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Global stability of the positive equilibrium for a non-cooperative model of nuclear reactors[1]

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Abstract: In this paper, we investigate the non-cooperative reaction-diffusion model of nuclear reactors subject to the homogeneous Neumann boundary condition. By establishing appropriate Lyapunov functions, we prove the global stability of the unique positive constant equilibrium solution.

Key words: non-cooperative model, diffusion, nuclear reactors, global stability

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1 Introduction and main result

In this work, we study the following non-cooperative reaction-diffusion system which is an extension of a model proposed by Kastenberg and Chambré in [3] to describe the reaction process of nuclear reactors. The mathematical form of the system we shall investigate satisfies

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = au - buv, & \text{in } \ \Omega \times (0, \infty), \\ \dfrac{\partial v}{\partial t} - \Delta v = cu - duv - ev, & \text{in } \ \Omega \times (0, \infty), \\ \partial_\nu u = \partial_\nu v = 0, & \text{on } \ \partial\Omega \times (0, \infty), \\ u(x,0) = u_0(x) \geq 0, \not\equiv 0, & \text{in } \ \Omega, \\ v(x,0) = v_0(x) \geq 0, \not\equiv 0, & \text{in } \ \Omega. \end{cases} \tag{1.1}$$

Here, Ω is a bounded domain in $\mathbf{R}^N (N \geq 1)$ with smooth boundary $\partial\Omega$, and ν is the outward unit normal vector on $\partial\Omega$ and $\partial_\nu = \frac{\partial}{\partial \nu}$. The admissible initial data $u_0(x)$ and $v_0(x)$ are continuous functions on the closure $\overline{\Omega}$ and all the parameters a, b, c, d and e which appearing in model (1.1) are assumed to be positive constants. The unknown functions $u(x,t)$ and $v(x,t)$ respectively stand for the density of fast neutrons and the fuel temperature. The homogeneous Neumann boundary condition means that the boundary of the closed container is heat insulation and neutron flux can not go through the boundary $\partial\Omega$. By the standard theory for parabolic equations, it is easy to see that $u(x,t), v(x,t)$ exists for all $t > 0$ and

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$u(x,t), v(x,t) > 0$ for $x \in \overline{\Omega}$ and $t > 0$. As for more discussions on this model and the related ones, one may refer to [1, 3–5, 10].

In the very recent two research works [1, 4], the model has received analytical study under the homogeneous Dirichlet boundary condition. G. Arioli in [1] provided existence results for nontrivial periodic solutions and a global attractor. In [4], López-Gómez was mainly concerned with the corresponding steady state problem. Peng etc. [8] and Zhou [12] completed and sharpened those derived in [1, 4].

In the present work, we consider the model (1.1) under the homogeneous Neumann boundary condition and completely determine the global stability of the unique positive constant equilibrium solution by establishing appropriate Lyapunov functions.

First of all, we note that (1.1) has a unique positive constant equilibrium (u^*, v^*) if and only if $a < \frac{bc}{d}$, where

$$(u^*, v^*) = \Big(\frac{ae}{bc - ad}, \ \frac{a}{b}\Big).$$

The main result of this paper is as follows.

Theorem 1 *Suppose that* $a < \frac{bc}{d}$. *Then the solution* (u^*, v^*) *for system* (1.1) *is globally asymptotically stable.*

Clearly, the global stability of (u^*, v^*) implies that (1.1) admits no positive non-constant equilibrium solutions.

2 Proof of Theorem 1

Before giving the proof of Theorem 1, we first investigate the corresponding steady-state problem of the reaction-diffusion system (1.1), which may display the dynamical behavior of solutions to (1.1) as time goes to infinity. This steady-state problem satisfies

$$\begin{cases} -\Delta u = au - buv, & \text{in } \Omega, \\ -\Delta v = cu - duv - ev, & \text{in } \Omega, \\ \partial_\nu u = \partial_\nu v = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

In order to derive the desired results, we need to set

$$w = u - \frac{b}{d}v. \tag{2.2}$$

Thus, the original system (2.1) is changed into the following one:

$$\begin{cases} -\Delta w + \Big(\dfrac{bc}{d} - a\Big)w = \dfrac{b}{d}\Big(a + e - \dfrac{bc}{d}\Big)v, & \text{in } \Omega, \\ -\Delta v + \Big(e - \dfrac{bc}{d}\Big)v = cw - dwv - bv^2, & \text{in } \Omega, \\ \partial_\nu w = \partial_\nu v = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

