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On the limit cycles for a class of discontinuous piecewise cubic polynomial differential systems

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Abstract. This paper presents new results on the bifurcation of medium and small limit cycles from the periodic orbits surrounding a cubic center or from the cubic center that have a rational first integral of degree 2 respectively, when they are perturbed inside the class of all discontinuous piecewise cubic polynomial differential systems with the straight line of discontinuity $y = 0$.

We obtain that the maximum number of medium limit cycles that can bifurcate from the periodic orbits surrounding the cubic center is 9 using the first order averaging method, and the maximum number of small limit cycles that can appear in a Hopf bifurcation at the cubic center is 6 using the fifth order averaging method. Moreover, both of the numbers can be reached by analyzing the number of simple zeros of the obtained averaged functions. In some sense, our results generalize the results in [*Appl. Math. Comput.* **250**(2015), 887–907], Theorems 1 and 2 to the piecewise systems class.

Keywords: averaging method, center, piecewise differential systems, limit cycle, periodic orbits.

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1 Introduction and main results

One of the main open problems in the qualitative theory of real planar differential systems is the determination and distribution of limit cycles. There are several methods for studying the bifurcation of limit cycles. One of the methods is by perturbing a differential system which has a center. In this case the perturbed system displays limit cycles that bifurcate, either from some of the periodic orbits surrounding the center, or from the center (having the so-called Hopf bifurcation), see the book of Christopher–Li [4], and references cited therein.

The problem of bounding the number of limit cycles for planar smooth differential systems has been exhaustively studied in the last century and is closed related to the 16th Hilbert's problem [10, 13]. Solving this problem even for the quadratic case seems to be out of reach at the present state of knowledge. In the last few years there has been an increasing interest in

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the study of discontinuous piecewise differential systems, see [3,7,11,14,18,21] for instance. This interest has been mainly motivated by their wider range of application in various fields of science (e.g., control theory, biology, chemistry, engineering, physics, etc.).

Our goal in this paper is to study the bifurcation of limit cycles for a class of cubic polynomial differential systems having a rational first integral of degree 2. We remark that the classification of all cubic polynomial differential systems having a center at the origin and a rational first integral of degree 2 can be found in [17]. Later on, the authors in [16] summarized this classification in six families of cubic polynomial differential systems. In particular they obtained the class

$$\dot{x} = 2y(x+\alpha)^2, \quad \dot{y} = -2(x+\alpha)(\alpha x - y^2), \tag{1.1}$$

where $\alpha \neq 0$. System (1.1) called class P_6 in [16], which has $H(x,y) = \frac{x^2+y^2}{(\alpha+x)^2}$ as its first integral with the integrating factor $\mu(x,y) = 1/(\alpha+x)^4$. See [16] for the phase portraits of system (1.1) in the Poincaré disk.

A natural question is: *What happens with the periodic orbits (or the center) of the system (1.1) when it is perturbed inside the class of all smooth cubic polynomial differential systems, or inside the class of all discontinuous piecewise cubic polynomial differential systems with a straight line of discontinuity?*

In this article we say a *medium limit cycle* is one which bifurcates from a periodic orbit surrounding a center, and a *small limit cycle* is one which bifurcates from a center equilibrium point. Remark that, for the piecewise cubic polynomial vector fields there are two recent works, see [8,9], obtaining at least 18 and 24 small limit cycles, respectively. Our objective in this paper is to study the maximal number of medium and small limit cycles for the cubic center (1.1), when they are perturbed inside the class of all discontinuous piecewise cubic polynomial differential systems with the straight line of discontinuity $y=0$. The main results are based on the averaging method. We remark that the method of averaging is a classic and mature tool for studying the behaviour of nonlinear differential systems in the presence of a small parameter. For more details about this method see the book of Sanders, Verhulst and Murdock [24] and Llibre, Moeckel and Simó [19].

More precisely, we consider the following discontinuous piecewise polynomial differential systems

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2y(x+\alpha)^2 \\ -2(x+\alpha)(\alpha x - y^2) \end{pmatrix} + \varepsilon \begin{cases} \begin{pmatrix} p_1(x,y) \\ q_1(x,y) \end{pmatrix}, & y>0, \\ \begin{pmatrix} p_2(x,y) \\ q_2(x,y) \end{pmatrix}, & y<0, \end{cases} \tag{1.2}$$

where

$$\begin{aligned} p_1(x,y) &= \sum_{0\le i+j\le 3} a_{i,j}x^iy^j, \quad q_1(x,y) = \sum_{0\le i+j\le 3} b_{i,j}x^iy^j, \\ p_2(x,y) &= \sum_{0\le i+j\le 3} c_{i,j}x^iy^j, \quad q_2(x,y) = \sum_{0\le i+j\le 3} d_{i,j}x^iy^j. \end{aligned} \tag{1.3}$$

Moveover, we consider the following smooth polynomial differential systems

$$\begin{cases} \dot{x} = 2y(x+\alpha)^2 + \displaystyle\sum_{s=1}^{5} \varepsilon^s \mu_s(x,y), \\ \dot{y} = -2(x+\alpha)(\alpha x - y^2) + \displaystyle\sum_{s=1}^{5} \varepsilon^s \nu_s(x,y), \end{cases} \tag{1.4}$$

and the discontinuous piecewise cubic polynomial differential systems

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2y(x+\alpha)^2 \\ -2(x+\alpha)(\alpha x - y^2) \end{pmatrix} + \sum_{s=1}^{5} \varepsilon^s \begin{cases} \begin{pmatrix} \mu_s(x,y) \\ \nu_s(x,y) \end{pmatrix}, & y > 0, \\ \begin{pmatrix} \psi_s(x,y) \\ \phi_s(x,y) \end{pmatrix}, & y < 0, \end{cases} \tag{1.5}$$

where

$$\mu_s(x,y) = \sum_{0 \le i+j \le 3} a_{s,i,j} x^i y^j, \quad \nu_s(x,y) = \sum_{0 \le i+j \le 3} b_{s,i,j} x^i y^j,$$
$$\psi_s(x,y) = \sum_{0 \le i+j \le 3} c_{s,i,j} x^i y^j, \quad \phi_s(x,y) = \sum_{0 \le i+j \le 3} d_{s,i,j} x^i y^j.$$

The main results of this paper are stated as follows.

Theorem 1.1. *For $|\varepsilon| > 0$ sufficiently small the maximum number of medium limit cycles of the discontinuous piecewise differential system (1.2) is 9 using the first order averaging method, and this number can be reached.*

If $a_{i,j} = c_{i,j}$ and $b_{i,j} = d_{i,j}$ (see (1.3)), then the perturbed system (1.2) is smooth. In this case, we obtain the following corollary of Theorem 1.1.

Corollary 1.2. *When $a_{i,j} = c_{i,j}$ and $b_{i,j} = d_{i,j}$, the maximum number of medium limit cycles of system (1.2) that bifurcate using the first order averaging method is 3 and it is reached.*

Remark 1.3. Theorem 1.1 gives the exact upper bound of the number of limit cycles bifurcated from the periodic orbits of the cubic center (1.1), which is challenging. Theorem 1.1 and Corollary 1.2 show that the maximum number of limit cycles for the piecewise case is 6 more than the smooth one. We note that the smooth case of system (1.2) has been studied in [16, Section 3.3] under the condition $a_{0,0} = b_{0,0} = c_{0,0} = d_{0,0} = 0$. Corollary 1.2 shows that the non-zero constant terms provide no more information on the limit cycles. However, in the piecewise case, with the non-zero constant terms the perturbed system (1.2) can produce at least one more limit cycle than the case without them (see Remark 3.1 in Section 3). This phenomenon coincides with the well-known pseudo-Hopf bifurcation (see [2,6]).

Theorem 1.4. *For $|\varepsilon| > 0$ sufficiently small using the fifth order averaging method, we obtain that*

(a) for any real constants $a_{s,i,j}$ and $b_{s,i,j}$ ($s = 1,\dots,5, 0 \le i+j \le 3$) with $a_{1,0,0} = b_{1,0,0} = 0$, system (1.4) has at most 2 small limit cycles bifurcating from the center (1.1), and this number can be reached;

(b) system (1.5) has at most 6 small limit cycles bifurcating from the center (1.1) under the condition $a_{1,0,0} = b_{1,0,0} = c_{1,0,0} = d_{1,0,0} = 0$, and this number can be reached.

More concretely, we provide in Table 1.1 the maximum number of limit cycles for systems (1.4) and (1.5) up to the i-th order averaging method for $i = 1,\dots,5$.

The outline of this paper is as follows. In Section 2, we introduce the basic theory of the averaging method for discontinuous piecewise planar differential systems. The averaged function associated to system (1.2) is obtained in Section 3. Section 4 focuses on the analysis of the exact upper bound for the number of zeros of the averaged function, and the theory of Chebyshev systems is used to prove Theorem 1.1. The objective of Section 5 is to study the small limit cycles of systems (1.4) and (1.5). Finally, we present the explicit formulae of the i-th order averaged function up to $i = 5$ in Appendix A for reference.

Averaging order	System (1.4)	System (1.5)
1	0	1
2	0	2
3	1	4
4	1	6
5	2	6

Table 1.1: Number of small limit cycles for systems (1.4) and (1.5).

