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On a nonlocal nonlinear Schrödinger equation with self-induced parity-time-symmetric potential

Jingjun Zhang ✉

College of Mathematics, Physics and Information Engineering, Jiaxing University,
No. 56 Yuexiu Road, Jiaxing 314001, China

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Abstract. We consider the Cauchy problem of a nonlocal nonlinear Schrödinger equation with self-induced parity-symmetric potential. Global existence of solution and decay estimates are obtained for suitably small initial data when the spatial dimension $d \geq 2$.

Keywords: nonlocal Schrödinger equation, global solution, decay estimate.

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1 Introduction

This paper is concerned with a nonlocal nolinear Schrödinger (NLS) equation which reads

$$i\psi_t(t,x) + \frac{1}{2}\Delta\psi(t,x) + g\psi(t,x)\overline{\psi}(t,\mathcal{P}x)\psi(t,x) = 0, \tag{1.1}$$

where $\psi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ is unknown, $\overline{\psi}$ is the complex conjugation of ψ, and g is a real constant ($g > 0$ and $g < 0$ denote the focusing and defocusing cases, respectively). In the above equation, $\mathcal{P}$ is a $d \times d$ matrix, which denotes a parity transformation with the determinant satisfying

$$\det \mathcal{P} = -1. \tag{1.2}$$

More precisely, in odd spatial dimensions, $\mathcal{P}x = -x$, that is, the sign of all the coordinates is changed, while in even spatial dimensions, a parity transformation means that the sign of only an odd number of coordinates can be reversed. In particular, in one dimensional case, equation (1.1) reduces to

$$i\psi_t(t,x) + \frac{1}{2}\psi_{xx}(t,x) + g\psi(t,x)\overline{\psi}(t,-x)\psi(t,x) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}. \tag{1.3}$$

Note that $\mathcal{P}$ is not unique in even dimensions. For example, if $d = 2$, $\mathcal{P}x$ can take as either $\mathcal{P}x = (-x_1, x_2)$ or $\mathcal{P}x = (x_1, -x_2)$.

✉ Email: zjj@zjxu.edu.cn

Equation (1.1) was first derived by Ablowitz and Musslimani [1] in one dimensional case, and by Sinha and Ghosh [9] in higher dimensional case. In the equation, the self-induced potential $V(t,x) := \psi(t,x)\overline{\psi}(t,\mathcal{P}x)$ is non-Hermitian but parity-time-symmetric ($\mathcal{PT}$-symmetric), that is, $\overline{V}(t,\mathcal{P}x) = V(t,x)$. Note that the value of the potential V at x depends not only on the information of ψ at x, but also on $\mathcal{P}x$, so it is a nonlocal potential. $\mathcal{PT}$-symmetric condition is weaker than the condition of self-adjointness, however, it was shown by Bender and Boettcher [3] that non-Hermitian Hamiltonians having $\mathcal{PT}$ symmetry may also exhibit real spectra, hence, a great deal of investigations on $\mathcal{PT}$-symmetric systems are carried out both theoretically and experimentally. Using a unified two-parameter model, equation (1.1) can be generalized to vector form [12]. If $\psi(t,-x) = \psi(t,x)$, equation (1.1) reduces to the classical NLS equation

$$i\psi_t(t,x) + \frac{1}{2}\Delta\psi(t,x) + g|\psi(t,x)|^2\psi(t,x) = 0. \tag{1.4}$$

When $d = 1$, Ablowitz and Musslimani [1] showed that the nonlocal NLS equation (1.1) is an integrable system. Exact soliton solutions were obtained in [1,2,6,8,9]. In particular, from the identity (22) in [1], we know the focusing nonlocal NLS equation (1.3) (i.e., $g > 0$) has the one-soliton solution

$$\psi^*(t,x) = \pm\frac{2(\eta_1+\eta_2)e^{i\theta_2}e^{i2g\eta_2^2 t}e^{-2\sqrt{g}\eta_2 x}}{1+e^{i(\theta_1+\theta_2)}e^{-i2g(\eta_1^2-\eta_2^2)t}e^{-2\sqrt{g}(\eta_1+\eta_2)x}},$$

where the four parameters $\eta_1, \eta_2, \theta_1, \theta_2$ are real, $\eta_1, \eta_2 > 0$ and $\eta_1 \neq \eta_2$. Note that ψ^* eventually develops a singularity in finite time T_n at $x = 0$,

$$\lim_{t\to T_n} |\psi^*(t,0)| = +\infty \text{ with } T_n = \frac{(2n+1)\pi - \theta_1 - \theta_2}{2g(\eta_2^2 - \eta_1^2)}, \quad n \in \mathbb{Z}.$$