We have the following result:

Proposition 1 *Assume that* $a < \frac{bc}{d}$ *and* $(u(x),\ v(x))$ *is a coexistence state (i.e.* $u(x) > 0,\ v(x) > 0$*) of* (2.1)*. We have*

(i) *if* $a + e - \frac{bc}{d} > 0$, *then* $u(x) > \frac{bv(x)}{d}$ *on* $\overline{\Omega}$*;*

(ii) *if* $a + e - \frac{bc}{d} = 0$, *then* $u(x) = \frac{bv(x)}{d}$ *on* $\overline{\Omega}$*;*

(iii) *if* $a + e - \frac{bc}{d} < 0$, *then* $u(x) < \frac{bv(x)}{d}$ *on* $\overline{\Omega}$*.*

Proof. In case (i), if (u, v) is a coexistence state of (2.1), then $v > 0$ in Ω, thanks to $a < \frac{bc}{d}$, the well-known maximum principle implies $w > 0$ by using the first equation in (2.3). For case (ii) and (iii), the argument is similar to that in case (i). $\square$

In the following, we give the proof of Theorem 1. In order to analyze the global stability of (u^*, v^*), we consider three different cases.

Case 1. $a < \frac{bc}{d} - e$

To our purpose, we need another change of variables $z = -w$, and consider the equations of (u, z). Thus, this means that the original system (1.1) is transformed into the following:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = (a - du - dz)u, & \text{in } \ \Omega \times (0, \infty), \\ \dfrac{\partial z}{\partial t} - \Delta z = \Big(\dfrac{bc}{d} - a - e\Big)u - ez, & \text{in } \ \Omega \times (0, \infty), \\ \partial_\nu u = \partial_\nu z = 0, & \text{on } \ \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x) \geq 0, \not\equiv 0, & \text{in } \ \Omega, \\ z(x, 0) = z_0(x) = \dfrac{bv(x, 0)}{d} - u(x, 0), & \text{in } \ \Omega. \end{cases} \tag{2.4}$$

When $\frac{bc}{d} - a - e > 0$, the system (2.4) has a unique equilibrium point (u^*, z^*), where

$$(u^*, z^*) = \Big(\frac{ae}{bc - ad}, \ \frac{a(bc - ad - de)}{d(bc - ad)}\Big).$$

For the system (2.4), we can derive that (u^*, z^*) is globally asymptotically stable, which thereby implies the global stability of (u^*, v^*). To show this, we first prove the following result.

Proposition 2 *Assume that* $a < \frac{bc}{d} - e$, *then for any small* $\epsilon > 0$, *there exists a* $T > 0$, *such that the solution* $(u(x, t), v(x, t))$ *of* (2.4) *satisfies,*

$$0 < u(x, t) < \frac{a}{d} + \epsilon, \qquad -\epsilon < z(x, t) < \frac{a(bc - ad - de)}{d^2 e} + \epsilon$$

for all $x \in \overline{\Omega}$ *and* $t > T$.

Proof. Thanks to $u(x, 0) = u_0(x) \geq 0, \not\equiv 0$, by the strong maximum principle for parabolic equation, we have $u(x, t) > 0$ for all $x \in \overline{\Omega}$ and $t > 0$. So (2.4) is a mixed quasi-monotonic system by section 8.2 in [7]. This property allows us to construct the following pair of lower and upper solutions:

$$(\underline{u}, \underline{z}) = (0, c_1^* \exp(-et)), \quad (\overline{u}, \overline{z}) = (M_1, M_2),$$

where $c_1^* = \min_{\overline{\Omega}} z_0(x)$, M_1 and M_2 are large and positive constants to be determined. Note that c_1^* may be negative or positive. Then, simple analysis shows that, if

$$a - dM_1 - dc_1^* \exp(-et) \leq 0, \qquad \Big(\frac{bc}{d} - a - e\Big) M_1 - eM_2 \leq 0,$$

then $(\underline{u}, \underline{z})$ and $(\overline{u}, \overline{z})$ are a pair of lower and upper solutions to (2.4) . This implies that

$$0 < u(x,t) < M_1, \qquad c_1^* \exp(-et) < z(x,t) < M_2$$

for all $x \in \overline{\Omega}$ and $t > 0$. By letting t be large, we have that, there is a large $T_1 > 0$ such that

$$0 < u(x,t) < M_1, \qquad -\epsilon < z(x,t) < M_2 \tag{2.5}$$

for all $x \in \overline{\Omega}$ and $t > T_1$.