2 Preliminary results

In this section we introduce the basic theory of the averaging method that we shall use in our study of the cubic center (1.1). The following result is due to Itikawa, Llibre and Novaes [14].

Consider the discontinuous piecewise differential systems of the form

$$\frac{dr}{d\theta} = r' = \begin{cases} F^{+}(\theta, r, \varepsilon), & \text{if } 0 \leq \theta \leq \gamma, \\ F^{-}(\theta, r, \varepsilon), & \text{if } \gamma \leq \theta \leq 2\pi, \end{cases} \tag{2.1}$$

where

$$F^{\pm}(\theta, r, \varepsilon) = \sum_{i=1}^{k} \varepsilon^{i} F_i^{\pm}(\theta, r) + \varepsilon^{k+1} R^{\pm}(\theta, r, \varepsilon),$$

and ε is a real small parameter. The set of discontinuity of system (2.1) is $\sum = \{\theta = 0\} \cup \{\theta = \gamma\}$ if $0 < \gamma < 2\pi$. Here $F_i^{\pm} : \mathbb{S}^1 \times D \to \mathbb{R}$ for $i = 1, \ldots, k$, and $R^{\pm} : \mathbb{S}^1 \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ are $\mathcal{C}^k$ functions, being D an open and bounded interval of $(0, \infty)$, ε_0 is a small parameter, and $\mathbb{S}^1 \equiv \mathbb{R}/(2\pi)$. This last convention implies that the functions involved in system (2.1) are 2π-periodic in the variable θ.

The averaging method consists in defining a collection of functions $f_i : D \to \mathbb{R}$, called the i-th order averaged function, for $i = 1, 2, \ldots, k$, which control (their simple zeros control), for $|\varepsilon| > 0$ sufficiently small, the isolated periodic solutions of the differential system (2.1). In Itikawa–Llibre–Novaes [14] it has been established that

$$f_i(z) = \frac{y_i^{+}(\gamma, z) - y_i^{-}(\gamma - 2\pi, z)}{i!}, \tag{2.2}$$

where $y_i^{\pm} : \mathbb{S}^1 \times D \to \mathbb{R}$, for $i = 1, 2, \ldots, k$, are defined recurrently by the following integral equations

$$\begin{aligned} y_1^{\pm}(\theta, z) &= \int_0^{\theta} F_1^{\pm}(\varphi, z) d\varphi, \\ y_i^{\pm}(\theta, z) &= i! \int_0^{\theta} \Big(F_i^{\pm}(\varphi, z) + \sum_{\ell=1}^{i} \sum_{S_\ell} \frac{1}{b_1! b_2! 2!^{b_2} \cdots b_\ell! \ell!^{b_\ell}} \cdot \partial^L F_{i-\ell}^{\pm}(\varphi, z) \prod_{j=1}^{\ell} y_j^{\pm}(\varphi, z)^{b_j} \Big) d\varphi, \end{aligned} \tag{2.3}$$

where S_ℓ is the set of all ℓ-tuples of non-negative integers $[b_1, b_2, \ldots, b_\ell]$ satisfying $b_1 + 2b_2 + \cdots + \ell b_\ell = \ell$ and $L = b_1 + b_2 + \cdots + b_\ell$. Here, $\partial^L F(\varphi, z)$ denotes the Fréchet's derivative with respect to the variable z. We remark that, the investigation in this paper is restricted to $F_0 = 0$ in expression (2.3). For the general form of the averaged functions see [20].

We point out that taking $\gamma = 2\pi$ system (2.1) becomes smooth. So the averaging method described above can also apply to smooth differential systems. In practical terms, the evaluation of the recurrence (2.3) is a computational problem that requires powerful computerized resources. Usually, the higher the averaging order is, the more complex are the computational operations to calculate the averaged function (2.2). Recently in [22] the Bell polynomials were used to provide a relatively simple alternative formula for the recurrence (2.3). And based on this new formula, an algorithmic approach to revisit the averaging method was introduced in [12] for the analysis of bifurcation of small limit cycles of planar differential systems. Moreover, we provide an upper bound of the number of zeros of the averaged functions for the general class of perturbed differential systems (see Theorem 3.1 in [12]).

The following k-th order averaging theorem gives a criterion for the existence of limit cycles. Its proof can be found in Section 2 of [14].

Theorem 2.1 ([14]). *Assume that, for some $j \in \{1, 2, \ldots, k\}$, $f_i = 0$ for $i = 1, 2, \ldots, j-1$ and $f_j \neq 0$. If there exists $z^* \in D$ such that $f_j(z^*) \neq 0$, then for $|\varepsilon| > 0$ sufficiently small, there exists a 2π-periodic solution $r(\theta, \varepsilon)$ of (2.1) such that $r(0, \varepsilon) \to z^*$ when $\varepsilon \to 0$.*

The following theorem (see Theorem 5.2 of [1] for a proof) provides an approach to transform a perturbed differential system into the standard form (2.1), which can be used to apply the first order averaging method.

Theorem 2.2 ([1]). *Consider the differential system*

$$\begin{aligned} \dot{x} &= P(x, y) + \varepsilon p(x, y), \\ \dot{y} &= Q(x, y) + \varepsilon q(x, y), \end{aligned} \tag{2.4}$$

where P, Q, p and q are continuous functions in the variables x and y, and ε is a small parameter. Suppose that system $(2.4)_{\varepsilon=0}$ has a continuous family of ovals $\{\Gamma_h\} \subset \{(x, y) | H(x, y) = h, h \in (h_1, h_2)\}$, where $H(x, y)$ is a first integral of $(2.4)_{\varepsilon=0}$, and h_1 and h_2 correspond to the center and the separatrix polycycle, respectively. For a given first integral $H = H(x, y)$, assume that $xQ(x, y) - yP(x, y) \neq 0$ for all (x, y) in the periodic annulus formed by the ovals $\{\Gamma_h\}$. Let $\rho : (\sqrt{h_1}, \sqrt{h_2}) \times [0, 2\pi) \to [0, +\infty)$ be a continuous function such that

$$H(\rho(R, \varphi) \cos \varphi, \rho(R, \varphi) \sin \varphi) = R^2,$$

for all $R \in (\sqrt{h_1}, \sqrt{h_2})$ and all $\varphi \in [0, 2\pi)$. Then the differential equation which describes the dependence between the square root of energy $R = \sqrt{h}$ and the angle φ for system (2.4) is

$$\frac{dR}{d\varphi} = \varepsilon \frac{\mu(x^2 + y^2)(Qp - Pq)}{2R(Qx - Py)} + \mathcal{O}(\varepsilon^2), \tag{2.5}$$

where $\mu = \mu(x, y)$ is the integrating factor of system $(2.4)_{\varepsilon=0}$ corresponding to the first integral H, and $x = \rho(R, \varphi) \cos \varphi$ and $y = \rho(R, \varphi) \sin \varphi$.

In general, it is not an easy thing to deal with zeros of the averaged function (2.2). The techniques and arguments to tackle this kind of problem are usually very long and technical. In what follows we present some effective results on obtaining the lower bound and the upper bound of the number of zeros for a complicated function. The next result is used to obtain a lower bound of the number of simple zeros for an averaged function.

Lemma 2.3 ([5]). *Consider $n+1$ linearly independent analytical functions $f_i(x) : A \to \mathbb{R}, i = 0,1,\dots,n$, where $A \subset \mathbb{R}$ is an interval. Suppose that there exists $k \in \{0,1,\dots,n\}$ such that $f_k(x)$ has constant sign. Then there exist $n+1$ constants $c_i, i = 0,1,\dots,n$ such that $c_0 f_0(x) + c_1 f_1(x) + \cdots + c_n f_n(x)$ has at least n simple zeros in A.*

It is important to point out that the classical theory of Chebyshev systems is useful to provide an upper bound for the number of zeros. Let $\mathcal{F} = [f_0, \dots, f_n]$ be an ordered set of functions of class C^n defined in the closed interval $[a,b]$. We consider only elements in $\text{Span}(\mathcal{F})$, that is, functions such as $f = a_0 f_0 + a_1 f_1 + \cdots + a_n f_n$ where a_i, for $i = 0,1,\dots,n$, are real numbers. When the maximum number of zeros, taking into account its multiplicity, is n, the set $\mathcal{F}$ is called an Extended Chebyshev system (ET-system) in $[a,b]$. We say that $\mathcal{F}$ is an Extended Complete Chebyshev system (ECT-system) in $[a,b]$, if any set $[f_0, f_1, \dots, f_k]$, for $k = 0,\dots,n$ is an ET-system. When all the Wronskians, $W_k := W(f_0, f_1, \dots, f_k) \neq 0$ for $0 \le k \le n$ in $[a,b]$ the set $\mathcal{F}$ is an ECT-system. For more details on ET-systems and ECT-system, see [15] for instance.

We remark that not always the standard study of ET-systems can be applied to bound the number of zeros of elements in $\text{Span}(\mathcal{F})$. Here we use an extension of this theory (see [23]). The following result provides an effective estimation for the number of isolated zeros of elements in $\text{Span}(\mathcal{F})$ when some Wronskians vanish.

