|_2\|\varphi\|_{H_0^1(\Omega)}$, where $c$ is the constant in the Hardy inequality of Remark 2.1.

**Remark 2.3.** (See e.g., [10]) $\lambda \in \mathbb{R}$ is called a principal eigenvalue for $-\Delta$ in $\Omega$, with homogeneous Dirichlet condition and weight function $b \in L^\infty(\Omega)$, if the problem $-\Delta\phi = \lambda b\phi$ in $\Omega$, $\phi = 0$ on $\partial\Omega$ has a solution $\varphi_1$ (called a principal eigenfunction) such that $\varphi_1 > 0$ in $\Omega$. It is a well known fact that, for any $C^{1,1}$ bounded domain $\Omega \subset \mathbb{R}^n$, $b \in L^\infty(\Omega)$, and $b^+ \not\equiv 0$, there exists a unique positive principal eigenvalue $\lambda_1(b)$, and its eigenspace $V_{\lambda_1}$ is a one dimensional subspace of $C^1(\overline{\Omega})$. Moreover, for each positive $\varphi_1 \in V_{\lambda_1}$, there are positive constants $c_1, c_2$ such that $c_1d_\Omega \le \varphi_1 \le c_2d_\Omega$ in $\Omega$. Consequently, $|\ln(\varphi_1)| \in L^1(\Omega)$; and $\varphi_1^t \in L^1(\Omega)$ if, and only if, $t > -1$.

The following lemma is an adaptation of Lemma 3.2 in [22]

**Lemma 2.4.** *Let $\psi \in L^\infty_{loc}(\Omega)$ be such that $|\psi| \in \left(H_0^1(\Omega)\right)'$, and let $u \in W^{1,2}_{loc}(\Omega) \cap C(\Omega)$ be a solution, in the sense of distributions, of the problem*

$$-\Delta u = \psi \ \text{ in } \Omega.$$

*If there exist constants $c > 0$ and $r > \frac{1}{2}$ such that $0 \le u \le cd_\Omega^r$ in $\Omega$, then $u \in H_0^1(\Omega) \cap C^1(\Omega) \cap C(\overline{\Omega})$, and $u$ is a weak solution of $-\Delta u = \psi$ in $\Omega$, $u = 0$ on $\partial\Omega$.*

*Proof.* Let $\varphi \in H_0^1(\Omega)$ such that $\operatorname{supp}(\varphi) \subset \Omega$. Then $\int_\Omega\langle\nabla u, \nabla\varphi\rangle = \int_\Omega\psi\varphi$. Indeed, let $\delta > 0$ be such that $\operatorname{supp}(\varphi) \subset \Omega_\delta$, and let $\{\varphi_j\}_{j\in\mathbb{N}}$ be a sequence in $C_c^\infty(\Omega)$ satisfying $\operatorname{supp}(\varphi_j) \subset \Omega_\delta$ for all $j$, and such that $\{\varphi_j\}_{j\in\mathbb{N}}$ converges to $\varphi$ in $H_0^1(\Omega_\delta)$. Now, $\nabla u_{|\Omega_\delta} \in L^2(\Omega_\delta, \mathbb{R}^n)$, and so $\zeta \to \int_{\Omega_\delta}\langle\nabla u, \nabla\zeta\rangle$ is continuous on $H_0^1(\Omega_\delta)$. Also, $\int_\Omega\langle\nabla u, \nabla\varphi_j\rangle = \int_\Omega\psi\varphi_j$ for all $j$. Then $\int_\Omega\langle\nabla u, \nabla\varphi\rangle = \lim_{j\to\infty}\int_\Omega\langle\nabla u, \nabla\varphi_j\rangle = \lim_{j\to\infty}\int_\Omega\psi\varphi_j = \int_\Omega\psi\varphi$.

For each $j \in \mathbb{N}$, let $\mu_j : \mathbb{R} \to \mathbb{R}$ be defined by $\mu_j(s) := 0$ if $s \leq \frac{1}{j}$, $\mu_j(s) := -3j^2s^3 + 14js^2 - 19s + \frac{8}{j}$ if $\frac{1}{j} < s < \frac{2}{j}$, and $\mu_j(s) := s$ if $\frac{2}{j} \leq s$. Then $\mu_j \in C^1(\mathbb{R})$, $\mu_j(s) = 0$ for $s < \frac{1}{j}$, $\mu_j'(s) \geq 0$ for $\frac{1}{j} < s < \frac{2}{j}$, and $\mu_j'(s) = 1$ for $\frac{2}{j} < s$. Also, $0 < \mu_j(s) < s$ for all $s \in \left(0, \frac{2}{j}\right)$.

Let $\mu_j(u) := \mu_j \circ u$. Then, for all $j$, $\nabla\left(\mu_j(u)\right) = \left(\mu_j' \circ u\right)\nabla u$ in $D'(\Omega)$. Since $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$, it follows that $\mu_j(u) \in W^{1,2}_{\mathrm{loc}}(\Omega)$. Since $\operatorname{supp}\left(\mu_j(u)\right) \subset \Omega$, we have $\mu_j(u) \in H^1_0(\Omega)$. Therefore, for all $j$, $\int_\Omega \langle \nabla u, \nabla\left(\mu_j(u)\right)\rangle = \int_\Omega \psi\mu_j(u)$, i.e.,

$$\int_{\{u>0\}} \left(\mu_j' \circ u\right) |\nabla u|^2 = \int_\Omega \psi\mu_j(u). \tag{2.1}$$

Now, $\left(\mu_j' \circ u\right) |\nabla u|^2$ is nonnegative and $\lim_{j\to\infty}\left(\mu_j' \circ u\right)|\nabla u|^2 = |\nabla u|^2$ *a.e.* in $\Omega$, and so, from (2.1) and Fatou's lemma, we have

$$\int_\Omega |\nabla u|^2 \leq \underline{\lim}_{j\to\infty} \int_\Omega \psi\mu_j(u).$$

Let $\varphi_1$ be the principal eigenfunction for $-\Delta$ in $\Omega$ with homogeneous Dirichlet condition, and with weight function $\mathbf{1}$, normalized by $\|\varphi_1\|_\infty = 1$. Since, for some positive constant $c'$, $u \leq c'\varphi_1^r$ in $\Omega$, and $\varphi_1^r \in H^1_0(\Omega)$, we have $\psi u \in L^1(\Omega)$. Now, $\lim_{j\to\infty}\psi\mu_j(u) = \psi u$ in $\Omega$, and, for any $j \in \mathbb{N}$, $\left|\psi\mu_j(u)\right| \leq |\psi u|$ in $\Omega$. Then, Lebesgue's dominated convergence theorem gives

$$\lim_{j\to\infty} \int_\Omega \psi\mu_j(u) = \int_\Omega \psi u < \infty.$$

Thus $\int_\Omega |\nabla u|^2 < \infty$, and so $u \in H^1(\Omega)$. As $-\Delta u = \psi$ in $D'(\Omega)$, $u \in L^\infty(\Omega)$ and $\psi \in L^\infty_{\mathrm{loc}}(\Omega)$, then the inner elliptic estimates (as stated e.g., in [5], Proposition 4.1.2, see also [20], Theorem 9.11) give that $u \in C^1(\Omega)$. From $0 \leq u \leq cd^r_\Omega$ in $\Omega$, and $u \in C(\Omega)$, we conclude that $u \in C\left(\overline{\Omega}\right)$. Since $u \in H^1(\Omega)$, $u \in C\left(\overline{\Omega}\right)$, and $u = 0$ on $\partial\Omega$, we get $u \in H^1_0(\Omega)$.