In the following, we will obtain the more exact upper bound of the solutions to (2.4). It is clear that $a - du - dz \leq a - du + d\epsilon$ in $\overline{\Omega} \times [T_1, \infty)$ from the second inequality of (2.5). Therefore, $u(x,t)$ is a lower solution of the following problem

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} - \Delta \varphi = (a - d\varphi + d\epsilon)\varphi, & \text{in } \Omega \times (T_1, \infty), \\ \partial_\nu \varphi = 0, & \text{on } \partial\Omega \times (T_1, \infty), \\ \varphi(x, T_1) = u(x, T_1), & \text{on } \overline{\Omega}. \end{cases} \tag{2.6}$$

Let $\phi(t)$ be the unique positive solution of the problem

$$\begin{cases} \phi_t = (a - d\phi + d\epsilon)\phi, & \text{in } (T_1, \infty), \\ \phi(T_1) = \max_{\overline{\Omega}} u(x, T_1). \end{cases}$$

Then $\phi(t)$ is a upper solution of (2.6). As $\lim_{t\to\infty} \phi(t) = \frac{a}{d} + \epsilon$, taking larger $T_2 \geq T_1$ if necessary, we can get from the comparison principle that

$$u(x,t) < \phi(t) + \epsilon < \frac{a}{d} + \epsilon$$

for all $x \in \overline{\Omega}$ and $t \geq T_2$. Hence, by the second equation of (2.4), $z(x,t)$ is a lower solution of the following problem

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} - \Delta \varphi = \Big(\dfrac{bc}{d} - a - e\Big)\Big(\dfrac{a}{d} + \epsilon\Big) - e\varphi, & \text{in } \ \Omega \times (T_2, \infty), \\ \partial_\nu \varphi = 0, & \text{on } \ \partial\Omega \times (T_2, \infty), \\ \varphi(x, T_2) = z(x, T_2), & \text{on } \ \overline{\Omega}. \end{cases} \tag{2.7}$$

Let $\phi(t)$ be the unique positive solution of the problem

$$\begin{cases} \phi_t = \Big(\dfrac{bc}{d} - a - e\Big)\Big(\dfrac{a}{d} + \epsilon\Big) - e\phi, & \text{in } (T_2, \infty), \\ \phi(T_2) = \max_{\overline{\Omega}} z(x, T_2). \end{cases}$$

Then $\phi(t)$ is a upper solution of (2.7). As $\lim_{t\to\infty}\phi(t) = \frac{a(bc-ad-de)}{d^2e} + \epsilon$, taking $T \geq T_2$ if necessary, we have

$$z(x,t) < \phi(t) + \epsilon < \frac{a(bc-ad-de)}{d^2e} + \epsilon, \quad \text{for all} \quad x \in \overline{\Omega}, \quad t \geq T.$$

The proof is complete. $\square$

Proposition 3 *Suppose that* $a < \frac{bc}{d} - e$. *Then the solution* (u^*, z^*) *for system* (2.4) *is globally asymptotically stable, which implies the global stability of* (u^*, v^*) *for system* (1.1).

Proof. To prove our result, we need to construct the following Lyapunov functional:

$$E(t) = \int_\Omega W(u(x,t), z(x,t))\mathrm{d}x,$$

with

$$W(u,z) = \int \frac{u-u^*}{u}\mathrm{d}u + k\int (z - z^*)\mathrm{d}z,$$

where k is a positive constant to be determined.