Theorem 2.4 ([23]). *Let $\mathcal{F} = [f_0, f_1, \dots, f_n]$ be an ordered set of analytic functions in $[a,b]$. Assume that all the ν_i zeros of the Wronskian W_i are simple for $i = 0,1,\dots,n$. Then the number of isolated zeros for every element of Span $(\mathcal{F})$ does not exceed*

$$n + \nu_n + \nu_{n-1} + 2(\nu_{n-2} + \cdots + \nu_0) + \lambda_{n-1} + \cdots + \lambda_3,$$

where $\lambda_i = \min(2\nu_i, \nu_{i-3} + \cdots + \nu_0)$, for $i = 3,\dots,n-1$.

3 Averaged function associated to system (1.2)

In this section we will get the first order averaged function associated to system (1.2) by using Theorem 2.1. We remark that the period annulus of the differential system (1.1) is formed by the ovals $\{\Gamma_h\} \subset \{(x,y) | H(x,y) = h, h \in (0,1)\}$. By solving implicitly the equation $H(\rho(R,\varphi)\cos\varphi, \rho(R,\varphi)\sin\varphi) = R^2$ we obtain the positive function $\rho(R,\varphi)$ given by

$$\rho(R,\varphi) = -\frac{\alpha R(\text{signum}(\alpha) + R\cos\varphi)}{R^2\cos^2\varphi - 1}$$

for $\varphi \in [0, 2\pi)$ and $R \in (0,1)$, where $\text{signum}(\alpha)$ is the sign function defined by

$$\text{signum}(\alpha) = \begin{cases} 1, & \alpha > 0, \\ -1, & \alpha < 0. \end{cases}$$

Using Theorem 2.2, we can transform system (1.2) into the form

$$\frac{dR}{d\varphi} = \begin{cases} \varepsilon \dfrac{-(Qp_1 - Pq_1)}{4\alpha R(x+\alpha)^5}\Big|_{x=\rho(R,\varphi)\cos\varphi, y=\rho(R,\varphi)\sin\varphi} + \mathcal{O}(\varepsilon^2), & 0 \le \varphi \le \pi, \\ \varepsilon \dfrac{-(Qp_2 - Pq_2)}{4\alpha R(x+\alpha)^5}\Big|_{x=\rho(R,\varphi)\cos\varphi, y=\rho(R,\varphi)\sin\varphi} + \mathcal{O}(\varepsilon^2), & \pi \le \varphi \le 2\pi. \end{cases} \tag{3.1}$$

Now the discontinuous piecewise differential system (3.1) is under the assumptions of Theorem 2.1. Next, we will study the zeros of the averaged function $f : (0,1) \to \mathbb{R}$ given by

$$
\begin{aligned}
f(R) &= \int_0^{\pi} \frac{-(Qp_1 - Pq_1)}{4\alpha R(x+\alpha)^5}\Big|_{x=\rho(R,\varphi)\cos\varphi, y=\rho(R,\varphi)\sin\varphi} d\varphi \\
&\quad + \int_{\pi}^{2\pi} \frac{-(Qp_2 - Pq_2)}{4\alpha R(x+\alpha)^5}\Big|_{x=\rho(R,\varphi)\cos\varphi, y=\rho(R,\varphi)\cos\varphi} d\varphi \\
&= \int_0^{\pi} \frac{A(\varphi;a,b)\cos\varphi + B(\varphi;a,b)}{2\alpha^3(\text{signum}(\alpha)\cdot R\cos\varphi - 1)} d\varphi + \int_{\pi}^{2\pi} \frac{A(\varphi;c,d)\cos\varphi + B(\varphi;c,d)}{2\alpha^3(\text{signum}(\alpha)\cdot R\cos\varphi - 1)} d\varphi,
\end{aligned}
$$

where

$$
\begin{aligned}
A(\varphi;a,b) = &-R^3\big[\alpha^3(a_{0,3} - a_{2,1} - b_{3,0} + b_{1,2}) + \alpha^2(-b_{0,2} + b_{2,0} + a_{1,1}) \\
&+ \alpha(-b_{1,0} - a_{0,1}) + b_{0,0}\big]S^3 + \text{signum}(\alpha)\cdot R^2\big[\alpha^3 R^2(a_{1,2} - a_{3,0}) \\
&+ \alpha^2(R^2(a_{2,0} - a_{0,2}) - a_{0,2} + a_{2,0} - b_{1,1}) + \alpha(-R^2 a_{1,0} - 2a_{1,0} + 2b_{0,1}) \\
&+ (R^2+3)a_{0,0}\big]S^2 - R\big[\alpha^3 R^2(a_{2,1} + b_{3,0}) - \alpha^2 R^2(2a_{1,1} + b_{2,0}) \\
&+ \alpha(R^2(3a_{0,1} + b_{1,0}) + a_{0,1} + b_{1,0}) - (R^2+3)b_{0,0}\big]S + \text{signum}(\alpha) \\
&\cdot \big[\alpha^3 R^4 a_{3,0} - \alpha^2 R^2(R^2+1)a_{2,0} + \alpha R^2(R^2+3)a_{1,0} - (R^4 + 6R^2 + 1)a_{0,0}\big], \\
B(\varphi;a,b) = \; &R^3[\alpha^3(-b_{0,3} + b_{2,1} + a_{1,2} - a_{3,0}) + \alpha^2(-b_{1,1} + a_{2,0} - a_{0,2}) \\
&+ \alpha(-a_{1,0} + b_{0,1}) + a_{0,0}]S^4 + \text{signum}(\alpha)\cdot R^2[\alpha^3 R^2(a_{0,3} - a_{2,1}) \\
&+ \alpha^2((R^2+1)a_{1,1} - b_{0,2} + b_{2,0}) - \alpha((R^2+2)a_{0,1} + 2b_{1,0}) + 3b_{0,0}]S^3 \\
&- R[\alpha^3 R^2(a_{1,2} - 2a_{3,0} + b_{2,1}) + \alpha^2 R^2(-2a_{0,2} + 3a_{2,0} - b_{1,1}) \\
&+ \alpha(R^2(-4a_{1,0} + b_{0,1}) - a_{1,0} + b_{0,1}) + (5R^2+3)a_{0,0}]S^2 \\
&+ \text{signum}(\alpha)\cdot[\alpha^3 R^4 a_{2,1} - \alpha^2 R^2((R^2+1)a_{1,1} + b_{2,0}) \\
&+ \alpha R^2((R^2+3)a_{0,1} + 2b_{1,0}) - (3R^2+1)b_{0,0}]S \\
&- R[\alpha^3 R^2 a_{3,0} - 2\alpha^2 R^2 a_{2,0} + \alpha(3R^2+1)a_{1,0} - 4(R^2+1)a_{0,0}]
\end{aligned}
$$

with $S = \sin\varphi$, and $a = (a_{i,j})$, $b = (b_{i,j})$, $c = (c_{i,j})$ and $d = (d_{i,j})$, with $a_{i,j}, b_{i,j}, c_{i,j}$ and $d_{i,j}$ are parameters appearing in the perturbed polynomials (1.3).

Computing the above integrals and making the transformation $R = \frac{2\omega}{1+\omega^2}$ for $0 < \omega < 1$ we obtain

$$
f(R) \overset{R=\frac{2\omega}{1+\omega^2}}{=} \frac{\tilde{f}(\omega)}{6\alpha^3\omega(\omega^2+1)^3} = \frac{\sum_{i=0}^{8} k_i f_i(\omega)}{6\alpha^3\omega(\omega^2+1)^3}, \tag{3.2}
$$

where

$$
\begin{aligned}
&f_0(\omega) = \omega^2, \quad f_1(\omega) = \omega^4, \quad f_2(\omega) = \omega^6, \\
&f_3(\omega) = \omega^8, \quad f_4(\omega) = \omega^5 + \omega^3, \quad f_5(\omega) = \omega^7 + \omega, \\
&f_6(\omega) = \omega^4 \ln\left(\frac{1+\omega}{1-\omega}\right), \quad f_7(\omega) = (\omega^8+1)\ln\left(\frac{1+\omega}{1-\omega}\right), \\
&f_8(\omega) = (\omega^6+\omega^2)\ln\left(\frac{1+\omega}{1-\omega}\right),
\end{aligned} \tag{3.3}
$$