In particular, by setting $\theta_1 = \theta_2 = 0$, $\eta_1 = \epsilon$, $\eta_2 = 2\epsilon$, it can be computed that

$$\|\psi^*(0,x)\|_{L^2(\mathbb{R})} \lesssim \epsilon^{\frac{1}{2}}, \quad \|\psi_x^*(0,x)\|_{L^2(\mathbb{R})} \lesssim \epsilon^{\frac{3}{2}}.$$

This implies that solutions of (1.1) may develop finite time blow up behavior even with H^1 small initial data. Therefore, compared to the classical NLS equation (1.4) where we know global solutions exist with arbitrarily large H^1 initial data and possesses a modified scattering behavior for small initial data [5,7], the nonlocal NLS equation exhibits a completely different picture in one spatial dimension due to the presence of the nonlocal nonlinearity. So a natural question is whether such phenomenon still occurs for higher space dimensions. This is the main motivation of the present work.

In this paper, the notation $A \lesssim B$ $(A, B \geq 0)$ means that there exists a constant $C > 0$ such that $A \leq CB$. For $1 \leq p \leq +\infty$, $L^p(\mathbb{R}^d)$ is the usual Lebesgue space. For $s \in \mathbb{R}$, $H^s(\mathbb{R}^d)$ denotes the inhomogeneous Sobolev space equipped with the norm

$$\|u\|_{H^s} := \|(1+|\xi|^2)^{s/2}\widehat{u}\|_{L^2},$$

where $\widehat{u} = \widehat{u}(\xi)$ is the Fourier transform of u, namely,

$$\widehat{u}(\xi) = \mathcal{F}u := \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} e^{-ix\cdot\xi}u(x)dx.$$

Now, we state the main result of the paper, see Theorems 1.1 and 1.2 below. The initial data of the equation (1.1) is endowed as

$$\psi(0,x) = \psi_0(x). \tag{1.5}$$

Theorem 1.1. *Let $d \geq 3$, $N > \frac{d}{2}$ be an integer. Then there exists a sufficiently small constant $\epsilon_0 > 0$ such that if the initial data ψ_0 satisfies*

$$\|\psi_0\|_{H^N(\mathbb{R}^d)} + \|\psi_0\|_{L^1(\mathbb{R}^d)} \leq \epsilon_0, \tag{1.6}$$

then the nonlocal NLS equation (1.1) admits a unique global solution $\psi \in C(\mathbb{R}; H^N(\mathbb{R}^d))$. Moreover, for all $t \in \mathbb{R}$, there hold that

$$\|\psi(t,x)\|_{H^N(\mathbb{R}^d)} \lesssim \epsilon_0, \quad \|\psi(t,x)\|_{L^\infty(\mathbb{R}^d)} \lesssim \frac{\epsilon_0}{(1+|t|)^{d/2}}. \tag{1.7}$$

Theorem 1.2. *Assume $d = 2$ and the initial data ψ_0 satisfies*

$$\|\psi_0\|_{H^N(\mathbb{R}^2)} + \||x|^2\psi_0\|_{L^2(\mathbb{R}^2)} \leq \epsilon_0, \tag{1.8}$$

where the integer $N > 1$ and $\epsilon_0 > 0$ is sufficiently small. Then the Cauchy problem (1.1) and (1.5) has a unique global solution $\psi \in C(\mathbb{R}; H^N(\mathbb{R}^2))$ satisfying for all $t \in \mathbb{R}$,

$$\|\psi(t,x)\|_{H^N(\mathbb{R}^2)} + \||x|^2 f(t,x)\|_{L^2(\mathbb{R}^2)} \lesssim \epsilon_0, \quad \|\psi(t,x)\|_{L^\infty(\mathbb{R}^2)} \lesssim \frac{\epsilon_0}{1+|t|}, \tag{1.9}$$

where $f(t,x) := e^{-\frac{it\Delta}{2}}\psi(t,x)$ is the profile of $\psi(t,x)$.

From the above theorems, we observe that small initial data still leads to global solution for the nonlocal NLS equation when $d \geq 2$, which is different from one dimensional case. This shows that for long time existence, the dispersive effect dominates the nonlocal effect in higher dimensions. By using the energy norm and the decay norm, Theorems 1.1 and 1.2 are proved in Sections 3 and 4, respectively.