$$\begin{aligned}
\frac{\mathrm{d}E(t)}{\mathrm{d}t} &= \int_\Omega \Big\{\frac{u-u^*}{u}u_t + k(z-z^*)z_t\Big\}\mathrm{d}x \\
&= \int_\Omega \frac{u-u^*}{u}\Delta u\mathrm{d}x + k\int_\Omega (z-z^*)\Delta z\mathrm{d}x + \int_\Omega (u-u^*)(a - du - dz)\mathrm{d}x \\
&\quad + k\int_\Omega (z-z^*)\left[\Big(\frac{bc}{d} - a - e\Big)u - ez\right]\mathrm{d}x \\
&= -\int_\Omega \frac{u^*|\nabla u|^2}{u^2}\mathrm{d}x - k\int_\Omega |\nabla z|^2\mathrm{d}x + \int_\Omega (u-u^*)(du^* + dz^* - du - dz)\mathrm{d}x \\
&\quad + k\int_\Omega (z-z^*)\left[\Big(\frac{bc}{d} - a - e\Big)(u-u^*) - e(z-z^*)\right]\mathrm{d}x \\
&= -\int_\Omega \frac{u^*|\nabla u|^2}{u^2}\mathrm{d}x - k\int_\Omega |\nabla z|^2\mathrm{d}x - d\int_\Omega (u-u^*)^2\mathrm{d}x \\
&\quad + \left[k\Big(\frac{bc}{d} - a - e\Big) - d\right]\int_\Omega (u-u^*)(z-z^*)\mathrm{d}x - ek\int_\Omega (z-z^*)^2\mathrm{d}x.
\end{aligned}$$

Set $\xi = u - u^*$, $\eta = z - z^*$, we have

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} \leq \int_\Omega \Big\{ -d\xi^2 + \left[k\Big(\frac{bc}{d} - a - e\Big) - d\right]\xi\eta - ek\eta^2\Big\}\mathrm{d}x. \tag{2.8}$$

If the inequality

$$\left[k\left(\frac{bc}{d}-a-e\right)-d\right]^2-4dek<0 \tag{2.9}$$

holds, from (2.8), it is easy to see that

$$-d\xi^2+\left[k\left(\frac{bc}{d}-a-e\right)-d\right]\xi\eta-ek\eta^2$$

takes negative values unless $u=u^*$ and $v=v^*$.

Next, we will show that it is possible to choose a suitable $k>0$ such that (2.9) holds. To this end, we rewrite (2.9) as

$$\left(\frac{bc}{d}-a-e\right)^2k^2-2(bc-ad+de)k+d^2<0, \tag{2.10}$$

through a simple computation, we can choose a fixed constant $k=\frac{d^2(bc-ad+de)}{(bc-ad-de)^2}>0$ such that (2.9) holds. Thus, combing Proposition 2.2 and the routine computation as that in the proof of Lemma 5.1 in [6] or Theorem 2.1 in [11], one can show that (u^*,z^*) is globally asymptotically stable and the proof is complete. □

Case 2. $a=\frac{bc}{d}-e$

Similarly, under the scaling $z=\frac{bv}{d}-u$, in our case, system (1.1) becomes the following:

$$\begin{cases}\dfrac{\partial u}{\partial t}-\Delta u=(a-du-dz)u, & \text{in } \Omega\times(0,\infty),\\ \dfrac{\partial z}{\partial t}-\Delta z=-ez, & \text{in } \Omega\times(0,\infty),\\ \partial_\nu u=\partial_\nu z=0, & \text{on } \partial\Omega\times(0,\infty),\\ u(x,0)=u_0(x)\ge 0,\not\equiv 0, & \text{in } \Omega,\\ z(x,0)=z_0(x)=\dfrac{bv(x,0)}{d}-u(x,0), & \text{in } \Omega,\end{cases} \tag{2.11}$$

the constant equilibrium for system (2.11) is $(u^*,z^*)=(\frac{a}{d},0)$.

Proposition 4 *Suppose that $a=\frac{bc}{d}-e$. Then the solution (u^*,z^*) for system (2.11) is globally asymptotically stable, which implies the global stability of (u^*,v^*) for system (1.1).*

Proof. The proof is similar to that of Theorem 1.1 in [9]. Basic analysis deduces that

$$z(x,t)\to 0 \quad \text{uniformly on } \overline{\Omega}, \quad \text{as } t\to\infty. \tag{2.12}$$

Obviously, $u(x,t)$ solves

$$\begin{cases}\dfrac{\partial u}{\partial t}-\Delta u=(a-du-dz)u, & \text{in } \Omega\times(0,\infty),\\ \partial_\nu u=0, & \text{on } \partial\Omega\times(0,\infty),\\ u(x,0)\ge 0,\not\equiv 0, & \text{in } \Omega.\end{cases} \tag{2.13}$$

By use of (2.12), for any small $\epsilon > 0$, we can find a large $T_3 > 0$ such that

$$-\epsilon < z < \epsilon, \quad \text{for all } x \in \overline{\Omega} \text{ and } t \geq T_3.$$

Consider the two auxiliary problems:

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial t} - \Delta \tilde{u} = (a - d\tilde{u} + d\epsilon)\tilde{u}, & \text{in } \ \Omega \times (T_3, \infty), \\ \partial_\nu \tilde{u} = 0, & \text{on } \ \partial\Omega \times (T_3, \infty), \\ \tilde{u}(x, T_3) = u(x, T_3) > 0, & \text{in } \ \Omega, \end{cases} \tag{2.14}$$

and

$$\begin{cases} \dfrac{\partial \hat{u}}{\partial t} - \Delta \hat{u} = (a - d\hat{u} - d\epsilon)\hat{u}, & \text{in } \ \Omega \times (T_3, \infty), \\ \partial_\nu \hat{u} = 0, & \text{on } \ \partial\Omega \times (T_3, \infty), \\ \hat{u}(x, T_3) = u(x, T_3) > 0, & \text{in } \ \Omega. \end{cases} \tag{2.15}$$

It is clear that $\tilde{u}(x,t)$ and $\hat{u}(x,t)$ respectively are the upper and lower solutions of (2.13). Thus, due to the comparison principle for parabolic equations, we get

$$\hat{u}(x,t) < u(x,t) < \tilde{u}(x,t), \quad \text{for all } x \in \overline{\Omega} \text{ and } t \geq T_3.$$

By Lemma 2.1 in [9], we have

$$\tilde{u}(x,t) \to \tilde{u}_\epsilon(x) \ \text{ and } \ \hat{u}(x,t) \to \hat{u}_\epsilon(x)$$

uniformly on $\overline{\Omega}$ as $t \to \infty$, where $\tilde{u}_\epsilon(x)$ and $\hat{u}_\epsilon(x)$ respectively are the corresponding unique steady state of (2.14) and (2.15). By a simple upper-lower solution argument for elliptic equations, together with the uniqueness of solution, it is easily proved that both $\tilde{u}_\epsilon(x)$ and $\hat{u}_\epsilon(x)$ converge to $\overline{u}(x)$ uniformly on $\overline{\Omega}$ as $\epsilon \to 0$, where $\overline{u}(x) = \frac{a}{d}$ is the unique positive solution of the following system:

$$\begin{cases} -\Delta u = (a - du)u, & \text{in } \ \Omega, \\ \partial_\nu u = 0, & \text{on } \ \partial\Omega. \end{cases} \tag{2.16}$$

Thus, our analysis shows $u(x,t) \to \frac{a}{d}$ uniformly on $\overline{\Omega}$ as $\epsilon \to 0$. Then, using the theory in [2], we can confirm the asymptotic stability of (u^*, z^*). This completes the proof. $\square$

Case 3. $\frac{bc}{d} - e < a < \frac{bc}{d}$

By (2.2), the original system (1.1) is transformed into the following system:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = (a - du + dw)u, & \text{in } \ \Omega \times (0, \infty), \\ \dfrac{\partial w}{\partial t} - \Delta w = \Big(a + e - \dfrac{bc}{d}\Big)u - ew, & \text{in } \ \Omega \times (0, \infty), \\ \partial_\nu u = \partial_\nu w = 0, & \text{on } \ \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x) \geq 0, \not\equiv 0, & \text{in } \ \Omega, \\ w(x, 0) = w_0(x) = u(x, 0) - \dfrac{bv(x, 0)}{d}, & \text{in } \ \Omega. \end{cases} \tag{2.17}$$

When $\frac{bc}{d} - a - e < 0$, the system (2.17) has a unique equilibrium point (u^*, w^*), where

$$(u^*, w^*) = \Big(\frac{ae}{bc - ad}, \ \frac{a(ad + de - bc)}{d(bc - ad)}\Big).$$

To prove the global stability of (u^*, w^*) for system (2.17), we also need to show $u(x, t), w(x, t)$ is bounded for all x and large t.

Proposition 5 *Assume that* $\frac{bc}{d} - e < a < \frac{bc}{d}$. *then for any small* $\epsilon > 0$, *there exists a large* $T > 0$ *and two positive constants* C_1 *and* C_2 *such that the solution* $(u(x, t), w(x, t))$ *of* (2.17) *satisfies:*

$$0 < u(x, t) < C_1, \quad -\epsilon < w(x, t) < C_2$$

for all $x \in \overline{\Omega}$ *and* $T > 0$.