As $u \in H^1_0(\Omega)$, we have that $\varphi \to \int_\Omega \langle \nabla u, \nabla\varphi\rangle$ is continuous on $H^1_0(\Omega)$. Therefore, since $C^\infty_c(\Omega)$ is dense in $H^1_0(\Omega)$, and since, for any $\varphi \in C^\infty_c(\Omega)$,

$$\int_\Omega \langle \nabla u, \nabla\varphi\rangle = \int_\Omega \psi\varphi, \tag{2.2}$$

we conclude that (2.2) holds for all $\varphi \in H^1_0(\Omega)$. □

**Remark 2.5.** i) Let $\omega_0$ be as in Definition 1.2, $\alpha < 1$, and $\rho < 2$. Let $z \in C([0, \omega_0])$ be such that $z(0) = 0$ and $\int_0^{\omega_0} t^{1-\rho}L_z(t)\,dt < \infty$, where $L_z(t) := \exp\left(\int_t^{\omega_0} \frac{z(s)}{s}ds\right)$. Let $\sigma \in (0,1)$, and let $a \in C^\sigma_{\mathrm{loc}}(\Omega)$ satisfy, for some constant $c > 0$,

$$\frac{1}{c}L_z \circ d_\Omega \leq d^\rho_\Omega a \leq cL_z \circ d_\Omega \quad \text{in } \Omega. \tag{2.3}$$

Then, Theorem 1 in [27] says that the problem

$$\begin{cases} -\Delta u = au^\alpha & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega \end{cases} \tag{2.4}$$

has a unique classical solution $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$; and that, for some positive constant $c'$, $u$ satisfies,

$$\left(c'\right)^{-1}\theta_\rho \circ d_\Omega \leq u \leq c'\theta_\rho \circ d_\Omega \quad \text{in } \Omega,$$

where

$$\theta_\rho(t) := \left(\int_0^{\omega_0} \frac{L_z(s)}{s} ds\right)^{\frac{1}{1-\alpha}} \quad \text{if } \rho = 2,$$
$$\theta_\rho(t) := t^{\frac{2-\rho}{1-\alpha}} (L_z(t))^{\frac{1}{1-\alpha}} \quad \text{if } 1+\alpha < \rho < 2,$$
$$\theta_\rho(t) := t \left(\int_t^{\omega_0} \frac{L_z(s)}{s} ds\right)^{\frac{1}{1-\alpha}} \quad \text{if } \rho = 1+\alpha,$$

and

$$\theta_\rho(t) := t \quad \text{if } \rho < 1+\alpha.$$

ii) Let $\beta \geq 0$, and let $\gamma < 2$. If in i) we take $\alpha := -\beta$, $z := \mathbf{0}$ (then $L_z = \mathbf{1}$), and $\rho := \gamma$, we get that the problem

$$\begin{cases} -\Delta v = d_\Omega^{-\gamma} v^{-\beta} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \\ v > 0 & \text{in } \Omega \end{cases} \tag{2.5}$$

has a unique classical solution $v_{\beta,\gamma} \in C^2(\Omega) \cap C(\overline{\Omega})$; and that there exists positive constants $c_1$ and $c_2$ such that

$$c_1 \vartheta_{\beta,\gamma} \leq v_{\beta,\gamma} \leq c_2 \vartheta_{\beta,\gamma} \text{ in } \Omega, \tag{2.6}$$

where $\vartheta_{\beta,\gamma}$ is as in Definition 1.2.

**Remark 2.6.** Let $\beta \geq 0$ and $\gamma \geq 0$ be such that $\beta + 2\gamma < 3$.

i) $d_\Omega^{1-\gamma} \vartheta_{\beta,\gamma}^{-\beta} \in L^2(\Omega)$. Indeed, if $\beta + \gamma < 1$ then $d_\Omega^{1-\gamma} \vartheta_{\beta,\gamma}^{-\beta} = d_\Omega^{1-\beta-\gamma} \in L^\infty(\Omega) \subset L^2(\Omega)$. If $\beta + \gamma = 1$, then $d_\Omega^{1-\gamma} \vartheta_{\beta,\gamma}^{-\beta} = \left(\ln\left(\frac{\omega_0}{d_\Omega}\right)\right)^\beta \in L^\infty(\Omega) \subset L^2(\Omega)$. If $\beta + \gamma > 1$, then $d_\Omega^{1-\gamma} \vartheta_{\beta,\gamma}^{-\beta} = d_\Omega^{1-\gamma-\beta\frac{2-\gamma}{1+\beta}} = d_\Omega^{-\frac{1}{\beta+1}(\beta+\gamma-1)}$ and, since $\beta + 2\gamma < 3$, $-\frac{2}{\beta+1}(\beta+\gamma-1) > -1$, and so, again in this case, $d_\Omega^{1-\gamma} \vartheta_{\beta,\gamma}^{-\beta} \in L^2(\Omega)$ (because, for $r \in \mathbb{R}$, $d_\Omega^r \in L^2(\Omega)$ whenever $2r + 1 > 0$).

ii) There exist positive constants $c$ and $\tau > \frac{1}{2}$ such that $\vartheta_{\beta,\gamma} \leq c d_\Omega^\tau$ in $\Omega$. Indeed, if $\beta + \gamma < 1$ then $\vartheta_{\beta,\gamma} = d_\Omega$, if $\beta + \gamma = 1$ then $\vartheta_{\beta,\gamma} = d_\Omega \ln\left(\frac{\omega_0}{d_\Omega}\right)$ and so, for any $\varepsilon > 0$, $\vartheta_{\beta,\gamma}(d_\Omega^{1-\varepsilon})^{-1} = d_\Omega^\varepsilon \ln\left(\frac{\omega_0}{d_\Omega}\right) \in L^\infty(\Omega)$; and if $\beta + \gamma > 1$, then $\vartheta_{\beta,\gamma} = d_\Omega^{\frac{2-\gamma}{1+\beta}}$ and, since $\beta + 2\gamma < 3$, $\frac{2-\gamma}{1+\beta} > \frac{1}{2}$.

iii) Let $v_{\beta,\gamma}$ be the solution of problem (2.5) given by Remark 2.5. From i) and (2.6), it follows that $d_\Omega^{1-\gamma} v_{\beta,\gamma}^{-\beta} \in L^2(\Omega)$.

iv) Let $v_{\beta,\gamma}$ be as in iii). Then, by iii) and Remark 2.2, $d_\Omega^{-\gamma} v_{\beta,\gamma}^{-\beta} \in \left(H_0^1(\Omega)\right)'$.

v) Let $v_{\beta,\gamma}$ be as in iii). Since $d_\Omega^{-\gamma} v_{\beta,\gamma}^{-\beta} \in \left(H_0^1(\Omega)\right)'$ and since, by Remark 2.5, $v_{\beta,\gamma} \in C^2(\Omega) \cap C(\overline{\Omega})$, Lemma 2.4 gives that $v_{\beta,\gamma} \in H_0^1(\Omega)$, and that $v_{\beta,\gamma}$ is a weak solution of problem (2.5). Moreover, since $v_{\beta,\gamma} \in L^\infty(\Omega)$ and $d_\Omega^{-\gamma} v_{\beta,\gamma}^{-\beta} \in L^\infty_{\text{loc}}(\Omega)$, the inner elliptic estimates give $v_{\beta,\gamma} \in W^{2,q}_{\text{loc}}(\Omega)$ for any $q \in [1, \infty)$.

**Remark 2.7.** Let $g : \Omega \times (0, \infty) \to \mathbb{R}$ be a Carathéodory function. We say that $w \in L^1_{\text{loc}}(\Omega)$ is a subsolution (supersolution) of the problem

$$-\Delta z = g(\cdot, z) \quad \text{in } \Omega \tag{2.7}$$

in the sense of distributions, if, and only if: $w > 0$ in $\Omega$, $g(\cdot, w) \in L^1_{\mathrm{loc}}(\Omega)$, and $\int_\Omega \langle \nabla w, \nabla \varphi \rangle \leq (\geq) \int_\Omega g(\cdot, w)\varphi$ for all nonnegative $\varphi \in C^\infty_c(\Omega)$. We say that $z \in L^1_{\mathrm{loc}}(\Omega)$ is a solution, in the sense of distributions, of (2.7) if, and only if, $z > 0$ *a.e.* in $\Omega$, and, $\int_\Omega \langle \nabla z, \nabla \varphi \rangle = \int_\Omega g(\cdot, z)\varphi$ for all $\varphi \in C^\infty_c(\Omega)$.