and

$$
\begin{aligned}
k_0 =& -3\pi\Big(-\alpha(a_{1,0}+c_{1,0})-\alpha(b_{0,1}+d_{0,1})+4(a_{0,0}+c_{0,0})\Big),\\
k_1 =& -3\pi\Big(-3\alpha^3(a_{3,0}+c_{3,0})-3\alpha^3(b_{0,3}+d_{0,3})-\alpha^3(a_{1,2}+c_{1,2})\\
&-\alpha^3(b_{2,1}+d_{2,1})+4\alpha^2(a_{0,2}+c_{0,2})+4\alpha^2(a_{2,0}+c_{2,0})-6\alpha(a_{1,0}+c_{1,0})\\
&-2\alpha(b_{0,1}+d_{0,1})+12(a_{0,0}+c_{0,0})\Big),\\
k_2 =& -3\pi\Big(2\alpha^3(a_{1,2}+c_{1,2})+2\alpha^3(a_{3,0}+c_{3,0})-2\alpha^3(b_{0,3}+d_{0,3})-2\alpha^3(b_{2,1}+d_{2,1})\\
&-\alpha(a_{1,0}+c_{1,0})-\alpha(b_{0,1}+d_{0,1})+4(a_{0,0}+c_{0,0})\Big),\\
k_3 =& 3\pi\alpha^3\Big((a_{1,2}+c_{1,2})-(a_{3,0}+c_{3,0})-(b_{0,3}+d_{0,3})+(b_{2,1}+d_{2,1})\Big),\\
k_4 =& \operatorname{signum}(\alpha)\cdot\Big[-2\alpha^3(a_{2,1}-c_{2,1})-22\alpha^3(a_{0,3}-c_{0,3})+2\alpha^3(b_{1,2}-d_{1,2})\\
&-26\alpha^3(b_{3,0}-d_{3,0})+8\alpha^2(b_{2,0}-d_{2,0})+8\alpha^2(a_{1,1}-c_{1,1})+16\alpha^2(b_{0,2}-d_{0,2})\\
&-32\alpha(a_{0,1}-c_{0,1})-8\alpha(b_{1,0}-d_{1,0})+26(b_{0,0}-d_{0,0})\Big],\\
k_5 =& \operatorname{signum}(\alpha)\cdot\Big[6\alpha^3(a_{0,3}-c_{0,3})+6\alpha^3(b_{1,2}-d_{1,2})-6\alpha^3(b_{3,0}-d_{3,0})\\
&-6\alpha^3(a_{2,1}-c_{2,1})+6(b_{0,0}-d_{0,0})\Big],\\
k_6 =& -\operatorname{signum}(\alpha)\cdot 6\alpha^3\Big(3(a_{0,3}-c_{0,3})+(a_{2,1}-c_{2,1})-(b_{1,2}-d_{1,2})-3(b_{3,0}-d_{3,0})\Big),\\
k_7 =& -\operatorname{signum}(\alpha)\cdot 3\alpha^3\Big((a_{0,3}-c_{0,3})-(a_{2,1}-c_{2,1})+(b_{1,2}-d_{1,2})-(b_{3,0}-d_{3,0})\Big),\\
k_8 =& \operatorname{signum}(\alpha)\cdot 12\alpha^3\Big((a_{0,3}-c_{0,3})+(b_{3,0}-d_{3,0})\Big).
\end{aligned}
$$

It follows directly from

$$
\frac{\partial(k_0,k_1,k_2,k_3,k_4,k_5,k_6,k_7,k_8)}{\partial(b_{0,0},a_{3,0},a_{1,2},a_{1,0},a_{1,1},a_{2,0},b_{3,0},a_{2,1},a_{0,3})}=\operatorname{signum}(\alpha)\cdot 107495424\pi^4\alpha^{20}\neq 0
$$

that the constants $k_0,k_1,\cdots,k_8$ are independent. That is to say, the coefficients of the functions $f_i(\omega)$, $i=0,1,\ldots,8$ can be chosen arbitrarily. Moreover, the functions $f_0(\omega),\ldots,f_8(\omega)$ are linearly independent. In fact, we obtain the following Taylor expansions in the variable ω around $\omega=0$ for these functions:

$$
\begin{aligned}
&f_0(\omega)=\omega^2,\quad f_1(\omega)=\omega^4,\quad f_2(\omega)=\omega^6,\\
&f_3(\omega)=\omega^8,\quad f_4(\omega)=\omega^5+\omega^3,\quad f_5(\omega)=\omega^7+\omega,\\
&f_6(\omega)=2\omega^5+\frac{2}{3}\omega^7+\frac{2}{5}\omega^9+\mathcal{O}(\omega^{11}),\\
&f_7(\omega)=2\omega+\frac{2}{3}\omega^3+\frac{2}{5}\omega^5+\frac{2}{7}\omega^7+\frac{20}{9}\omega^9+\mathcal{O}(\omega^{11}),\\
&f_8(\omega)=2\omega^3+\frac{2}{3}\omega^5+\frac{12}{5}\omega^7+\frac{20}{21}\omega^9+\mathcal{O}(\omega^{11}).
\end{aligned}
\tag{3.4}
$$

The determinant of the coefficient matrix of the variables $\omega,\omega^2,\ldots,\omega^9$ is equal to 8388608/496125. Hence, by Lemma 2.3 it follows that there exists a linear combination of $f_i(\omega)$, $i=0,1,\ldots,8$ with at least 8 simple zeros, which means that system (1.2) has at least 8 limit cycles bifurcating from the period orbits surrounding the origin.

Remark 3.1. We notice that when the constant terms $a_{0,0}$, $b_{0,0}$, $c_{0,0}$, $d_{0,0}$ are identically zeros. In a similar way, we can prove that system (1.2) has at least 7 limit cycles bifurcating from the period orbits surrounding the origin. In fact, $k_5 + 2k_7 = 0$ in this case, and the function $\tilde{f}(\omega)$ in (3.2) is a linear combination of 8 linearly independent functions $f_0, \dots, f_4, f_6, f_7 - 2f_5, f_8$. Therefore, by Lemma 2.3, the perturbed system (1.2) with the non-zero constant terms can produce at least one more limit cycle than the case without them.

Proof of Corollary 1.2. Obviously, when $a_{i,j} = c_{i,j}$ and $b_{i,j} = d_{i,j}$, the coefficients $k_4, k_5, \dots, k_8$ are identically zeros. It is easy to check that $(f_0, \dots, f_3)$ is an ECT-system. Therefore, the averaged function f in this case has at most 3 simple zeros and this number can be reached. Hence, by Theorem 2.1, Corollary 1.2 is proved. □

In what follows, we first provide an upper bound of the number of zeros of the function $\tilde{f}(\omega)$ in (3.2). We eliminate the logarithmic function by taking the ninth derivative of $\tilde{f}(\omega)$ and obtain

$$\tilde{f}^{(9)}(\omega) = \text{signum}(\alpha) \cdot \frac{110592\alpha^3}{(1+\omega)^9(-1+\omega)^9}(H_1\omega^8 + H_2\omega^6 + H_3\omega^4 + H_2\omega^2 + H_1),$$

where

$$\begin{aligned}
H_1 &= -14(a_{2,1} - c_{2,1}) + 14(b_{1,2} - d_{1,2}) + 8(a_{0,3} - c_{0,3}) - 83(b_{3,0} - d_{3,0}),\\
H_2 &= -24(a_{2,1} - c_{2,1}) + 24(b_{1,2} - d_{1,2}) - 32(a_{0,3} - c_{0,3}) - 1988(b_{3,0} - d_{3,0}),\\
H_3 &= 76(a_{2,1} - c_{2,1}) - 76(b_{1,2} - d_{1,2}) + 48(a_{0,3} - c_{0,3}) - 4818(b_{3,0} - d_{3,0}).
\end{aligned}$$

As a result of the symmetry of coefficients of the function $\tilde{f}^{(9)}(\omega)$ with respect to ω, it is easy to know that the zeros of the function $\tilde{f}^{(9)}(\omega)$ appear in pairs. Recalling this property, we obtain that $\tilde{f}^{(9)}(\omega)$ has at most 2 zeros in $(0,1)$. Thus, by using Rolle's theorem and noting the fact that $\tilde{f}(0) = 0$, we conclude that $\tilde{f}(\omega)$ has at most $2 + 9 - 1 = 10$ zeros in $(0,1)$, which means that system (1.2) has at most 10 limit cycles bifurcating from the period orbits surrounding the origin. In next section, we will show that the bound of the number of limit cycles can be reduced to 9 by Theorem 2.4. Moreover, this number can be reached.

4 Proof of Theorem 1.1

In this section we will study the maximum number of simple zeros of the averaged function (3.2). The main effort is based largely on algebraic calculations with the theory of Chebyshev systems used to deal with the Wronskian determinants.

First, we denote by $W_i(\omega)$ the Wronskian for the functions $f_0, f_1, \dots, f_i$ depending on ω:

$$W_i(\omega) = W(f_0, \dots, f_i), \quad i = 0, 1, \dots, 8.$$

Next, we will show that all the Wronskians have no zeros except $W_7(\omega)$ which vanishes at a unique zero in $(0,1)$, which is simple. Using the expressions in (3.3), we perform the calculation and obtain

$$
\begin{aligned}
&W_0(\omega) = \omega^2, \quad W_1(\omega) = 2\omega^5, \quad W_2(\omega) = 16\omega^9, \\
&W_3(\omega) = 768\omega^{14}, \quad W_4(\omega) = 2304\omega^{13}(3\omega^2 - 5), \\
&W_5(\omega) = 69120\omega^9(1-\omega^2)(3\omega^6 - 7\omega^4 - 7\omega^2 + 35), \\
&W_6(\omega) = -\frac{3317760\omega^8(\omega^2+1)}{(1-\omega^2)^5} T_6(\omega), \\
&W_7(\omega) = -\frac{133772083200\omega(\omega^2+1)^3 T_{7,0}(\omega)}{(1-\omega^2)^4}\left(\ln\left(\frac{1+\omega}{1-\omega}\right) - \frac{2\omega T_{7,1}(\omega)}{105(1-\omega^2)^6 T_{7,0}(\omega)}\right), \\
&W_8(\omega) = \frac{821895679180800(\omega^2+1)^6}{(1-\omega^2)^{10}}\left(T_{8,0}(\omega)\cdot\ln\left(\frac{1+\omega}{1-\omega}\right) + \frac{2\omega T_{8,1}(\omega)}{105(1-\omega^2)^4}\right),
\end{aligned}
\tag{4.1}
$$

where

$$
\begin{aligned}
T_6(\omega) &= 15\omega^{14} - 195\omega^{12} - 89\omega^{10} + 1149\omega^8 + 421\omega^6 - 4305\omega^4 + 805\omega^2 - 105 < 0, \\
T_{7,0}(\omega) &= 15\omega^8 - 140\omega^6 + 1018\omega^4 - 140\omega^2 + 15 > 0, \\
T_{7,1}(\omega) &= 160\omega^{20} - 8569\omega^{18} + 105687\omega^{16} - 547324\omega^{14} + 1437092\omega^{12} - 2101414\omega^{10} \\
&\quad + 1752730\omega^8 - 839580\omega^6 + 210980\omega^4 - 23625\omega^2 + 1575, \\
T_{8,0}(\omega) &= 35\omega^8 - 1100\omega^6 + 2898\omega^4 - 1100\omega^2 + 35, \\
T_{8,1}(\omega) &= 45477\omega^{14} - 444465\omega^{12} + 1433397\omega^{10} - 2210985\omega^8 + 1803095\omega^6 \\
&\quad - 745675\omega^4 + 128975\omega^2 - 3675,
\end{aligned}
\tag{4.2}
$$

by Sturm's theorem. It is easy to judge that $W_i(\omega)$ for $i = 0, \dots, 6$ does not vanish in the open interval $(0,1)$. The difficulties mainly focus on the determination of $W_7(\omega)$ and $W_8(\omega)$.