Finally, we remark that the total charge $\mathcal{N}$ and the Hamiltonian $\mathcal{H}$ of the equation (1.1) are conserved (see [9]), namely, $\mathcal{N}(t) = \mathcal{N}(0)$ and $\mathcal{H}(t) = \mathcal{H}(0)$ with

$$\begin{aligned}
\mathcal{N}(t) &:= \int_{\mathbb{R}^d} \psi(t,x)\overline{\psi}(t,\mathcal{P}x)dx, \\
\mathcal{H}(t) &:= \int_{\mathbb{R}^d} \big[\frac{1}{2}\nabla\psi(t,x)\cdot\nabla\overline{\psi}(t,\mathcal{P}x) - \frac{g}{2}\big(\psi(t,x)\overline{\psi}(t,\mathcal{P}x)\big)^2\big]dx.
\end{aligned}$$

Although each term in $\mathcal{N}$ and $\mathcal{H}$ is real-valued, it is not semipositive-definite. Hence, unlike the classical NLS equation, it is not known clearly how to use these conserved quantities in our mathematical analysis, especially in the study of the blow up problems for the nonlocal NLS equation (1.1). Such issues will be exploited in the further research.

2 Preliminaries

In this section, we collect preparatory materials, including some basic inequalities, linear decay estimates for the Schrödinger operator and the local well-posedness result. Firstly, from (1.2) and the definition of the parity transformation $\mathcal{P}$, it is easy to see for any function $u(x)$, there hold

$$\begin{aligned}
\|u(\mathcal{P}x)\|_{L^p(\mathbb{R}^d)} &= \|u(x)\|_{L^p(\mathbb{R}^d)}, && 1 \leq p \leq +\infty, \\
\mathcal{F}[u(\mathcal{P}x)](\xi) &= \widehat{u}(\mathcal{Q}\xi), && \mathcal{Q} := \mathcal{P}^{-1}, \\
\|u(\mathcal{P}x)\|_{H^s(\mathbb{R}^d)} &= \|u(x)\|_{H^s(\mathbb{R}^d)}, && s \geq 0.
\end{aligned} \tag{2.1}$$

Lemma 2.1. *Assume $u, v \in H^s(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ with $s \geq 0$, then there holds*

$$\|uv\|_{H^s} \lesssim \|u\|_{H^s}\|v\|_{L^\infty} + \|u\|_{L^\infty}\|v\|_{H^s}. \tag{2.2}$$

The proof of this lemma can be found, for example, in [11, Lemma A.8].

Lemma 2.2. *There hold that*

$$\|f\|_{L^1(\mathbb{R}^2)} \lesssim \|f\|_{L^2(\mathbb{R}^2)}^{1/2} \||x|^2 f\|_{L^2(\mathbb{R}^2)}^{1/2}, \tag{2.3}$$

$$\|f\|_{L^{4/3}(\mathbb{R}^2)} \lesssim \|f\|_{L^2(\mathbb{R}^2)}^{1/2} \|xf\|_{L^2(\mathbb{R}^2)}^{1/2}. \tag{2.4}$$

Proof. Let $a > 0$ be determined later. Using the basic estimate $\int_{|x|\geq a} |x|^{-4} dx \lesssim a^{-2}$, we deduce by the Cauchy–Schwarz inequality that

$$\|f\|_{L^1} \leq \int_{|x|\leq a} |f(x)| \cdot 1 dx + \int_{|x|\geq a} |x|^2|f(x)| \cdot |x|^{-2} dx \lesssim \|f\|_{L^2} a + \||x|^2 f\|_{L^2} a^{-1}.$$

Then (2.3) follows easily if we choose $a = \||x|^2 f\|_{L^2}^{1/2} \|f\|_{L^2}^{-1/2}$. Here we may assume $\|f\|_{L^2} \neq 0$, otherwise the estimate (2.3) holds obviously.