According to Theorem 2.4 in [25], if (2.7) has a subsolution $\underline{z}$ and a supersolution $\overline{z}$ (in the sense of distributions), both in $L^\infty_{\mathrm{loc}}(\Omega) \cap W^{1,2}_{\mathrm{loc}}(\Omega)$, and such such that $0 < \underline{z} \leq \overline{z}$ in $\Omega$, and if there exists $\psi \in L^\infty_{\mathrm{loc}}(\Omega)$ such that $|g(x,s)| \leq \psi(x)$ *a.e.* $x \in \Omega$ for all $s \in [\underline{z}(x), \overline{z}(x)]$; then (2.7) has a solution $z$ in the sense of distributions, which satisfies $\underline{z} \leq z \leq \overline{z}$ in $\Omega$.

## 3 Proof of the main results

**Remark 3.1.** Let $\varphi_1$ be a positive principal eigenfunction of $-\Delta$ in $\Omega$, with homogeneous Dirichlet boundary condition. If $r > \frac{1}{2}$, then $\varphi_1^r \in H^1_0(\Omega)$. Indeed, $\varphi_1^r \in L^2(\Omega)$. Also, $\varphi_1^{r-1} \in L^2(\Omega)$ and $|\nabla \varphi_1| \in L^\infty(\Omega)$, thus $\nabla(\varphi_1^r) \in L^2(\Omega)$.

**Lemma 3.2.** *Let $\eta$ and $p$ be as in the condition h3) of Theorem 1.3. Then:*

*i)* $d_\Omega^{\gamma-\eta} \vartheta_{\beta,\gamma}^{p+\beta} \in L^\infty(\Omega)$.

*ii) If, in addition, $\beta + 2\gamma < 3$, then $d_\Omega^{1-\eta} \vartheta_{\beta,\gamma}^{p} \in L^2(\Omega)$ and $d_\Omega^{1-\gamma} \vartheta_{\beta,\gamma}^{-\beta} \in L^2(\Omega)$.*

*Proof. i)* follows directly from the definition of $\vartheta_{\beta,\gamma}$ and the facts that $\eta < \gamma + p + \beta$ when $\beta + \gamma \leq 1$, and that $\eta < \gamma + (p+\beta)\frac{2-\gamma}{1+\beta}$ when $\beta + \gamma > 1$, and using, when $\beta + \gamma = 1$, that $d_\Omega^\varepsilon \ln\left(\frac{\omega_0}{d_\Omega}\right) \in L^\infty(\Omega)$ for any $\varepsilon > 0$.

To see the first assertion of *ii)* note that, by *h3)*, $2(1 - \eta + p) > 0$ when $\beta + \gamma \leq 1$. Now, $\left(d_\Omega^{1-\eta}\vartheta_{\beta,\gamma}^p\right)^2 = d_\Omega^{2(1-\eta+p)}$ when $\beta + \gamma < 1$, and $\left(d_\Omega^{1-\eta}\vartheta_{\beta,\gamma}^p\right)^2 = d_\Omega^{2(1-\eta+p)}\left(\ln\left(\frac{\omega_0}{d_\Omega}\right)\right)^{2p}$ when $\beta + \gamma = 1$. Thus, in both cases, $d_\Omega^{1-\eta}\vartheta_{\beta,\gamma}^p \in L^\infty(\Omega) \subset L^2(\Omega)$. If $\beta + \gamma > 1$, then $d_\Omega^{1-\eta}\vartheta_{\beta,\gamma}^p = d_\Omega^{1-\eta+p\frac{2-\gamma}{1+\beta}}$ and, by *h3)*,

$$2\left(1 - \eta + p\frac{2-\gamma}{1+\beta}\right) + 1 > 2\left(1 - \left(\gamma + (p+\beta)\frac{2-\gamma}{1+\beta}\right) + p\frac{2-\gamma}{1+\beta}\right) + 1$$
$$= \frac{3 - \beta - 2\gamma}{\beta + 1} > 0.$$

Thus, again in this case, $d_\Omega^{1-\eta}\vartheta_{\beta,\gamma}^p \in L^2(\Omega)$.

Finally, $d_\Omega^{1-\gamma}\vartheta_{\beta,\gamma}^{-\beta} = d_\Omega^{1-\beta-\gamma} \in L^\infty(\Omega) \subset L^2(\Omega)$ when $\beta + \gamma < 1$, and $d_\Omega^{1-\gamma}\vartheta_{\beta,\gamma}^{-\beta} = \left(\ln\left(\frac{\omega_0}{d_\Omega}\right)\right)^{-\beta} \in L^\infty(\Omega) \subset L^2(\Omega)$ when $\beta + \gamma = 1$. If $\beta + \gamma > 1$, then $d_\Omega^{1-\gamma}\vartheta_{\beta,\gamma}^{-\beta} = d_\Omega^{1-\gamma-\beta\frac{2-\gamma}{1+\beta}}$ and, since $2\left(1 - \gamma - \beta\frac{2-\gamma}{1+\beta}\right) + 1 = \frac{3-\beta-2\gamma}{\beta+1} > 0$, we have, again in this case, $d_\Omega^{1-\gamma}\vartheta_{\beta,\gamma}^{-\beta} \in L^2(\Omega)$. □

**Remark 3.3.** Assume the conditions k1), k3), h1), and h3) of Theorem 1.3. Assume also that $\beta + 2\gamma < 3$. Then, for any $\varepsilon > 0$, $k(\cdot, \varepsilon v_{\beta,\gamma})$ and $h(\cdot, \varepsilon v_{\beta,\gamma})$ belong to $\left(H^1_0(\Omega)\right)'$. Indeed, by k1) and h1), $k(\cdot, \varepsilon v_{\beta,\gamma})$ and $h(\cdot, \varepsilon v_{\beta,\gamma})$ are measurable functions, and by k3) and h3),

$$d_\Omega k(\cdot, \varepsilon v_{\beta,\gamma}) \leq \varepsilon^{-\beta} B_2 d_\Omega^{1-\gamma} v_{\beta,\gamma}^{-\beta} \leq \varepsilon^{-\beta} B_2 c_2^{-\beta} d_\Omega^{1-\gamma} \vartheta_{\beta,\gamma}^{-\beta} \quad \text{a.e. in } \Omega,$$
$$d_\Omega h(\cdot, \varepsilon v_{\beta,\gamma}) \leq \varepsilon^{p} B_3 d_\Omega^{1-\eta} v_{\beta,\gamma}^{p} \leq \varepsilon^{p} B_3 c_3^{p} d_\Omega^{1-\eta} \vartheta_{\beta,\gamma}^{p} \quad \text{a.e. in } \Omega.$$

Then, by Lemma 3.2, $d_\Omega k\left(\cdot, \varepsilon v_{\beta,\gamma}\right)$ and $d_\Omega h\left(\cdot, \varepsilon v_{\beta,\gamma}\right)$ belong to $L^2(\Omega)$, and so, by Remark 2.2, $k\left(\cdot, \varepsilon v_{\beta,\gamma}\right)$ and $h\left(\cdot, \varepsilon v_{\beta,\gamma}\right)$ belong to $\left(H_0^1(\Omega)\right)'$.

**Lemma 3.4.** *Assume the conditions k1), k3), k4), h1), and h3), of Theorem 1.3, and let $v_{\beta,\gamma}$ be the solution, given by Remark 2.5, of problem (2.5). Then, for any $\varepsilon$ positive and small enough, $\varepsilon v_{\beta,\gamma}$ is a subsolution, in the sense of distributions, of problem (1.4) and, if in addition, $\beta + 2\gamma < 3$, then $\varepsilon v_{\beta,\gamma}$ is a weak subsolution of (1.4).*