Proposition 4.1. *$W_7(\omega)$ has a unique zero in $\omega \in (0,1)$ and this zero is simple.*

Proof. Denote the function in the parentheses of $W_7(\omega)$ by $Q_7(\omega)$, then

$$
Q_7'(\omega) = \frac{64\omega^6(\omega^2+1)(5\omega^8 + 172\omega^6 - 1122\omega^4 + 172\omega^2 + 5)T_6(\omega)}{105(1-\omega^2)^7 T_{7,0}^2(\omega)}
$$

has a unique simple zero ω^* in $(0,1)$ and can be easily isolated (e.g. by using the command realroot(%,1/10000) in Maple) as $\omega^* \in \left[\frac{112087}{262144}, \frac{14011}{32768}\right]$. It follows that $Q_7(\omega)$ decreases in $(0,\omega^*)$ and increases in $(\omega^*,1)$. Note also that $\lim_{\omega\to 0^+} Q_7(\omega) = 0$ and $\lim_{\omega\to 1^-} Q_7(\omega) = +\infty$. Thus, $Q_7(\omega)$ has a unique simple zero in $(0,1)$, equivalently, $W_7(\omega)$ has a simple zero in $(0,1)$. This ends the proof. □

Proposition 4.2. *$W_8(\omega)$ does not vanish in $\omega \in (0,1)$.*

Proof. First, using Sturm's theorem, we get that $T_{8,0}(\omega)$ has two simple zeros ω_1 and ω_2 in $(0,1)$ and $T_{8,1}(\omega)$ has three simple zeros ω_3, ω_4 and ω_5 in $(0,1)$, and these zeros can be respectively isolated as

$$
\begin{aligned}
0.18709157 &\approx \omega_1 \in \left[\tfrac{6277751}{33554432}, \tfrac{784719}{4194304}\right], \\
0.64417845 &\approx \omega_2 \in \left[\tfrac{337735}{524288}, \tfrac{5403761}{8388608}\right], \\
0.18709131 &\approx \omega_3 \in \left[\tfrac{3138871}{16777216}, \tfrac{6277743}{33554432}\right], \\
0.66278355 &\approx \omega_4 \in \left[\tfrac{5559831}{8388608}, \tfrac{694979}{1048576}\right], \\
0.75595958 &\approx \omega_5 \in \left[\tfrac{792681}{1048576}, \tfrac{6341449}{8388608}\right].
\end{aligned}
$$

We denote the function in the parenthesis of $W_8(\omega)$ by $Q_8(\omega)$, it is easy to verify that $Q_8(\omega_1) \neq 0$ and $Q_8(\omega_2) \neq 0$. In order to study the number of zeros of $Q_8(\omega)$ in $(0,1)$ we define a function $Z_8(\omega)$ as follows

$$Z_8(\omega) := \frac{Q_8(\omega)}{T_{8,0}(\omega)} = \ln\left(\frac{1+\omega}{1-\omega}\right) + \frac{2\omega T_{8,1}(\omega)}{105(1-\omega^2)^4 T_{8,0}(\omega)}, \quad \omega \in (0,1)\backslash\{\omega_1, \omega_2\}.$$

It is obvious that the function $Z_8(\omega)$ has the following properties (see Fig. 4.1):

$$\lim_{\omega\to\omega_1^-} Z_8(\omega) = +\infty, \quad \lim_{\omega\to\omega_1^+} Z_8(\omega) = -\infty,$$
$$\lim_{\omega\to\omega_2^-} Z_8(\omega) = -\infty, \quad \lim_{\omega\to\omega_2^+} Z_8(\omega) = +\infty.$$

A direct calculation shows that

$$Z_8'(\omega) = \frac{32768\omega^8(\omega^2+1)H_8(\omega)}{35(1-\omega^2)^5 T_{8,0}^2(\omega)},$$

where

$$H_8(\omega) = 35\omega^{14} + 85\omega^{12} - 129\omega^{10} - 503\omega^8 - 119\omega^6 + 1855\omega^4 - 875\omega^2 + 35.$$

Obviously, $H_8(\omega)$ has two simple zeros ω_1^* and ω_2^* in $(0,1)$ and can be respectively isolated as

$$\begin{aligned} 0.21002672 \approx \omega_1^* &\in \left[\tfrac{451028943}{2147483648}, \tfrac{902057887}{4294967296}\right], \\ 0.69221454 \approx \omega_2^* &\in \left[\tfrac{185814925}{268435456}, \tfrac{743259701}{1073741824}\right]. \end{aligned} \tag{4.3}$$

Therefore $Z_8(\omega)$ increases when $\omega \in (0,\omega_1) \cup (\omega_1, \omega_1^*)$ and $\omega \in (\omega_2^*, 1)$; decreases when $\omega \in (\omega_1^*, \omega_2) \cup (\omega_2, \omega_2^*)$ (see Fig. 4.1). Notice that

$$\lim_{\omega\to 0^+} Z_8(\omega) = 0, \quad \lim_{\omega\to 1^-} Z_8(\omega) = +\infty.$$

It follows from (4.3) that

$$Z_8(\omega_1^*) \approx -0.0000126678 < 0, \quad Z_8(\omega_2^*) \approx 1.126483743 > 0.$$

Taking into account the above results, we conclude that $Z_8(\omega)$ does not vanish for $\omega \in (0,1)\backslash\{\omega_1, \omega_2\}$. Thus the desired result follows. □

Proof of Theorem 1.1. It follows from equation (4.1), Propositions 4.1 and 4.2 that $W_i(\omega)$, $i = 0, 1, \ldots 6$ and $W_8(\omega)$ do not vanish in the interval $(0,1)$, and $W_7(\omega)$ has exactly 1 simple zero in $(0,1)$. Thus $\mathcal{F} = [f_0, f_1, \ldots, f_8]$ defined in (3.3) satisfies the assumptions of Theorem 2.4, which implies that any linear combination of $f_0, f_1, \ldots, f_8$ can possess at most 9 zeros in $(0,1)$, counting with multiplicities. But the authors in [23] do not prove that the upper bound can be reached in the general cases. In what follows we will show that the upper bound 9 can be reached in our system.

Following the ideas of [23], we first look for an element in Span $(\mathcal{F})$ with a zero of the highest multiplicity, then we perturb it inside Span $(\mathcal{F})$ in order to have the prescribed configuration of zeros. We remark that since the Wronskian determinant $W_8(\omega)$ does not vanish,

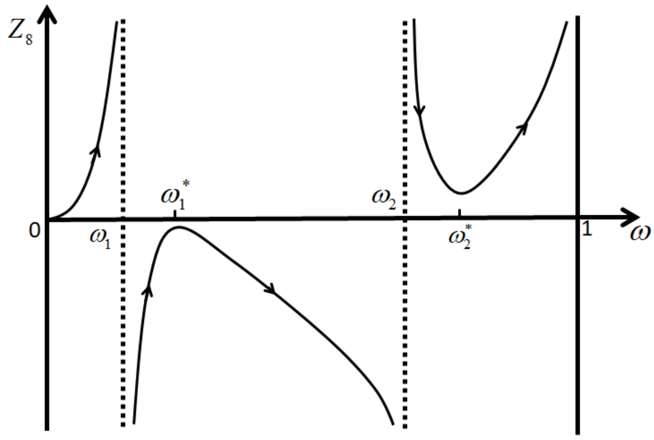

Figure 4.1: The curve $Z_8(\omega)$ does not vanish for $\omega \in (0,1)\backslash\{\omega_1, \omega_2\}$.