The proof for (2.4) is similar. In fact, using Hölder's inequality, we have

$$\begin{aligned} \|f\|_{L^{4/3}}^{4/3} &\leq \int_{|x|\leq b} |f(x)|^{4/3} \cdot 1 dx + \int_{|x|\geq b} |xf(x)|^{4/3} \cdot |x|^{-4/3} \\ &\lesssim \|f\|_{L^2}^{4/3} b^{2/3} + \|xf\|_{L^2}^{4/3} b^{-2/3}, \end{aligned}$$

which gives the desired estimate (2.4) provided that we set $b = \|xf\|_{L^2}\|f\|_{L^2}^{-1}$. □

For the Schrödinger operator $e^{\frac{it\Delta}{2}}$, it is known that (see e.g., [10])

$$\|e^{\frac{it\Delta}{2}} u\|_{L^p(\mathbb{R}^d)} \lesssim \frac{1}{|t|^{d(\frac{1}{2}-\frac{1}{p})}} \|u\|_{L^{p'}(\mathbb{R}^d)}, \qquad \frac{1}{p} + \frac{1}{p'} = 1, \quad 2 \leq p \leq +\infty. \tag{2.5}$$

Using Duhamel's formula, the solution $\psi(t,x)$ of (1.1) can be expressed by

$$\psi(t,x) = e^{\frac{it\Delta}{2}} \psi_0(x) - ig \int_0^t e^{\frac{i(t-s)\Delta}{2}} \psi(s,x)\overline{\psi}(s,\mathcal{P}x)\psi(s,x) ds. \tag{2.6}$$

Equation (2.6) is the main identity that we will discuss later.

Finally, we end this section with a local well-posedness result.

Proposition 2.3. *For any $\psi_0 \in H^N(\mathbb{R}^d)$ with $N > \frac{d}{2}$, $d \geq 1$, there exists $T_0 = T_0(\|\psi_0\|_{H^N}) > 0$ such that the Cauchy problem (1.1) and (1.5) has a unique solution $\psi \in C([0,T_0]; H^N)$ satisfying (2.6). Moreover, if $T^* < +\infty$ is the maximal existence time for this solution, then*

$$\limsup_{t\uparrow T^*} \|\psi(t,x)\|_{H^N} = +\infty. \tag{2.7}$$

This proposition can be proved by applying the Banach fixed-point theorem, since the argument is standard, we skip the details.

3 Proof of Theorem 1.1

From now on, we focus on the case $t \geq 0$ for simplicity. To prove Theorem 1.1, we first introduce the work space A_T as follows,

$$\|\psi\|_{A_T} := \sup_{t\in[0,T)} \left(\|\psi(t,x)\|_{H^N(\mathbb{R}^d)} + (1+t)^{\frac{d}{2}} \|\psi(t,x)\|_{L^\infty(\mathbb{R}^d)} \right), \tag{3.1}$$

where $T \in (0, +\infty]$. The result of Theorem 1.1 relies essentially on the following proposition.

Proposition 3.1. *Let $d \geq 3$, $N > \frac{d}{2}$ be an integer and $\psi_0 \in H^N(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. Assume $\psi(t,x) \in C([0,T]; H^N(\mathbb{R}^d))$ is the solution of (1.1) and (1.5). Then we have*

$$\|\psi\|_{A_T} \lesssim \|\psi_0\|_{H^N \cap L^1} + \|\psi\|_{A_T}^3, \tag{3.2}$$

where the implicit constant is independent of T.

Proof. The start point is the identity (2.6). Using Lemma 2.1, (2.1) and the definition of $\|\cdot\|_{A_T}$, we have for any $t \in [0,T]$,

$$\begin{aligned}
\|\psi(t,x)\|_{H^N} &\leq \|\psi_0(x)\|_{H^N} + |g| \int_0^t \|\psi(s,x)\overline{\psi}(s,\mathcal{P}x)\psi(s,x)\|_{H^N} ds \\
&\lesssim \|\psi_0(x)\|_{H^N} + \int_0^t \|\psi^2(s,x)\|_{H^N} \|\overline{\psi}(s,\mathcal{P}x)\|_{L^\infty} ds \\
&\qquad + \int_0^t \|\psi^2(s,x)\|_{L^\infty} \|\overline{\psi}(s,\mathcal{P}x)\|_{H^N} ds \\
&\lesssim \|\psi_0(x)\|_{H^N} + \int_0^t \|\psi(s,x)\|_{H^N} \|\psi(s,x)\|_{L^\infty}^2 ds \\
&\lesssim \|\psi_0(x)\|_{H^N} + \|\psi\|_{A_T}^3 \int_0^t (1+s)^{-d} ds \\
&\lesssim \|\psi_0(x)\|_{H^N} + \|\psi\|_{A_T}^3.
\end{aligned} \tag{3.3}$$