Proof. The proof is similar to that of Proposition 2. In this case, we notice that the system (2.17) is a quasimonotone increasing system (see section 8.2 in [7]). We construct the similar pair of lower and upper solutions

$$(\underline{u}, \underline{w}) = (0, c_2^* \exp{(-et)}), \quad (\overline{u}, \overline{w}) = (C_1, C_2),$$

in the present case, $c_2^* = \min_{\overline{\Omega}} w_0(x)$ and C_1, C_2 should satisfy

$$\frac{a}{d} + C_2 \leq C_1, \quad (a + e - \frac{bc}{d})C_1 \leq eC_2. \tag{2.18}$$

In view of $a < \frac{bc}{d}$, we can choose $M \geq \frac{ae}{bc - ad}$, and C_1, C_2 satisfies

$$C_1 = M, \qquad \frac{d(a + e) - bc}{de}M \leq C_2 \leq M - \frac{a}{d}.$$

Then the inequalities of (2.18) hold. This implies that

$$0 < u(x, t) < C_1, \qquad c_2^* \exp(-et) < w(x, t) < C_2$$

for all $x \in \overline{\Omega}$ and $t > 0$. By letting t be large, we have that, there is a large $T > 0$ such that

$$0 < u(x, t) < C_1, \qquad -\epsilon < w(x, t) < C_2$$

for all $x \in \overline{\Omega}$ and $t > T$. This ends the proof. □

Now, based on Proposition 5, we can claim the following result:

Proposition 6 *Suppose that* $\frac{bc}{d} - e < a < \frac{bc}{d}$*. Then the solution* (u^*, w^*) *for system* (2.17) *is globally asymptotically stable, which implies the global stability of* (u^*, v^*) *for system* (1.1).

Proof. The proof is similar to that of Proposition 2.3. We first construct the following Lyapunov functional:

$$E(t) = \int_\Omega E(u(x,t), w(x,t))\mathrm{d}x,$$

with

$$E(u, w) = \int \frac{u - u^*}{u}\mathrm{d}u + \lambda \int (w - w^*)\mathrm{d}z,$$

where λ is a positive constant to be determined.

$$\begin{aligned}
\frac{\mathrm{d}E(t)}{\mathrm{d}t} &= \int_\Omega \Big\{\frac{u-u^*}{u}u_t + \lambda(w - w^*)w_t\Big\}\mathrm{d}x \\
&= \int_\Omega \frac{u-u^*}{u}\Delta u\mathrm{d}x + \lambda\int_\Omega (w-w^*)\Delta w\mathrm{d}x + \int_\Omega (u-u^*)(a - du + dw)\mathrm{d}x \\
&\quad +\lambda \int_\Omega (w-w^*)\left[\Big(a + e - \frac{bc}{d}\Big)u - ew\right]\mathrm{d}x \\
&= -\int_\Omega \frac{u^*|\nabla u|^2}{u^2}\mathrm{d}x - \lambda\int_\Omega |\nabla w|^2\mathrm{d}x + \int_\Omega (u-u^*)(du^* - dw^* - du + dw)\mathrm{d}x \\
&\quad +\lambda \int_\Omega (w-w^*)\left[\Big(a + e - \frac{bc}{d}\Big)(u-u^*) - e(w-w^*)\right]\mathrm{d}x \\
&= -\int_\Omega \frac{u^*|\nabla u|^2}{u^2}\mathrm{d}x - \lambda\int_\Omega |\nabla w|^2\mathrm{d}x - d\int_\Omega (u-u^*)^2\mathrm{d}x \\
&\quad + \left[\lambda\Big(a + e - \frac{bc}{d}\Big) + d\right]\int_\Omega (u-u^*)(w-w^*)\mathrm{d}x - e\lambda\int_\Omega (w-w^*)^2\mathrm{d}x.
\end{aligned}$$

Applying the similar argument to the last part in the proof of Proposition 3, we can choose a fixed constant $\lambda = \frac{d^2(de+bc-ad)}{(ad+de-bc)^2} > 0$ such that for any $t > 0$, $E(t)$ is a Lyapunov function for system (2.17), namely, for $t > 0$, $E'(t) < 0$ along trajectories except at (u^*, w^*) where $E'(t) = 0$. The equivalence between (1.1) and (2.17) implies the global stability of $(u(x,t), v(x,t))$ under the condition $\frac{bc}{d} - e < a < \frac{bc}{d}$.

Now from Propositions 3, 4 and 6, we arrive at the conclusion of Theorem 1 and the proof of Theorem 1 is completed. $\square$

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