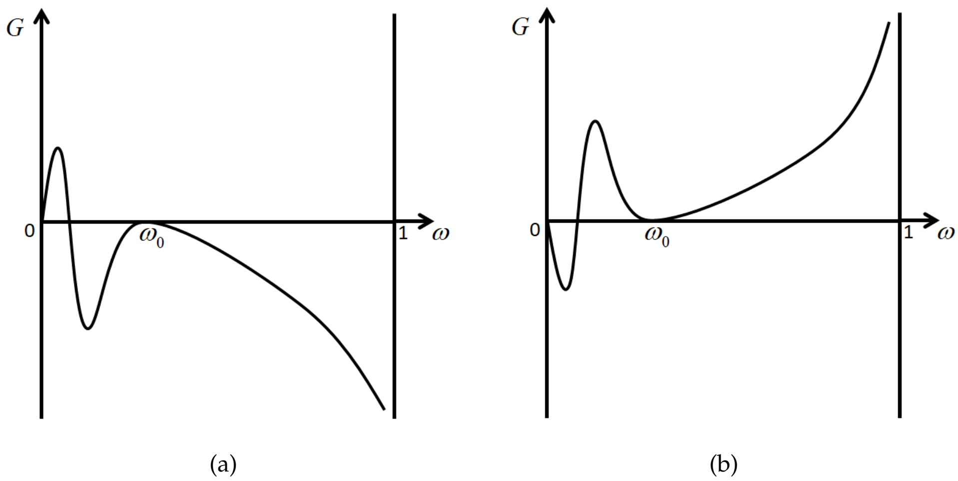

Figure 4.2: Two cases for $G(\omega)$ having 9 zeros in $(0,1)$ taking into account multiplicity. In particular ω_0 has multiplicity 8.

the averaged function (an element in Span $(\mathcal{F})$) can not have a zero in $(0,1)$ with multiplicity 9. Then we try to find an element $G(\omega) = \sum_{i=0}^{7} a_i f_i + k f_8 \in$ Span $(\mathcal{F})$, of which has a zero $\omega_0 \in (0,1)$ with multiplicity 8. Note that $G(\omega)$ has 9 zeros in $(0,1)$ with ω_0 of multiplicity 8 may have two cases as shown in Fig. 4.2. For the generation of such ω_0 we provide an algorithm (Maple program) in Appendix B.

Now let $\omega_0 = 781/10001$, $K_0 = \ln\left(\frac{1+\omega_0}{1-\omega_0}\right)$ and $k = 10^8$. Consider the function

$$G(\omega) = a_0 f_0(\omega) + a_1 f_1(\omega) + \cdots + a_7 f_7(\omega) + k f_8(\omega), \quad \omega \in (0,1). \tag{4.4}$$

By direct calculation we get the power series of G around the point ω_0:

$$G(\omega) = e_0 + e_1(\omega - \omega_0) + \cdots + e_7(\omega - \omega_0)^7 + e_8(\omega - \omega_0)^8 + \cdots,$$

where e_i is the linear combination of $a_0, a_1, \ldots, a_7$. We solve the equations

$$e_0 = 0, \quad e_1 = 0, \quad \ldots, \quad e_7 = 0,$$

and find the values of $a_0, a_1, \ldots, a_7$ which have the form

$$a_i = \frac{\sum_{j=0}^{j_i} L_{i,j} K_0^j}{k_1 K_0 + k_2}, \quad i = 0, \ldots, 7, \tag{4.5}$$

where

$$\begin{aligned} k_1 = {} & 585397408871072540089139375831993705697245971 \\ & 45302237853421492432853598362240000000, \\ k_2 = {} & -\,916164764498521481287490087182092157549 \\ & 2097096776449170037730387965998807150160399, \end{aligned}$$

and

$$j_i = \begin{cases} 2, & i \in \{0, 1, 2\}, \\ 1, & i \in \{3, 4, 5, 6\}, \\ 0, & i \in \{7\}, \end{cases}$$

and each $L_{i,j}$ in (4.5) is an integer or rational with high number of digits in numerators and denominators. We will not write down here the explicit expression of a_i for the sake of brevity. It turns out that

$$G(\omega) = e_8(\omega - \omega_0)^8 + \mathcal{O}((\omega - \omega_0)^9), \quad \omega \to \omega_0, \tag{4.6}$$

where

$$e_8 = -\frac{k_3 \cdot (B_1 K_0 + B_0)}{625678681207969855947716482401 \cdot (k_1 K_0 + k_2)},$$

with

$$\begin{aligned} k_3 &= 6373960409705365063968756422951747001176840429758709070500, \\ B_1 &= 2371833114839857298494412882156005750986234376264757348752800000, \\ B_0 &= -371199090602328323784582373340236998424005450432934748931637759, \end{aligned}$$

and $e_8 \approx 6.468110730 \times 10^7$. On the other hand, the Taylor expansion of $G(\omega)$ near $\omega = 0$ is

$$G(\omega) = C_1 \omega + \mathcal{O}(\omega^2), \tag{4.7}$$

where

$$C_1 = \frac{k_4 \cdot \big(k_5 K_0 - k_6\big)}{55588252797009 \cdot (k_1 K_0 + k_2)} \approx -3.242325599$$

with

$$\begin{aligned} k_4 &= 227096370975140733661254232304854673313104068100000, \\ k_5 &= 864359913055284073500033389565682256669487378000, \\ k_6 &= 135274953622915880496646897785052547295533923181. \end{aligned}$$

By the way, we would like to point out that our purpose of choosing such a k in (4.4) is to make the expressions of the numbers e_8 and C_1 to be relative simple. Equations (4.6) and (4.7) mean that (i) G has a zero at ω_0 with multiplicity 8, (ii) there exists an ε_0 with $0 < \varepsilon_0 < \omega_0$ such that $G(\omega)$ is positive in $[\varepsilon_0, \omega_0)$, and (iii) $G(\omega)$ is negative near $\omega = 0$. Moreover, $G(\omega)$

is positive in $(\omega_0, 1)$ because $\lim\limits_{\omega\to 1^-} G(\omega) = +\infty$ (otherwise $G(\omega)$ would has 10 zeros in $(0,1)$ counting multiplicity, which leads to a contradiction).

Fixing the numbers $a_0, a_1, \ldots, a_7$ and k, we consider the function

$$G_\varepsilon(\omega) = G(\omega) + \sum_{i=0}^{8} \varepsilon_i f_i(\omega), \quad \omega \in (0,1). \tag{4.8}$$

We note that f_i can be extended analytically to $[0,1)$. Thus there exists a small number $M > 0$ such that

$$\begin{aligned} G_\varepsilon(\varepsilon_0) &> \frac{1}{2} G(\varepsilon_0) > 0, \\ G_\varepsilon(\omega) &< \frac{1}{2} C_1 \omega < 0, \quad \text{when } \omega \to 0^+, \\ \lim_{\omega\to 1^-} G_\varepsilon(\omega) &= +\infty, \end{aligned}$$

for all $|\varepsilon_i| < M$, $i = 0, 1, \ldots, 8$. Moreover, near ω_0 we find

$$\sum_{i=0}^{8} \varepsilon_i f_i(\omega) = \mu_0 + \mu_1(\omega - \omega_0) + \cdots + \mu_8(\omega - \omega_0)^8 + \cdots,$$

where $\mu_i = \mu_i(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_8)$ is linear combination of ε_0, ε_1, ..., ε_8. One can directly check that the matrix of the coefficients of $\mu_0, \mu_1, \ldots, \mu_8$ with respect to $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_8$ has rank 9, and hence $\mu_0, \mu_1, \ldots, \mu_8$ are independent.

Consequently, since f_i is analytic at ω_0 and $G(\omega)$ has a zero at ω_0 with multiplicity 8, it follows that there exists some small $|\varepsilon_i| \ll M$ $(i = 0, 1, \ldots 8)$ (and hence μ_i is small) such that G_ε has exactly 8 simple zeros in a small enough neighborhood of ω_0. In view of (4.8) $G(\omega)$ has an extra zero in $(0, \varepsilon_0)$. According to the result of [23], this zero is simple. That is to say, G_ε has 9 simple zeros.

Finally, taking into account the above analysis, we see that system (1.2), up to the first order averaging method, has at most 9 limit cycles, and the upper bound can be reached. The proof of Theorem 1.1 is finished. □

Remark 4.3. If $\bar{R}$ is a simple zero of the averaged function $f(R)$ (see (3.2)), by Theorem 2.1 we have a limit cycle $R(\varphi, \varepsilon)$ of the differential system (3.1) such that $R(0, \varepsilon) = \bar{R} + \mathcal{O}(\varepsilon)$. Then, going back through the changes of variables (see (3.1)) we have for the discontinuous piecewise differential system (1.2) the medium limit cycle $(x(t,\varepsilon), y(t,\varepsilon)) = \big(\rho(\bar{R}, \cos\theta), \rho(\bar{R}, \sin\theta)\big) + \mathcal{O}(\varepsilon)$.

5 Proof of Theorem 1.4

In this section, we will present the k-th order averaged functions up to $k = 5$ associated to systems (1.4) and (1.5) respectively, and then we use them to prove Theorem 1.4.