Next, we turn to estimate the L^∞ norm of $\psi(t,x)$. Note that

$$\|e^{\frac{it\Delta}{2}}\psi_0(x)\|_{L^\infty} \lesssim \frac{1}{(1+t)^{\frac{d}{2}}} \|\psi_0(x)\|_{L^1\cap H^N}, \tag{3.4}$$

which is a consequence of (2.5) for large t and the Sobolev embedding $H^N \hookrightarrow L^\infty$ for small t. Hence, using (3.4), (2.1), Lemma 2.1 and Hölder's inequality, it follows from (2.6) that

$$\begin{aligned}
\|\psi(t,x)\|_{L^\infty} &\leq \|e^{\frac{it\Delta}{2}}\psi_0(x)\|_{L^\infty} + |g| \int_0^t \|e^{\frac{i(t-s)\Delta}{2}} (\psi^2(s,x)\overline{\psi}(s,\mathcal{P}x))\|_{L^\infty} ds \\
&\lesssim \frac{1}{(1+t)^{\frac{d}{2}}} \|\psi_0(x)\|_{L^1\cap H^N} + \int_0^t \frac{1}{(1+t-s)^{\frac{d}{2}}} \|\psi^2(s,x)\overline{\psi}(s,\mathcal{P}x)\|_{L^1\cap H^N} ds \\
&\lesssim \frac{1}{(1+t)^{\frac{d}{2}}} \|\psi_0(x)\|_{L^1\cap H^N} + \int_0^t \frac{1}{(1+t-s)^{\frac{d}{2}}} \|\psi(s,x)\|_{L^2}^2 \|\psi(s,x)\|_{L^\infty} ds \\
&\qquad + \int_0^t \frac{1}{(1+t-s)^{\frac{d}{2}}} \|\psi(s,x)\|_{H^N} \|\psi(s,x)\|_{L^\infty}^2 ds \\
&\lesssim \frac{1}{(1+t)^{\frac{d}{2}}} \|\psi_0(x)\|_{L^1\cap H^N} + \|\psi\|_{A_T^3} \int_0^t \frac{1}{(1+t-s)^{\frac{d}{2}}} \cdot \frac{1}{(1+s)^{\frac{d}{2}}} ds \\
&\lesssim \frac{1}{(1+t)^{\frac{d}{2}}} \|\psi_0(x)\|_{L^1\cap H^N} + \frac{1}{(1+t)^{\frac{d}{2}}} \|\psi\|_{A_T^3}.
\end{aligned} \tag{3.5}$$

Therefore, the desired estimate (3.2) follows easily from (3.3) and (3.5). $\square$

Based on Proposition 3.1, we now present the proof of Theorem 1.1.

Proof of Theorem 1.1. By Proposition 2.3, we know there exists a unique solution ψ to (1.1) and (1.5) such that $\psi \in C([0, T^*); H^N)$ with T^* the maximal existence time of the solution. In order to obtain Theorem 1.1, we shall show $T^* = +\infty$ if the initial data is small enough. Define $\phi(t) := \|\psi\|_{A_t}$ for $t \geq 0$, where A_t is given by (3.1). Then from the condition (1.6) and Proposition 3.1, we obtain

$$\phi(t) \leq C\epsilon_0 + C\phi^3(t), \quad t \in [0, T^*). \tag{3.6}$$

where $C > 1$ is independent of T^*.

The bound (1.6) implies $\phi(0) \leq \epsilon_0$, then by the continuity of the solution, there exists a time T such that $\phi(t) \leq 2C\epsilon_0$ for all $t \in [0, T]$. Here, C is the same as (3.6). Let

$$T' := \sup\{T;\ \phi(t) \leq 2C\epsilon_0 \text{ for all } t \in [0, T]\}$$

Using (3.6) and the continuity of ψ, there holds

$$\phi(T') \leq C\epsilon_0 + C\phi^3(T'). \tag{3.7}$$

Now we claim $T' = T^*$ provided that $\epsilon_0^2 \leq (16C^3)^{-1}$. Indeed, if $T' < T^*$, (3.7) gives

$$2C\epsilon_0 \leq C\epsilon_0 + 8C^4\epsilon_0^3,$$

which is a contradiction for sufficiently small ϵ_0. Therefore, we conclude that if $\epsilon_0 \leq (16C^3)^{-\frac{1}{2}}$, then $\phi(T^*) \leq 2C\epsilon_0$. This bound and the blowup criterion (2.7) in turn yield $T^* = +\infty$. Hence, we obtain $\psi \in C(\mathbb{R}^+; H^N)$ and the bound (1.7) holds for $t \geq 0$. The case $t \leq 0$ can be proved similarly. This ends the proof of Theorem 1.1. □