*Proof.* By Lemma 3.2 *i)*, there exists a positive constant $c_1$ such that $d_\Omega^{-\eta}\vartheta_{\beta,\gamma}^p \leq c_1 d_\Omega^{-\gamma}\vartheta_{\beta,\gamma}^{-\beta}$ in $\Omega$, and by Remark 2.5, there exist positive constants $c_2$ and $c_3$ such that $c_2\vartheta_{\beta,\gamma} \leq v_{\beta,\gamma} \leq c_3\vartheta_{\beta,\gamma}$ in $\Omega$. Thus, for some positive constant $c_4$, $d_\Omega^{-\eta}v_{\beta,\gamma}^p \leq c_1 d_\Omega^{-\gamma}v_{\beta,\gamma}^{-\beta}$ in $\Omega$. Then, for any $\varepsilon$ positive and small enough, $\frac{1}{2}\varepsilon^{-\beta}B_1 d_\Omega^{-\gamma}v_{\beta,\gamma}^{-\beta} \geq \varepsilon^p B_3 d_\Omega^{-\eta}v_{\beta,\gamma}^p$ in $\Omega$. By diminishing $\varepsilon$ if necessary, we can assume that, in addition, $\varepsilon < \min\left\{1, \frac{\delta}{\|v_{\beta,\gamma}\|_\infty}, \frac{1}{2}B_1\right\}$. By Remark 2.5, $v_{\beta,\gamma}$ satisfies, in the sense of distributions,

$$-\Delta v_{\beta,\gamma} = d_\Omega^{-\gamma}v_{\beta,\gamma}^{-\beta} \quad \text{in } \Omega. \tag{3.1}$$

Then, in the sense of distributions,

$$\begin{aligned} -\Delta\left(\varepsilon v_{\beta,\gamma}\right) &= \varepsilon d_\Omega^{-\gamma}v_{\beta,\gamma}^{-\beta} \leq \frac{1}{2}\varepsilon^{-\beta}B_1 d_\Omega^{-\gamma}v_{\beta,\gamma}^{-\beta} \\ &\leq \varepsilon^{-\beta}B_1 d_\Omega^{-\gamma}v_{\beta,\gamma}^{-\beta} - \varepsilon^p B_3 d_\Omega^{-\eta}v_{\beta,\gamma}^p \leq k\left(\cdot, \varepsilon v_{\beta,\gamma}\right) - h\left(\cdot, \varepsilon v_{\beta,\gamma}\right) \quad \text{in } \Omega; \end{aligned} \tag{3.2}$$

where, in the last inequality, we have used that, since $\varepsilon < \frac{\delta}{\|v_{\beta,\gamma}\|_\infty}$ we have $\varepsilon v_{\beta,\gamma} \leq \delta$ in $\Omega$, and then, by *k4)*, $k\left(\cdot, \varepsilon v_{\beta,\gamma}\right) \geq \varepsilon^{-\beta}B_1 d_\Omega^{-\gamma}v_{\beta,\gamma}^{-\beta}$ *a.e.* in $\Omega$. Thus, for any nonnegative $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega \left\langle \nabla\left(\varepsilon v_{\beta,\gamma}\right), \nabla\varphi \right\rangle \leq \int_\Omega \left(k\left(\cdot, \varepsilon v_{\beta,\gamma}\right) - h\left(\cdot, \varepsilon v_{\beta,\gamma}\right)\right)\varphi. \tag{3.3}$$

and so $\varepsilon v_{\beta,\gamma}$ is a subsolution, in the sense of distributions, of problem (1.4). Now suppose $\beta + 2\gamma < 3$. By Remark 2.6 *v)*, $\varepsilon v_{\beta,\gamma} \in H_0^1(\Omega)$, and by Remark 3.3, $k\left(\cdot, \varepsilon v_{\beta,\gamma}\right)$ and $h\left(\cdot, \varepsilon v_{\beta,\gamma}\right)$ belong to $\left(H_0^1(\Omega)\right)'$. Thus $\left(k\left(\cdot, \varepsilon v_{\beta,\gamma}\right) - h\left(\cdot, \varepsilon v_{\beta,\gamma}\right)\right)\varphi \in L^1(\Omega)$ for any $\varphi \in H_0^1(\Omega)$ and, since $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, it follows that (3.3) holds for any nonnegative $\varphi \in H_0^1(\Omega)$. □

**Remark 3.5.** Let us recall the following well known result: Let $g : \Omega \times (0, \infty) \to \mathbb{R}$ be a Carathéodory function such that $s \to g(x, s)$ is nonincreasing for *a.e.* $x \in \Omega$, and consider the problem

$$\begin{cases} -\Delta u = g(\cdot, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega. \end{cases} \tag{3.4}$$

Let $\underline{u} \in H_0^1(\Omega)$ be a weak subsolution of problem (3.4) and let $\overline{u} \in H_0^1(\Omega)$ be a weak supersolution of the same problem. Then $\underline{u} \leq \overline{u}$ *a.e.* in $\Omega$. Indeed, we have, in weak sense, $-\Delta\left(\underline{u} - \overline{u}\right) \leq g\left(\cdot, \underline{u}\right) - g\left(\cdot, \overline{u}\right)$, $\underline{u} - \overline{u} = 0$ on $\partial\Omega$. Taking $\left(\underline{u} - \overline{u}\right)^+$ as test function, and noting that $\left(g\left(\cdot, \underline{u}\right) - g\left(\cdot, \overline{u}\right)\right)\left(\underline{u} - \overline{u}\right)^+ \leq 0$ *a.e.* in $\Omega$, we conclude that $\underline{u} \leq \overline{u}$.

**Lemma 3.6.** *Assume the conditions k1), k4), h1), and h3) of Theorem 1.3. If problem (1.4) has a weak solution $u \in H_0^1(\Omega) \cap C\left(\overline{\Omega}\right)$, then $\gamma \leq \frac{3}{2}$.*

*Proof.* Let $u \in H_0^1(\Omega) \cap C(\overline{\Omega})$ be a weak solution of problem (1.4). For $\rho > 0$, let $A_\rho := \{x \in \Omega : d_\Omega(x) \leq \rho\}$. Since $u \in C(\overline{\Omega})$ and $u = 0$ on $\partial\Omega$, there exists $\rho > 0$ such that $u \leq \delta$ in $A_\rho$. Then, by *k4)*,

$$k(\cdot, u) \geq B_1 d_\Omega^{-\gamma} u^{-\beta} \quad a.e. \text{ in } A_\rho. \tag{3.5}$$

Let $\varphi_1$ be a positive principal eigenfunction for $-\Delta$ in $\Omega$, with homogeneous Dirichlet condition and weight function 1. Let $\varepsilon > 0$, and let $\varphi := \varphi_1^{\frac{1}{2}+\varepsilon}$. Then $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Note that, by *k1)* and *h1)*, $k(\cdot, u)\varphi$ and $h(\cdot, u)\varphi$ are nonnegative measurable functions, and that, by *h3)*,

$$\begin{aligned} h(\cdot, u)\varphi = d_\Omega h(\cdot, u) d_\Omega^{-1}\varphi &\leq B_3 d_\Omega^{2-\eta} \|u\|_\infty^{p-1} \left|d_\Omega^{-1}u\right| \left|d_\Omega^{-1}\varphi\right| \\ &\leq B_3 \|d_\Omega\|_\infty^{2-\eta} \|u\|_\infty^{p-1} \left|d_\Omega^{-1}u\right| \left|d_\Omega^{-1}\varphi\right| \ a.e. \text{ in } \Omega; \end{aligned}$$

and so, by the Hölder and the Hardy inequalities, $h(\cdot, u)\varphi \in L^1(\Omega)$. Now, taking into account (3.5) and that $k$ is nonnegative,

$$\begin{aligned} B_1 \|u\|_\infty^{-\beta} \int_{A_\rho} d_\Omega^{-\gamma}\varphi \leq B_1 \int_{A_\rho} d_\Omega^{-\gamma} u^{-\beta}\varphi &\leq \int_{A_\rho} k(\cdot, u)\varphi \\ &\leq \int_\Omega k(\cdot, u)\varphi = \int_\Omega \langle \nabla u, \nabla \varphi \rangle + \int_\Omega h(\cdot, u)\varphi < \infty. \end{aligned} \tag{3.6}$$

Thus $\int_\Omega d_\Omega^{-\gamma}\varphi < \infty$, therefore $\int_\Omega d_\Omega^{-\gamma+\frac{1}{2}+\varepsilon} < \infty$. Then $-\gamma + \frac{1}{2} + \varepsilon > -1$, i.e., $\gamma < \frac{3}{2} + \varepsilon$. Since this holds for any $\varepsilon > 0$, the lemma follows. □

**Remark 3.7.** Let us mention that, if the following condition k7) holds:

k7) There exists $B_1 > 0$ such that, for any $s > 0$, $k(\cdot, s) \geq B_1 d_\Omega^{-\gamma} s^{-\beta}$ *a.e.* in $\Omega$,

then the conclusion of Lemma 3.6 remains valid when the assumption $u \in H_0^1(\Omega) \cap C(\overline{\Omega})$ is weakened to $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Indeed, define $\varphi$ as in the proof of Lemma 3.6, and observe that (3.6) holds with $A_\rho$ replaced by $\Omega$.