5.1 Proof of Theorem 1.4 (a)

In order to analyze the Hopf bifurcation for system (1.4), applying Theorem 2.1, we set $\gamma = 2\pi$ in (2.2) and we introduce a small parameter ε doing the change of coordinates $x = \varepsilon X$, $y = \varepsilon Y$. After that we perform the polar change of coordinates $X = r\cos\theta$, $Y = r\sin\theta$, and by doing a

Taylor expansion truncated at the 5-th order in ε we obtain the following expression for $dr/d\theta$ of the form (2.1):

$$\frac{dr}{d\theta} = \sum_{i=0}^{5} \varepsilon^i F_i(\theta, r) + \mathcal{O}(\varepsilon^6), \tag{5.1}$$

where

$$F_0(\theta, r) = \frac{r(a_{1,0,0}\cos\theta + b_{1,0,0}\sin\theta)}{b_{1,0,0}\cos\theta - a_{1,0,0}\sin\theta - 2r\alpha^2}. \tag{5.2}$$

The explicit expressions of $F_i(\theta, r)$ for $i = 1, \dots, 5$ are quite large so we omit them. To make $F_0(\theta, r) = 0$ we take $a_{1,0,0} = b_{1,0,0} = 0$. From now on, for each $j = 1, 2, \dots, 5$, we will perform the calculation of the averaged function f_j under the hypothesis $f_k \equiv 0$ for $k = 1, \dots, j-1$.

Now computing f_1 we obtain

$$f_1(r) = -\frac{\pi r}{2\alpha^2}(a_{1,1,0} + b_{1,0,1}).$$

Clearly equation $f_1(r) = 0$ has no positive zeros. Then the first order averaging theorem does not provide information about the limit cycles of system (1.4).

To apply the second order averaging theorem we take $b_{1,0,1} = -a_{1,1,0}$. Computing f_2 we obtain

$$f_2(r) = -\frac{\pi r}{2\alpha^3}\Big(\alpha(a_{2,1,0} + b_{2,0,1}) - 4a_{2,0,0}\Big).$$

As for the first function f_1, the second one also does not provide information on the bifurcating limit cycles.

Setting $a_{2,0,0} = \alpha(a_{2,1,0} + b_{2,0,1})/4$ we get $f_2(r) = 0$, and then we can apply the third order averaging theorem, and its corresponding function f_3 is

$$f_3(r) = -\frac{\pi r}{16\alpha^5}\Big(D_{3,2}r^2 + D_{3,0}\Big),$$

where

$$\begin{aligned}
D_{3,2} &= 2\alpha\Big(\alpha^2(a_{1,1,2} + 3a_{1,3,0} + 3b_{1,0,3} + b_{1,2,1}) - 4\alpha(a_{1,0,2} + a_{1,2,0}) + 4a_{1,1,0}\Big),\\
D_{3,0} &= 8\alpha^3(a_{3,1,0} + b_{3,0,1}) - \alpha^2(a_{1,1,1}a_{2,1,0} + a_{1,1,1}b_{2,0,1} + 2a_{2,1,0}b_{1,0,2} + 2b_{1,0,2}b_{2,0,1} + 32a_{3,0,0})\\
&\quad + 4\alpha(a_{1,0,1}a_{2,1,0} + a_{1,0,1}b_{2,0,1} + 2a_{1,2,0}b_{2,0,0} + b_{1,1,1}b_{2,0,0}) - 16a_{1,1,0}b_{2,0,0}.
\end{aligned}$$

Then there exists at most one positive simple zero of f_3. From Theorem 2.1 it follows that the third order averaging provides the existence of at most one small limit cycle of system (1.4) and this number can be reached by Lemma 2.3 ($D_{3,2}$ and $D_{3,0}$ are linearly independent constants). In order to apply the fourth order averaging theorem, we need to have $f_3(r) = 0$ so we let $a_{1,0,2} = D_{3,2}/8\alpha^2 + a_{1,0,2}$ and $a_{3,0,0} = D_{3,0}/32\alpha^2 + a_{3,0,0}$. The resulting fourth order averaged function is

$$f_4(r) = -\frac{\pi r}{128\alpha^7}\Big(D_{4,2}r^2 + D_{4,0}\Big),$$

where

$$D_{4,2} = 2\alpha\Big(\alpha^4(8a_{2,1,2} + 8b_{2,2,1} + 24a_{2,3,0} + 24b_{2,0,3}) + \alpha^3(-a_{1,1,1}a_{1,1,2} - 3a_{1,1,1}a_{1,3,0} - 3a_{1,1,1}b_{1,0,3} - a_{1,1,1}b_{1,2,1} - 2b_{1,0,2}a_{1,1,2} - 6b_{1,0,2}a_{1,3,0} - 6b_{1,0,2}b_{1,0,3} - 2b_{1,0,2}b_{1,2,1} - 32a_{2,0,2} - 32a_{2,2,0}) + \alpha^2(4a_{1,0,1}a_{1,1,2} + 24a_{1,0,1}a_{1,3,0} + 12a_{1,0,1}b_{1,0,3} + 8a_{1,0,1}b_{1,2,1} - 8a_{1,1,0}a_{1,2,1} - 8a_{1,1,0}b_{1,1,2} + 8b_{1,0,2}a_{1,2,0} + 8a_{1,2,0}b_{1,2,0} + 12a_{1,3,0}b_{1,1,0} + 4b_{1,0,2}b_{1,1,1} + 4b_{1,1,0}b_{1,2,1} + 4b_{1,1,1}b_{1,2,0} + 24a_{2,1,0} - 8b_{2,0,1}) + \alpha(-24a_{1,0,1}a_{1,2,0} - 4a_{1,0,1}b_{1,1,1} + 20a_{1,1,0}a_{1,1,1} - 8a_{1,1,0}b_{1,0,2} - 16a_{1,1,0}b_{1,2,0} - 24a_{1,2,0}b_{1,1,0} - 4b_{1,1,0}b_{1,1,1}) + 32a_{1,1,0}b_{1,1,0}\Big),$$

$$D_{4,0} = 64\alpha^5(b_{4,0,1} + a_{4,1,0}) + \alpha^4(-8a_{1,1,1}a_{3,1,0} - 8a_{1,1,1}b_{3,0,1} - 8a_{2,1,1}a_{2,1,0} - 16b_{2,0,2}a_{2,1,0} - 8a_{2,1,1}b_{2,0,1} - 16b_{1,0,2}a_{3,1,0} - 16b_{1,0,2}b_{3,0,1} - 16b_{2,0,2}b_{2,0,1} - 256a_{4,0,0}) + \alpha^3(a_{1,1,1}^2a_{2,1,0} + a_{1,1,1}^2b_{2,0,1} + 4a_{1,1,1}b_{1,0,2}a_{2,1,0} + 4a_{1,1,1}b_{1,0,2}b_{2,0,1} + 4b_{1,0,2}^2a_{2,1,0} + 4b_{1,0,2}^2b_{2,0,1} + 32a_{1,0,1}a_{3,1,0} + 32a_{1,0,1}b_{3,0,1} + 64a_{1,2,0}b_{3,0,0} + 32a_{2,0,1}a_{2,1,0} + 32a_{2,0,1}b_{2,0,1} + 64a_{2,2,0}b_{2,0,0} + 32b_{1,1,1}b_{3,0,0} + 32b_{2,0,0}b_{2,1,1}) + \alpha^2(-4a_{1,0,1}a_{1,1,1}a_{2,1,0} - 4a_{1,0,1}a_{1,1,1}b_{2,0,1} - 8a_{1,0,1}b_{1,0,2}a_{2,1,0} - 8a_{1,0,1}b_{1,0,2}b_{2,0,1} + 8a_{1,2,0}a_{1,1,0}a_{2,1,0} + 8a_{1,2,0}a_{1,1,0}b_{2,0,1} + 4a_{1,1,0}b_{1,1,1}a_{2,1,0} + 4a_{1,1,0}b_{1,1,1}b_{2,0,1} - 8a_{1,1,1}a_{1,2,0}b_{2,0,0} - 4a_{1,1,1}b_{1,1,1}b_{2,0,0} - 16b_{1,0,2}a_{1,2,0}b_{2,0,0} - 8b_{1,0,2}b_{1,1,1}b_{2,0,0} - 128a_{1,1,0}b_{3,0,0} - 96a_{2,1,0}b_{2,0,0} + 32b_{2,0,0}b_{2,0,1}) + \alpha(32a_{1,0,1}a_{1,2,0}b_{2,0,0} + 16a_{1,0,1}b_{1,1,1}b_{2,0,0} - 16a_{1,1,0}^2a_{2,1,0} - 16a_{1,1,0}^2b_{2,0,1} + 32a_{1,2,0}b_{1,1,0}b_{2,0,0} + 16b_{1,1,0}b_{1,1,1}b_{2,0,0}) - 64a_{1,1,0}b_{1,1,0}b_{2,0,0}.$$

Then there exists at most one positive simple zero of f_4. From Theorem 2.1 it follows that the fourth order averaging provides the existence of at most one small limit cycle of system (1.4) and this number can be reached.