4 Proof of Theorem 1.2

Since the decay rate is only t^{-1} in dimension two, the argument used in Section 3 can not be applied to prove Theorem 1.2. Inspired from the work [4,7] on the method of space-time resonances, here we would like to work on the space B_T defined by

$$\|\psi\|_{B_T} := \sup_{t \in [0,T)} \left(\|\psi(t,x)\|_{H^N(\mathbb{R}^2)} + \||x|^2 f(t,x)\|_{L^2(\mathbb{R}^2)} \right), \tag{4.1}$$

where $T \in (0, +\infty]$, and

$$f(t,x) := e^{-\frac{it\Delta}{2}} \psi(t,x) \tag{4.2}$$

is the profile of a solution $\psi(t,x)$ of (1.1). Notice that (4.1) implies

$$\begin{aligned} \|xf(t,x)\|_{L^2} &\leq \|f(t,x)\|_{L^2} + \||x|^2 f(t,x)\|_{L^2} \\ &= \|\psi(t,x)\|_{L^2} + \||x|^2 f(t,x)\|_{L^2} \\ &\leq 2\|\psi\|_{B_T}. \end{aligned} \tag{4.3}$$

Moreover, using (2.3), (2.5), (4.1) and (4.2), we have

$$\|\psi(t,x)\|_{L^\infty(\mathbb{R}^2)} = \|e^{\frac{it\Delta}{2}} f(t,x)\|_{L^\infty(\mathbb{R}^2)} \lesssim \frac{1}{1+t} \|\psi\|_{B_T}, \qquad t \in [0, T], \tag{4.4}$$

which shows that the decay rate of the solution ψ is bounded by the norm $\|\psi\|_{B_T}$.

Proposition 4.1. *Assume $\psi(t,x) \in C([0,T];H^N(\mathbb{R}^2))$ $(N > 1)$ is the solution of (1.1) with the initial data satisfying $\psi_0 \in H^N(\mathbb{R}^2)$ and $|x|^2\psi_0 \in L^2(\mathbb{R}^2)$, then we have $xf(t,x), |x|^2 f(t,x) \in C([0,T];L^2(\mathbb{R}^2))$ and*

$$\|\psi\|_{B_T} \lesssim \|\psi_0\|_{H^N} + \||x|^2\psi_0\|_{L^2} + \|\psi\|_{B_T}^3, \tag{4.5}$$

where the implicit constant is independent of T.

Proof. We first show the continuity for $xf(t,x)$ and $|x|^2 f(t,x)$. Recall the definition (4.2), it follows from (1.1) that

$$f_t(t,x) = ige^{-\frac{it\Delta}{2}}[\psi(t,x)\overline{\psi}(t,\mathcal{P}x)\psi(t,x)]. \tag{4.6}$$

Using the identity

$$x(e^{\pm\frac{it\Delta}{2}}u(x)) = e^{\pm\frac{it\Delta}{2}}(xu(x)) \mp ite^{\pm\frac{it\Delta}{2}}\nabla u(x), \tag{4.7}$$

which can be verified by taking Fourier transform on both sides of (4.7), then we can obtain

$$(xf)_t = ige^{-\frac{it\Delta}{2}}[x\psi(t,x)\overline{\psi}(t,\mathcal{P}x)\psi(t,x)] - gte^{-\frac{it\Delta}{2}}\nabla[\psi(t,x)\overline{\psi}(t,\mathcal{P}x)\psi(t,x)].$$

Integrating this equality with respect to time over $[0,t]$ gives (using also the fact $f(0,x) = \psi_0(x)$, and $\psi_0 \in L^2$, $|x|^2\psi_0 \in L^2$ implies $x\psi_0 \in L^2$)

$$\sup_{s\in[0,t]} \|xf(s,x)\|_{L^2} \leq \|x\psi_0\|_{L^2} + Ct \sup_{s\in[0,T]} \|\psi(s,x)\|_{H^N}^2 \sup_{s\in[0,t]} \|xf(s,x)\|_{L^2} + Ct^2 \sup_{s\in[0,T]} \|\psi(s,x)\|_{H^N}^3.$$

This implies $xf(t,x) \in L^2$ for $t \leq T_0 := [2C\sup_{s\in[0,T]} \|\psi(s,x)\|_{H^N}^2]^{-1}$. Moreover, with the same arguments as above, it is easy to obtain

$$\|xf(t_2,x) - xf(t_1,x)\|_{L^2} \lesssim |t_2 - t_1| \sup_{s\in[0,T]} \|\psi(s,x)\|_{H^N}^3, \quad t_1,t_2 \in [0,T_0],$$

which gives $xf \in C([0,T_0];L^2)$. Note that T_0 depends only on $\sup_{s\in[0,T]} \|\psi(s,x)\|_{H^N}$, so a standard bootstrap argument clearly yields that the continuity of xf holds in the whole interval $[0,T]$. The continuity of $|x|^2 f$ can be proved similarly but with more complicated computation, we thus omit the detailed proof for simplicity.