*Proof of Theorem 1.3..* We first prove *i)*. Let $v_{\beta,\gamma}$ be the solution, given by Remark 2.5, of problem (2.5). By Lemma 3.4, for $\varepsilon$ positive and small enough, $\underline{z} := \varepsilon v_{\beta,\gamma}$ is a weak subsolution of (1.4). Let $\overline{z} := B_2^{\frac{1}{1+\beta}} v_{\beta,\gamma}$. By Remark 3.3, $k(\cdot, \overline{z}) - h(\cdot, \overline{z}) \in \left(H_0^1(\Omega)\right)'$ and so, taking into account Remark 2.6 *v)*, $\overline{z} := B_2^{\frac{1}{1+\beta}} v_{\beta,\gamma}$ is a weak supersolution of problem (1.4). In particular $\underline{z}$ and $\overline{z}$ are a subsolution and a supersolution, respectively, in the sense of distributions, of the problem $-\Delta u = k(\cdot, u) - h(\cdot, u)$ in $\Omega$. By diminishing $\varepsilon$ if necessary we can assume that $\underline{z} \leq \overline{z}$ in $\Omega$. Moreover, by (2.6), there exist positive constants $c_1$ and $c_2$ such that $\underline{z} \geq c_1 \vartheta_{\beta,\gamma}$ and $\overline{z} \leq c_2 \vartheta_{\beta,\gamma}$ in $\Omega$. Thus, for *a.e.* $x \in \Omega$ and for $s \in [\underline{z}(x), \overline{z}(x)]$,

$$\begin{aligned} |k(x,s) - h(x,s)| &\leq k(x, \underline{z}(x)) + h(x, \overline{z}(x)) \\ &\leq B_2 d_\Omega^{-\gamma}(x) c_1^{-\beta} \vartheta_{\beta,\gamma}^{-\beta}(x) + B_3 d_\Omega^{-\eta}(x) c_2^p \vartheta_{\beta,\gamma}^p(x). \end{aligned}$$

Also, $d_\Omega^{-\gamma} \vartheta_{\beta,\gamma}^{-\beta}$ and $d_\Omega^{-\eta} \vartheta_{\beta,\gamma}^p$ belong to $L_{\text{loc}}^\infty(\Omega)$. Then Remark 2.7 gives a solution $u$, in the sense of distributions, of the problem $-\Delta u = k(\cdot, u) - h(\cdot, u)$ in $\Omega$, that satisfies $\underline{z} \leq u \leq \overline{z}$ *a.e.* in $\Omega$. Consequently, for some positive constants $c_1$ and $c_2$,

$$c_1 \vartheta_{\beta,\gamma} \leq u \leq c_2 \vartheta_{\beta,\gamma} \quad a.e. \text{ in } \Omega. \tag{3.7}$$

Therefore, $|k(\cdot,u) - h(\cdot,u)| \leq B_2 d_\Omega^{-\gamma} u^{-\beta} + B_3 d_\Omega^{-\eta} u^p \leq B_2 c_1^{-\beta} d_\Omega^{-\gamma} \vartheta_{\beta,\gamma}^{-\beta} + B_3 c_2^p d_\Omega^{-\eta} \vartheta_{\beta,\gamma}^p$ *a.e.* in $\Omega$. By Remark 2.6, $d_\Omega^{1-\gamma} \vartheta_{\beta,\gamma}^{-\beta} \in L^2(\Omega)$ and by Lemma 3.2, $d_\Omega^{1-\eta} \vartheta_{\beta,\gamma}^p \in L^2(\Omega)$, then, by Remark 2.2, $|k(\cdot,u) - h(\cdot,u)| \in \left(H_0^1(\Omega)\right)'$. Also, from (3.7), $k(\cdot,u) - h(\cdot,u) \in L_{loc}^\infty(\Omega)$ and $u \in L^\infty(\Omega)$. Then, by the inner elliptic estimates, $u \in W_{\mathrm{loc}}^{2,q}(\Omega)$ for any $q \in [1,\infty)$. Therefore $u \in C(\Omega)$. By (3.7), $u$ is also continuous on $\partial\Omega$. Then $u \in C\left(\overline{\Omega}\right)$. Notice that (from the definition of $\vartheta_{\beta,\gamma}$) there exist positive constants $\widetilde{c}_1$, $\widetilde{c}_2$ and $\tau$, such that $\widetilde{c}_1 d_\Omega \leq \vartheta_{\beta,\gamma} \leq \widetilde{c}_2 d_\Omega^\tau$ in $\Omega$, and so, by (3.7), an estimate of the same kind holds for $u$. Then, by Lemma 2.4, $u \in H_0^1(\Omega)$, and $u$ is a weak solution of problem (1.4). The uniqueness assertion of the theorem follows from Remark 3.5.

Now we prove *ii)*. Suppose that $u \in H_0^1(\Omega) \cap C\left(\overline{\Omega}\right)$ is a weak solution of problem (1.4). As $u \in C\left(\overline{\Omega}\right)$, $u > 0$ in $\Omega$, and $0 \leq k(\cdot,u) \leq B_2 d_\Omega^{-\gamma} u^{-\beta}$, we have $k(\cdot,u) \in L_{\mathrm{loc}}^\infty(\Omega)$. Also $h(\cdot,u) \in L_{\mathrm{loc}}^\infty(\Omega)$. Therefore, by the inner elliptic estimates,

$$u \in C\left(\overline{\Omega}\right) \cap W_{\mathrm{loc}}^{2,q}(\Omega) \quad \text{for any } q \in [1,\infty). \tag{3.8}$$

Note that, by Lemma 3.6, $\gamma < 2$. By Lemma 3.4, for $\varepsilon$ positive and small enough, $\varepsilon v_{\beta,\gamma}$ is a subsolution, in the sense of distributions, of (1.4). For $s > 0$, let $g(\cdot,s) := k(\cdot,s) - h(\cdot,s)$. Then

$$-\Delta\left(u - \varepsilon v_{\beta,\gamma}\right) \geq g(\cdot,u) - g\left(\cdot,\varepsilon v_{\beta,\gamma}\right) \quad \text{in } D'(\Omega); \tag{3.9}$$

and, by (3.8) and Remarks 2.5 and 2.6, $u - \varepsilon v_{\beta,\gamma} \in C\left(\overline{\Omega}\right) \cap W_{\mathrm{loc}}^{2,q}(\Omega)$ for any $q \in [1,\infty)$. Then, from (3.9), using a suitable family of mollifiers $\{\varphi_\varepsilon\}_{\varepsilon>0}$, we get

$$-\Delta\left(u - \varepsilon v_{\beta,\gamma}\right) \geq g(\cdot,u) - g\left(\cdot,\varepsilon v_{\beta,\gamma}\right) \quad a.e. \text{ in } \Omega. \tag{3.10}$$

Let us see that the open set $V := \left\{x \in \Omega : u(x) < \varepsilon v_{\beta,\gamma}(x)\right\}$ is empty. By way of contradiction, suppose $V \neq \varnothing$. From *h2)* and *k2)*, $s \to g(\cdot,s)$ is nonincreasing *a.e.* in $\Omega$, and then (3.10) gives that $-\Delta\left(u - \varepsilon v_{\beta,\gamma}\right) \geq 0$ *a.e.* in $V$. Moreover, from $u - \varepsilon v_{\beta,\gamma} \in C\left(\overline{\Omega}\right)$ and $u - \varepsilon v_{\beta,\gamma} = 0$ on $\partial\Omega$, it follows easily that $u - \varepsilon v_{\beta,\gamma} = 0$ on $\partial V$. Then, by the Aleksandrov maximum principle, (as stated, e.g., in [20], Theorem 9.1), $u \geq \varepsilon v_{\beta,\gamma}$ in $V$, which is impossible. Therefore, as stated, $V = \varnothing$, and so

$$u \geq \varepsilon v_{\beta,\gamma} \quad \text{in } \Omega. \tag{3.11}$$

On the other hand, from the definition of $v_{\beta,\gamma}$, *k3)*, and *h2)*,

$$\begin{aligned} -\Delta\left(B_2^{\frac{1}{1+\beta}} v_{\beta,\gamma}\right) &= B_2 d_\Omega^{-\gamma}\left(B_2^{\frac{1}{1+\beta}} v_{\beta,\gamma}\right)^{-\beta} \geq k\left(\cdot, B_2^{\frac{1}{1+\beta}} v_{\beta,\gamma}\right) \\ &\geq k\left(\cdot, B_2^{\frac{1}{1+\beta}} v_{\beta,\gamma}\right) - h\left(\cdot, B_2^{\frac{1}{1+\beta}} v_{\beta,\gamma}\right) = g\left(\cdot, B_2^{\frac{1}{1+\beta}} v_{\beta,\gamma}\right) \text{ in } D'(\Omega) \end{aligned}$$

and so $-\Delta\big(B_2^{\frac{1}{1+\beta}} v_{\beta,\gamma} - u\big) \geq g\big(\cdot, B_2^{\frac{1}{1+\beta}} v_{\beta,\gamma}\big) - g(\cdot,u)$ in $D'(\Omega)$, which gives

$$-\Delta\left(B_2^{\frac{1}{1+\beta}} v_{\beta,\gamma} - u\right) \geq g\left(\cdot, B_2^{\frac{1}{1+\beta}} v_{\beta,\gamma}\right) - g(\cdot,u) \quad a.e. \text{ in } \Omega.$$

Proceeding exactly as in the above proof of $V = \varnothing$ (with $u$ and $\varepsilon v_{\beta,\gamma}(x)$ replaced by $B_2^{\frac{1}{1+\beta}} v_{\beta,\gamma}$ and $u$, respectively) we get that $\widetilde{V} := \left\{x \in \Omega : B_2^{\frac{1}{1+\beta}} v_{\beta,\gamma} < u\right\}$ is empty, and consequently,

$$u \leq B_2^{\frac{1}{1+\beta}} v_{\beta,\gamma} \quad \text{in } \Omega. \tag{3.12}$$

Thus, taking into account (3.11) and (3.12), by Remark 2.5, there exist positive constants $c_1$ and $c_2$ such that

$$c_1\vartheta_{\beta,\gamma} \leq u \leq c_2\vartheta_{\beta,\gamma} \quad \text{in } \Omega. \tag{3.13}$$

In order to conclude the proof, we take $u$ as test function in problem (1.1), to get $\int_\Omega B_2 d_\Omega^{-\gamma} u^{1-\beta} \leq \int_\Omega k(\cdot,u)\, u = \int_\Omega |\nabla(u)|^2 + \int_\Omega h(\cdot,u)\, u < \infty$. Let us consider the three possible cases.

*Case* $\beta+\gamma \leq 1$. Since $\gamma \leq \frac{3}{2}$ we have $\beta+2\gamma = \beta+\gamma+\gamma \leq 1+\gamma < 3$.

*Case* $\beta+\gamma > 1$ *and* $\beta \leq 1$. In this case, by (3.13), $d_\Omega^{-\gamma} u^{1-\beta} \geq c_1^{1-\beta} d_\Omega^{-\gamma} \vartheta_{\beta,\gamma}^{1-\beta} = c_1^{1-\beta} d_\Omega^{-\gamma} d_\Omega^{(1-\beta)\frac{2-\gamma}{1+\beta}}$ in $\Omega$. Since $\int_\Omega d_\Omega^{-\gamma} u^{1-\beta} < \infty$ we get that $-\gamma + (1-\beta)\frac{2-\gamma}{1+\beta} > -1$, i.e., $\beta+2\gamma < 3$.

*Case* $\beta+\gamma > 1$ *and* $\beta > 1$. Here we have, by (3.13), $d_\Omega^{-\gamma} u^{1-\beta} \geq (c'')^{1-\beta} d_\Omega^{-\gamma} \vartheta_{\beta,\gamma}^{1-\beta}$ in $\Omega$ and so, as in the second case, we get $\beta+2\gamma < 3$. □

*Proof of Theorem 1.4.* Assume $\beta+\gamma < \frac{3}{2}$. Let $\delta$ and $E$ be given by *k5)*, let $m_0 := \inf_{E\times(0,\delta)}$, and let $\lambda_E$ and $\phi_E$ be the principal eigenvalue and the positive principal eigenfunction respectively, for $-\Delta$ in $\Omega$, with homogeneous Dirichlet condition and weight function $m_0\chi_E$, with $\phi_E$ normalized by $\|\phi_E\|_\infty = 1$. Let $\varepsilon \in (0, \min\{\delta, \lambda_E^{-1}\})$, and let $\underline{z} := \varepsilon\phi_E$. Then $\underline{z} \in W^{2,q}(\Omega)$ for any $q \in [1,\infty)$ and thus, in particular, $\underline{z} \in C(\overline{\Omega})$. Also,

$$-\Delta\underline{z} = \varepsilon\lambda_E m_0 \chi_E \phi_E \leq m_0\chi_E \leq k(\cdot,\underline{z}) \quad \text{in } D'(\Omega). \tag{3.14}$$

By the properties of the principal eigenfunctions (see Remark 2.3) there exist positive constants $\widetilde{c}_1$ and $\widetilde{c}_2$ such that $\widetilde{c}_1 d_\Omega \leq \underline{z} \leq \widetilde{c}_2 d_\Omega$ in $\Omega$. Let $v_{\beta,\gamma}$ be the solution, given by Remark 2.5, of problem (2.5), and let $\overline{z} := B_2^{\frac{1}{1+\beta}} v_{\beta,\gamma}$. By diminishing $\varepsilon$ if necessary, we can assume $\underline{z} \leq \overline{z}$ in $\Omega$. As in the proof of Theorem 1.3, Remark 2.7 applies to obtain a solution $u$, in the sense of distributions, of the problem $-\Delta u = k(\cdot,u)$ in $\Omega$, such that $\underline{z} \leq u \leq \overline{z}$ *a.e.* in $\Omega$. Then, for some positive constants $c$ and $c'$, $c d_\Omega \leq u \leq c'\vartheta_{\beta,\gamma}$ *a.e.* in $\Omega$. As in the proof of Theorem 1.3, we have $u \in W^{2,q}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$ for any $q \in [1,\infty)$. Note also that, by *k3)*, $d_\Omega k(\cdot,\underline{z}) \leq B_2 \widetilde{c}_1^{-\beta} d_\Omega^{1-\gamma-\beta}$ *a.e.* in $\Omega$, and so, since $d_\Omega^{1-\gamma-\beta} \in L^2(\Omega)$ (because $\beta+\gamma < \frac{3}{2}$), Remark 2.2 gives $k(\cdot,\underline{z}) \in \left(H_0^1(\Omega)\right)'$, and so, as in the proof of Theorem 1.3, using Lemma 2.4, we get $u \in H_0^1(\Omega)$, and that $u$ is a weak solution of problem (1.1). Finally, the uniqueness of the weak solution $u$ is proved, as in Theorem 1.3, using Remark 3.5 . □

**Remark 3.8.** Assume the hypothesis of Theorem 1.5, and let $\eta$ and $p$ be as in the condition *h4) there.* Then, $\eta < p-1$ if $\beta \leq 1$, and $\eta < \frac{2(p-1)}{1+\beta}$ if $\beta > 1$. Indeed, if $\beta+\gamma \leq 1$, then $\eta < p-1$. If $\beta \leq 1$ and $\beta+\gamma > 1$, then $\eta < (p-1)\frac{2-\gamma}{1+\beta} < p-1$ (because $\frac{2-\gamma}{1+\beta} < 1$ when $\beta+\gamma > 1$). If $\beta > 1$, then $\beta+\gamma > 1$, and so $\eta < (p-1)\frac{2-\gamma}{1+\beta} \leq \frac{2(p-1)}{1+\beta}$.

**Lemma 3.9.** *Assume the hypothesis of Theorem 1.5, and let $\eta$ and $p$ be as in the condition h4) there. Then:*

*i)* $d_\Omega^{1-\eta}\vartheta_{\beta,0}^p \in L^2(\Omega)$.