Next, we prove the bound (4.5). For the H^N norm part, one can use (4.4) and similar treatment as (3.3) to obtain

$$\|\psi(t,x)\|_{H^N} \lesssim \|\psi_0\|_{H^N} + \|\psi\|_{B_T}^3 \int_0^t (1+s)^{-2}ds \lesssim \|\psi_0\|_{H^N} + \|\psi\|_{B_T}^3. \tag{4.8}$$

So it remains to estimate the weighted norm. To this end, we integrate the equation (4.6) with respect to time and rewrite the resulted identity in the form of Fourier space, then we obtain (using also (4.2) and (2.1))

$$\widehat{f}(t,\xi) = \widehat{f}(0,\xi) + \frac{ig}{(2\pi)^2}\int_0^t \int_{\mathbb{R}^2\times\mathbb{R}^2} e^{is\Phi(\xi,\eta,\sigma)}\widehat{f}(s,\xi-\eta)\widehat{f}(s,\eta-\sigma)\widehat{\overline{f}}(s,\mathcal{Q}\sigma)d\eta d\sigma ds, \tag{4.9}$$

where the phase $\Phi(\xi,\eta,\sigma)$ is given by

$$\Phi(\xi,\eta,\sigma) := \frac{1}{2}(|\xi|^2 - |\xi-\eta|^2 - |\eta-\sigma|^2 + |\sigma|^2) = \xi\cdot\eta - |\eta|^2 + \eta\cdot\sigma. \tag{4.10}$$

Using Plancharel's identity, we know $\||x|^2 f\|_{L^2} = \|\Delta_\xi \widehat{f}\|_{L^2}$. Now applying Δ_ξ to (4.9) and recalling the fact $f(0,x) = \psi_0(x)$, we have

$$\Delta_\xi \widehat{f}(t,\xi) = \Delta_\xi \widehat{\psi_0} + I_1 + I_2 + I_3 \tag{4.11}$$

with

$$I_1 := ig(2\pi)^{-2} \int_0^t \int_{\mathbb{R}^2\times\mathbb{R}^2} e^{is\Phi(\xi,\eta,\sigma)} \Delta_\xi \widehat{f}(s,\xi-\eta) \widehat{f}(s,\eta-\sigma) \widehat{\overline{f}}(s,\mathcal{Q}\sigma) d\eta d\sigma ds,$$
$$I_2 := 2ig(2\pi)^{-2} \int_0^t \int_{\mathbb{R}^2\times\mathbb{R}^2} e^{is\Phi(\xi,\eta,\sigma)} (is\nabla_\xi\Phi) \nabla_\xi \widehat{f}(s,\xi-\eta) \widehat{f}(s,\eta-\sigma) \widehat{\overline{f}}(s,\mathcal{Q}\sigma) d\eta d\sigma ds,$$
$$I_3 := ig(2\pi)^{-2} \int_0^t \int_{\mathbb{R}^2\times\mathbb{R}^2} e^{is\Phi(\xi,\eta,\sigma)} (is)^2 |\nabla_\xi\Phi|^2 \widehat{f}(s,\xi-\eta) \widehat{f}(s,\eta-\sigma) \widehat{\overline{f}}(s,\mathcal{Q}\sigma) d\eta d\sigma ds.$$