*ii)* $d_\Omega \vartheta_{\beta,0}^{-\beta} \in L^2(\Omega)$.

*iii)* $\vartheta_{\beta,0}^{p+\beta} d_\Omega^{-\eta} \in L^\infty(\Omega)$.

*Proof.* To see *i)*, note that, by *h4)*, $1-\eta+p>0$ when $\beta\leq 1$. If $\beta<1$ then $d_\Omega^{1-\eta}\vartheta_{\beta,0}^{p}=d_\Omega^{1-\eta+p}\in L^\infty(\Omega)\subset L^2(\Omega)$. If $\beta=1$ then $d_\Omega^{1-\eta}\vartheta_{\beta,0}^{p}=d_\Omega^{1-\eta+p}\left(\ln\left(\frac{\omega_0}{d_\Omega}\right)\right)^p\in L^\infty(\Omega)\subset L^2(\Omega)$. If $\beta>1$ then $d_\Omega^{1-\eta}\vartheta_{\beta,0}^{p}=d_\Omega^{1-\eta+\frac{2p}{1+\beta}}$ and, by *h4)*, $1-\eta+\frac{2p}{1+\beta}>1-\left(\frac{2(p-1)}{1+\beta}\right)+\frac{2p}{1+\beta}>0$. Therefore, again in this case, $d_\Omega^{1-\eta}\vartheta_{\beta,0}^{p}\in L^\infty(\Omega)\subset L^2(\Omega)$.

To prove ii), observe that $d_\Omega\vartheta_{\beta,0}^{-\beta}=d_\Omega^{1-\beta}\in L^\infty(\Omega)$ when $\beta<1$, and that $d_\Omega\vartheta_{\beta,0}^{-\beta}=\left(\ln\left(\frac{\omega_0}{d_\Omega}\right)\right)^{-1}\in L^\infty(\Omega)$ when $\beta=1$. Note also that, if $\beta>1$, then $d_\Omega\vartheta_{\beta,0}^{-\beta}=d_\Omega^{1-\beta\frac{2}{1+\beta}}$ and $2\left(1-\beta\frac{2}{1+\beta}\right)+1=\frac{3-\beta}{\beta+1}>0$. Then $d_\Omega\vartheta_{\beta,0}^{-\beta}\in L^\infty(\Omega)\subset L^2(\Omega)$ when $\beta>1$.

To see *iii)*, note that $\vartheta_{\beta,0}^{p+\beta}d_\Omega^{-\eta}=d_\Omega^{p+\beta-\eta}$ when $\beta<1$, and $\vartheta_{\beta,0}^{p+\beta}d_\Omega^{-\eta}=d_\Omega^{-\eta+p+1}\left(\ln\left(\frac{\omega_0}{d_\Omega}\right)\right)^{p+1}$ when $\beta=1$. Since, in both cases, $p+\beta-\eta>\beta+1>0$, we obtain $\vartheta_{\beta,0}^{p+\beta}d_\Omega^{-\eta}\in L^\infty(\Omega)\subset L^2(\Omega)$ when $\beta\leq 1$. If $\beta>1$, then $\vartheta_{\beta,0}^{p+\beta}d_\Omega^{-\eta}=d_\Omega^{-\eta+(p+\beta)\frac{2}{1+\beta}}$ and, by *h4)*, $-\eta+(p+\beta)\frac{2}{1+\beta}>-\left(\frac{2(p-1)}{1+\beta}\right)+(p+\beta)\frac{2}{1+\beta}=2>0$. Thus, again in this case, $\vartheta_{\beta,0}^{p+\beta}d_\Omega^{-\eta}\in L^\infty(\Omega)\subset L^2(\Omega)$. □

**Lemma 3.10.** *Assume the hypothesis of Theorem 1.5 and let $\eta$ and $p$ be as in the condition h4) there. Let $B$ and $D$ be positive constants, let $F:\Omega\times(0,\infty)\to\mathbb{R}$ be defined by*

$$F(\cdot,s):=Ds^{-\beta}-Bd_\Omega^{-\eta}s^p, \tag{3.15}$$

*and let $v_{\beta,0}$ be the solution, provided by Remark 2.5, of problem (2.5) taking there $\gamma=0$. Then, for $\varepsilon$ positive and small enough, $\varepsilon v_{\beta,0}$ is a weak subsolution of the problem*

$$\begin{cases}-\Delta u=F(\cdot,u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\\ u>0 & \text{in } \Omega.\end{cases} \tag{3.16}$$

*Proof.* Note that, for $x\in\Omega$ and $s>0$, $F(x,s)=Ds^{-\beta}-Bd_\Omega^{-\eta}(x)s^p\geq\frac{1}{2}Ds^{-\beta}$ whenever $s^{p+\beta}\leq\frac{1}{2}B^{-1}Dd_\Omega^{\eta}(x)$. Let $Q:=\frac{1}{2}B^{-1}D$. Then

$$F(x,s)\geq\frac{1}{2}Ds^{-\beta}\quad\text{for } x\in\Omega \text{ and } s\in\left(0,Q^{\frac{1}{p+\beta}}d_\Omega^{\frac{\eta}{p+\beta}}(x)\right). \tag{3.17}$$

By Remark 2.5, $v_{\beta,0}\leq c_1\vartheta_{\beta,0}$, with $c_1$ a positive constant. By Lemma 3.9, there exists a positive constant $c_2$ such that $\vartheta_{\beta,0}^{p+\beta}d_\Omega^{-\eta}\leq c_2$ in $\Omega$. Then, for $\varepsilon>0$,

$$\left(\varepsilon v_{\beta,0}\right)^{p+\beta}\leq(\varepsilon c_1)^{p+\beta}c_2d_\Omega^{\eta}\quad\text{in } \Omega. \tag{3.18}$$

Thus, from (3.17) and (3.18), we have, for $\varepsilon$ positive and small enough,

$$F\left(\cdot,\varepsilon v_{\beta,0}\right)\geq\frac{1}{2}D\left(\varepsilon v_{\beta,0}\right)^{-\beta}\quad\text{in } \Omega. \tag{3.19}$$

We have also, for some positive constant $c$,

$$\left|F\left(\cdot,\varepsilon v_{\beta,0}\right)\right|\leq D\varepsilon^{-\beta}v_{\beta,0}^{-\beta}+B\varepsilon^p d_\Omega^{-\eta}v_{\beta,0}^{p}\leq c\left(\vartheta_{\beta,0}^{-\beta}+d_\Omega^{-\eta}\vartheta_{\beta,0}^{p}\right)\quad\text{in } \Omega,$$

the last inequality by Remark 2.5. By Lemma 3.9, $d_\Omega\vartheta_{\beta,0}^{-\beta}\in L^2(\Omega)$ and $d_\Omega^{1-\eta}\vartheta_{\beta,0}^{p}\in L^2(\Omega)$. Then, by Remark 2.2,

$$F\left(\cdot,\varepsilon v_{\beta,0}\right)\in(H_0^1(\Omega))'. \tag{3.20}$$

By diminishing $\varepsilon$, if necessary, we can assume that $\varepsilon \leq \frac{1}{2} D \varepsilon^{-\beta}$ Then (3.19) and (3.20) give that $\varepsilon v_{\beta,0}$ satisfies, in weak sense,

$$\begin{cases} [c]c - \Delta\left(\varepsilon v_{\beta,0}\right) = \varepsilon v_{\beta,0}^{-\beta} \leq \frac{1}{2} D\left(\varepsilon v_{\beta,0}\right)^{-\beta} \leq F\left(\cdot, \varepsilon v_{\beta,0}\right) & \text{in } \Omega, \\ \varepsilon v_{\beta,0} = 0 & \text{on } \partial\Omega. \end{cases}$$

□

**Lemma 3.11.** *Assume the hypothesis of Theorem 1.5 and let $\eta$ and $p$ be as in the condition h4) there. Then:*

*i)* $d_\Omega^{1-\gamma} \vartheta_{\beta,0}^{-\beta} \in L^2(\Omega)$.

*ii)* $d_\Omega^{1-\eta} \vartheta_{\beta,\gamma}^{p} \in L^2(\Omega)$.

*Proof.* To prove *i)*, observe that $d_\Omega^{1-\gamma} \vartheta_{\beta,0}^{-\beta} = d_\Omega^{1-\beta-\gamma}$ when $\beta < 1$, and that $2(1-\beta-\gamma)+1 > 0$ (because $\beta+\gamma < \frac{3}{2}$). Then $d_\Omega^{1-\gamma} \vartheta_{\beta,0}^{-\beta} \in L^2(\Omega)$ when $\beta < 1$. If $\beta = 1$, then $d_\Omega^{1-\gamma} \vartheta_{\beta,0}^{-\beta} = d_\Omega^{-\gamma}\left(\ln\left(\frac{\omega_0}{d_\Omega}\right)\right)^{-1}$ and $\gamma < \frac{3}{2} - \beta = \frac{1}{2}$. Therefore, since $\left(\ln\left(\frac{\omega_0}{d_\Omega}\right)\right)^{-1}$ is bounded, we obtain that $d_\Omega^{1-\gamma} \vartheta_{\beta,0}^{-\beta} \in L^2(\Omega)$ when $\beta = 1$. If $\beta > 1$ then $d_\Omega^{1-\gamma} \vartheta_{\beta,0}^{-\beta} = d_\Omega^{1-\gamma-\beta\frac{2}{1+\beta}}$ and $2\left(1-\gamma-\beta\frac{2}{1+\beta}\right)+1 > 2\left(1-\left(\frac{3}{2}-\beta\right)-\beta\frac{2}{1+\beta}\right)+1 = 2\beta\frac{\beta-1}{\beta+1} > 0$. Thus $d_\Omega^{1-\gamma} \vartheta_{\beta,0}^{-\beta} \in L^2(\Omega)$ when $\beta > 1$.

To prove *ii)*, note that, by *h4)*, $2(1-\eta+p)+1 > 2(1-(p-1)+p)+1 > 0$ when $\beta+\gamma \leq 1$. Then, taking into account that $d_\Omega^{1-\eta} \vartheta_{\beta,\gamma}^{p} = d_\Omega^{1-\eta+p}$ when $\beta+\gamma < 1$, and that $d_\Omega^{1-\eta} \vartheta_{\beta,\gamma}^{p} = d_\Omega^{1-\eta+p}\left(\ln\left(\frac{\omega_0}{d_\Omega}\right)\right)^{p}$ when $\beta+\gamma = 1$, we conclude that $d_\Omega^{1-\eta} \vartheta_{\beta,\gamma}^{p} \in L^2(\Omega)$ when $\beta+\gamma \leq 1$. If $\beta+\gamma > 1$ then $d_\Omega^{1-\eta} \vartheta_{\beta,\gamma}^{p} = d_\Omega^{1-\eta+p\frac{2-\gamma}{1+\beta}}$ and, by *h4)*,

$$\begin{aligned} 2\left(1-\eta+p\frac{2-\gamma}{1+\beta}\right)+1 &> 2\left(1-(p-1)\frac{2-\gamma}{\beta+1}+p\frac{2-\gamma}{1+\beta}\right)+1 = \frac{3\beta-2\gamma+7}{\beta+1} \\ &> \frac{1}{\beta+1}\left(3\beta-2\left(\frac{3}{2}-\beta\right)+7\right) = \frac{5\beta+4}{\beta+1} > 0 \end{aligned}$$

and so, again in this case, $d_\Omega^{1-\eta} \vartheta_{\beta,\gamma}^{p} \in L^2(\Omega)$. □

*Proof of Theorem 1.5.* Let $B_1$, $B_3$, $\eta$, and $p$, be given by *k6)* and *h4)*. Let $v_{\beta,\gamma}$ be the unique classical solution (given by Remark 2.5) of problem (2.5). Let $F$ be defined by (3.15), taking there $D$ such that $0 < D < B_1 \inf_\Omega\left(d_\Omega^{-\gamma}\right)$ and $B = B_3$; and let $\varepsilon > 0$ be small enough such that $\varepsilon v_{\beta,0}$ is a weak subsolution of problem (3.16). Then $\underline{u} := \varepsilon v_{\beta,0}$ is a weak subsolution of problem (1.4). Since $\beta + 2\gamma < \beta + 2\left(\frac{3}{2}-\beta\right) = 3-\beta \leq 3$, by Remark 2.6, $v_{\beta,\gamma} \in H_0^1(\Omega)$, and $v_{\beta,\gamma}$ is a weak solution of problem (2.5). Note that $\overline{u} := B_2^{\frac{1}{1+\beta}} v_{\beta,\gamma}$ is a weak supersolution of problem (1.4), thus, by Remark 3.5, $\underline{u} \leq \overline{u}$ in $\Omega$. Also, by (2.6), there exist positive constants $c_1$ and $c_2$ such that $\underline{u} \geq c_1 \vartheta_{\beta,0}$ and $\overline{u} \leq c_2 \vartheta_{\beta,\gamma}$ in $\Omega$. Then, for *a.e.* $x \in \Omega$ and $s \in \left[\underline{u}(x), \overline{u}(x)\right]$,

$$\begin{aligned} |k(x,s) - h(x,s)| &\leq k\left(x, \underline{u}(x)\right) + h\left(x, \overline{u}(x)\right) \\ &\leq B_2 d_\Omega^{-\gamma}(x)\, c_1^{-\beta} \vartheta_{\beta,0}^{-\beta}(x) + B_3 d_\Omega^{-\gamma}(x)\, c_2^{p} \vartheta_{\beta,\gamma}^{p}(x). \end{aligned}$$

Since $d_\Omega^{-\gamma} \vartheta_{\beta,\gamma}^{-\beta}$ and $d_\Omega^{-\gamma} \vartheta_{\beta,\gamma}^{p}$ belong to $L_{\mathrm{loc}}^\infty(\Omega)$, Remark 2.7 gives a solution $u$, in the sense of distributions, of the problem $-\Delta u = k(\cdot, u) - h(\cdot, u)$ in $\Omega$, that satisfies $\underline{u} \leq u \leq \overline{u}$ in $\Omega$. Then, for some positive constants $c_1$ and $c_2$,

$$c_1 \vartheta_{\beta,0} \leq u \leq c_2 \vartheta_{\beta,\gamma} \quad a.e. \text{ in } \Omega. \tag{3.21}$$

Therefore, $|k(\cdot,u)-h(\cdot,u)| \leq B_2 d_\Omega^{-\gamma} u^{-\beta} + B_3 d_\Omega^{-} u^p \leq B_2 c_1^{-\beta} d_\Omega^{-\gamma} \vartheta_{\beta,0}^{-\beta} + B_3 c_2^p d_\Omega^{-\eta} \vartheta_{\beta,\gamma}^p$ *a.e.* in $\Omega$. By Lemma 3.11, $d_\Omega^{1-\gamma} \vartheta_{\beta,0}^{-\beta} \in L^2(\Omega)$ and $d_\Omega^{1-\eta} \vartheta_{\beta,\gamma}^p \in L^2(\Omega)$; then, by Remark 2.2, $|k(\cdot,u)-h(\cdot,u)| \in \left(H_0^1(\Omega)\right)'$. Also, from (3.21), we get $k(\cdot,u)-h(\cdot,u) \in L_{\mathrm{loc}}^\infty(\Omega)$ and $u \in L^\infty(\Omega)$. Then, by the inner elliptic estimates, $u \in W_{loc}^{2,q}(\Omega)$ for any $q \in [1,\infty)$. Therefore $u \in C(\Omega)$. By (3.21), $u$ is also continuous on $\partial\Omega$. Then $u \in C(\overline{\Omega})$. Notice that, by the definition of $\vartheta_{\beta,\gamma}$, there exist positive constants $\widetilde{c}_2$ and $\tau$, such that $\vartheta_{\beta,\gamma} \leq \widetilde{c}_2 d_\Omega^\tau$ in $\Omega$, and so, by (3.21), $u \leq c' d_\Omega^\tau$ in $\Omega$ for some positive constant $c'$. Then, by Lemma 2.4, $u \in H_0^1(\Omega)$, and $u$ is a weak solution of problem (1.4). Finally, the uniqueness assertion of the theorem follows from Remark 3.5. □

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