Note that both I_2 and I_3 contain growth factor of s. However, the factor will not cause any difficulty for small s such as $s \in [0,1]$. Hence, the contribution of the time integral from 0 to 1 in I_2 and I_3 can be easily estimated by using only the energy bound and the weighted norm. In the following, we mainly deal with the time integral from 1 to t (still denoted by I_2 and I_3). In order to eliminate the growth factor s in the term I_2, we use the following crucial relation for Φ (see (4.10))

$$\nabla_\xi \Phi = \eta = \nabla_\sigma \Phi \tag{4.12}$$

to integrate by part in σ, then $I_2 = I_{2,1} + I_{2,2}$ with

$$I_{2,1} := -2ig(2\pi)^{-2} \int_1^t \int_{\mathbb{R}^2\times\mathbb{R}^2} e^{is\Phi(\xi,\eta,\sigma)} \nabla_\xi \widehat{f}(s,\xi-\eta) \nabla_\sigma \widehat{f}(s,\eta-\sigma) \widehat{\overline{f}}(s,\mathcal{Q}\sigma) d\eta d\sigma ds,$$
$$I_{2,2} := -2ig(2\pi)^{-2} \int_1^t \int_{\mathbb{R}^2\times\mathbb{R}^2} e^{is\Phi(\xi,\eta,\sigma)} \nabla_\xi \widehat{f}(s,\xi-\eta) \widehat{f}(s,\eta-\sigma) \nabla_\sigma \widehat{\overline{f}}(s,\mathcal{Q}\sigma) d\eta d\sigma ds.$$

Similarly, using (4.12) to integrate I_3 by part twice, then $I_3 = I_{3,1} + I_{3,2} + I_{3,3}$ with

$$I_{3,1} := ig(2\pi)^{-2} \int_1^t \int_{\mathbb{R}^2\times\mathbb{R}^2} e^{is\Phi(\xi,\eta,\sigma)} \widehat{f}(s,\xi-\eta) \Delta_\sigma \widehat{f}(s,\eta-\sigma) \widehat{\overline{f}}(s,\mathcal{Q}\sigma) d\eta d\sigma ds,$$
$$I_{3,2} := ig(2\pi)^{-2} \int_1^t \int_{\mathbb{R}^2\times\mathbb{R}^2} e^{is\Phi(\xi,\eta,\sigma)} \widehat{f}(s,\xi-\eta) \widehat{f}(s,\eta-\sigma) \Delta_\sigma \widehat{\overline{f}}(s,\mathcal{Q}\sigma) d\eta d\sigma ds,$$
$$I_{3,3} := 2ig(2\pi)^{-2} \int_1^t \int_{\mathbb{R}^2\times\mathbb{R}^2} e^{is\Phi(\xi,\eta,\sigma)} \widehat{f}(s,\xi-\eta) \nabla_\sigma \widehat{f}(s,\eta-\sigma) \nabla_\sigma \widehat{\overline{f}}(s,\mathcal{Q}\sigma) d\eta d\sigma ds.$$

Returning back to the physical space and using Hölder's inequality and (4.4), then

$$\begin{aligned} \|I_1\|_{L^2} + \|I_{3,1}\|_{L^2} + \|I_{3,2}\|_{L^2} &\lesssim \int_0^t \|\psi(t,x)\|_{L^\infty}^2 \||x|^2 f(s,x)\|_{L^2} ds \\ &\lesssim \|\psi\|_{B_T}^3 \int_0^t (1+s)^{-2} ds \\ &\lesssim \|\psi\|_{B_T}^3. \end{aligned} \tag{4.13}$$

For the remaining terms, we should use the inequality

$$\|e^{\frac{is\Delta}{2}}(xf(s,x))\|_{L^4} \lesssim s^{-\frac{1}{2}} \|\psi\|_{B_T}.$$

which follows from (2.4), (2.5) and (4.3), then

$$\begin{aligned}\|I_{2,1}\|_{L^2} + \|I_{2,2}\|_{L^2} + \|I_{3,3}\|_{L^2} &\lesssim \int_1^t \|\psi(t,x)\|_{L^\infty} \|e^{\frac{is\Delta}{2}}(xf(s,x))\|_{L^4}^2 ds \\ &\lesssim \|\psi\|_{B_T}^3 \int_1^t (1+s)^{-2} ds \\ &\lesssim \|\psi\|_{B_T}^3. \end{aligned} \tag{4.14}$$

Now, combing (4.11), (4.13) and (4.14) together yields

$$\||x|^2 f(t,x)\|_{L^2} \lesssim \||x|^2 \psi_0\|_{L^2} + \|\psi\|_{B_T}^3. \tag{4.15}$$

Therefore, the desired bound (4.5) follows immediately from (4.8) and (4.15). □

Finally, based on Proposition 4.1, one can argue analogously as the proof of Theorem 1.1 and obtain global existence of solution as stated in Theorem 1.2. The L^∞ decay bound in (1.9) follows also by using (4.4). Since the proof is similar as Theorem 1.1, we thus omit further details.

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