Letting $a_{2,0,2} = D_{4,2}/64\alpha^4 + a_{2,0,2}$ and $a_{4,0,0} = D_{4,0}/256\alpha^4 + a_{4,0,0}$ we obtain $f_4(r) = 0$, so we can apply the fifth order averaging theorem, and its corresponding function is of the form

$$f_5(r) = \frac{\pi r}{1024\alpha^9}\left(D_{5,4}r^4 + D_{5,2}r^2 + D_{5,0}\right),$$

where $D_{5,4} = 64\alpha^5(a_{1,1,2} + a_{1,3,0} - b_{1,0,3} - b_{1,2,1})$. Here we do not explicitly provide the expressions of $D_{5,2}$ and $D_{5,0}$, because they are very long. Moreover $D_{5,4}$, $D_{5,2}$ and $D_{5,0}$ are linearly independent constants. In fact only $D_{5,2}$ has the parameter $a_{3,0,2}$, and $D_{5,0}$ is the only one with parameters $a_{5,0,0}$ and $b_{5,0,1}$. We claim that $D_{5,4}$ is also linearly independent of the other coefficients. Suppose that this is false. Then there exist real numbers m_1, m_2 not all zero such that $D_{5,4} = m_1D_{5,0} + m_2D_{5,2}$. But $D_{5,0}$ is the only one with the variables $a_{5,0,0}$ and $b_{5,0,1}$, so in order to $D_{5,4}$ does not present these variables we must set $m_1 = 0$. Since the other function $D_{5,2}$ also has variable which uniquely appears in its expression, the same argument holds so $m_2 = 0$. But then $D_{5,4} \equiv 0$, which is a contradiction. Therefore $D_{5,4}$, $D_{5,2}$ and $D_{5,0}$ are linearly independent constants. Hence f_5 has at most two positive simple zeros. From Theorem 2.1 it follows that the fifth order averaging provides the existence of at most two small limit cycle of system (1.4) and this number can be reached by Lemma 2.3.

5.2 Proof of Theorem 1.4 (b)

In order to analyze the Hopf bifurcation for this case, applying Theorem 2.1, we set $\gamma = \pi$ in (2.2). By using similar arguments to those presented for the proof of Theorem 1.4 (a), we can transform system (1.5) into the form

$$\frac{dr}{d\theta} = \begin{cases} \sum_{i=1}^{5} \varepsilon^i F_i^+(\theta, r) + \mathcal{O}(\varepsilon^6), & \text{if } 0 \le \theta \le \pi, \\ \sum_{i=1}^{5} \varepsilon^i F_i^-(\theta, r) + \mathcal{O}(\varepsilon^6), & \text{if } \pi \le \theta \le 2\pi, \end{cases} \tag{5.3}$$

where

$$\begin{aligned} F_1^+(\theta, r) = -\frac{1}{2\alpha^2}\Big[&\big(r(a_{1,0,1} + b_{1,1,0}) \sin\theta + a_{2,0,0}\big) \cos\theta \\ &+ r(-a_{1,1,0} + b_{1,0,1}) \sin^2\theta + (2\alpha r^2 + b_{2,0,0}) \sin\theta + r a_{1,1,0}\Big], \end{aligned} \tag{5.4}$$

and $F_1^-(\theta, r)$ is an expression by taking $a = c$, $b = d$ in $F_1^+(\theta, r)$. The explicit expressions of $F_i^{\pm}(\theta, r)$ for $i = 2, \dots, 5$ are quite large so we omit them here for brevity. We remark that we have used the condition $a_{1,0,0} = b_{1,0,0} = c_{1,0,0} = d_{1,0,0} = 0$ for the vanishing of the unperturbed terms $F_0^+(\theta, r)$ and $F_0^-(\theta, r)$.

Now applying Theorem 2.1 we obtain the first order averaged function

$$f_1(r) = -\frac{1}{4\alpha^2}\left(Y_{1,1} r + Y_{1,0}\right),$$

where

$$Y_{1,1} = \pi(a_{1,1,0} + c_{1,1,0} + b_{1,0,1} + d_{1,0,1}), \quad Y_{1,0} = 4(b_{2,0,0} - d_{2,0,0}).$$

It is obvious that the coefficients $Y_{1,1}$ and $Y_{1,0}$ are independent. Thus $f_1(r)$ can have one positive zero. From Theorem 2.1 it follows that the first order averaging provides the existence of at most one small limit cycle of system (1.5) and this number can be reached.

To consider the second order averaging theorem we take $d_{1,0,1} = -Y_{1,1}/\pi + d_{1,0,1}$ and $d_{2,0,0} = Y_{1,0}/4 + d_{2,0,0}$. Computing f_2 we obtain

$$f_2(r) = -\frac{1}{48\alpha^4}\left(Y_{2,2} r^2 + Y_{2,1} r + Y_{2,0}\right),$$

where

$$Y_{2,2} = 16\alpha\Big((a_{1,1,1} - c_{1,1,1} + 2b_{1,0,2} - 2d_{1,0,2} + b_{1,2,0} - d_{1,2,0})\alpha - 4(a_{1,0,1} - c_{1,0,1}) - (b_{1,1,0} - d_{1,1,0})\Big),$$

$$\begin{aligned} Y_{2,1} = -3\pi\Big(&-4(a_{2,1,0} + c_{2,1,0} + b_{2,0,1} + d_{2,0,1})\alpha^2 + 16(a_{2,0,0} + c_{2,0,0})\alpha \\ &+ a_{1,1,0}(a_{1,0,1} - c_{1,0,1}) + b_{1,0,1}(a_{1,0,1} - c_{1,0,1}) - a_{1,1,0}(b_{1,1,0} - d_{1,1,0}) - b_{1,0,1}(b_{1,1,0} - d_{1,1,0})\Big), \end{aligned}$$

$$Y_{2,0} = 24\Big(2(b_{3,0,0} - d_{3,0,0})\alpha^2 - a_{1,1,0}(a_{2,0,0} + c_{2,0,0}) - b_{1,0,1}(a_{2,0,0} + c_{2,0,0}) + b_{2,0,0}(b_{1,1,0} - d_{1,1,0})\Big).$$

Since $f_2(r)$ can have at most two positive zeros, we conclude that system (1.5) has at most two small limit cycles and this number can be reached.

To consider the third order averaging theorem we take $d_{1,0,2} = Y_{2,2}/32\alpha^2 + d_{1,0,2}$, $d_{2,0,1} = -Y_{2,1}/12\pi\alpha^2 + d_{2,0,1}$ and $d_{3,0,0} = Y_{2,0}/48\alpha^2 + d_{3,0,0}$. Computing f_3 we obtain

$$r f_3(r) = -\frac{1}{1152\alpha^6}\Big(Y_{3,4} r^4 + Y_{3,3} r^3 + Y_{3,2} r^2 + Y_{3,1} r + Y_{3,0}\Big),$$

where

$$Y_{3,4} = 72\pi\alpha^2\Big((a_{1,1,2} + c_{1,1,2} + 3a_{1,3,0} + 3c_{1,3,0} + 3b_{1,0,3} + 3d_{1,0,3} + b_{1,2,1} + d_{1,2,1})\alpha^2 - 4(a_{1,0,2} + c_{1,0,2} + a_{1,2,0} + c_{1,2,0})\alpha + 4(a_{1,1,0} + c_{1,1,0})\Big),$$
$$Y_{3,0} = 72\pi(a_{2,0,0} - c_{2,0,0})(a_{2,0,0} + c_{2,0,0})(a_{1,1,0} + b_{1,0,1}).$$

We do not explicitly provide the expressions of $Y_{3,i}$ for $i = 1, 2, 3$, since they are very long. Since $f_3(r)$ can have at most four positive zeros, we conclude that system (1.5) has at most four small limit cycles and this number can be reached.

To consider the fourth order averaging theorem, we need to have $f_3(r) = 0$ so we let $d_{1,0,3} = -Y_{3,4}/216\pi\alpha^4 + d_{1,0,3}$, $d_{2,0,2} = Y_{3,3}/768\alpha^4 + d_{2,0,2}$, $d_{3,0,1} = -Y_{3,2}/288\pi\alpha^4 + d_{3,0,1}$, $d_{4,0,0} = Y_{3,1}/1152\alpha^4 + d_{4,0,0}$. Note that in order to make $Y_{3,0} = 0$, we consider the following three cases.

CASE 1. $a_{2,0,0} = c_{2,0,0}$, $a_{2,0,0} \neq -c_{2,0,0}$ and $a_{1,1,0} \neq -b_{1,0,1}$.

In this case, computing f_4 we obtain

$$r^2 f_4(r) = -\frac{1}{23040\alpha^8}\Big(Y^1_{4,6}r^6 + Y^1_{4,5}r^5 + Y^1_{4,4}r^4 + Y^1_{4,3}r^3 + Y^1_{4,2}r^2 + Y^1_{4,1}r + Y^1_{4,0}\Big),$$

where

$$Y^1_{4,6} = -1536\alpha^5\Big(8(a_{1,0,3} - c_{1,0,3}) + 2(a_{1,2,1} - c_{1,2,1}) - 2(b_{1,1,2} - d_{1,1,2}) - 3(b_{1,3,0} - d_{1,3,0})\Big),$$
$$Y^1_{4,1} = -720\pi(a_{1,1,0} + b_{1,0,1})c_{2,0,0}\Big(-4(a_{3,0,0} - c_{3,0,0})\alpha^2 + (a_{1,0,1} - c_{1,0,1})c_{2,0,0} - 2(a_{1,1,0} - c_{1,1,0})b_{2,0,0} - (b_{1,1,0} - d_{1,1,0})c_{2,0,0}\Big),$$
$$Y^1_{4,0} = -1920(a_{1,1,0} + b_{1,0,1})c^3_{2,0,0}.$$

We do not explicitly provide the expressions of $Y^1_{4,i}$ for $i = 2, 3, \ldots, 5$, since they are very long. Then $f_4(r)$ can have at most six positive zeros, we conclude that system (1.5) has at most six small limit cycles and this number can be reached.

To consider the fifth order averaging theorem, we need to have $f_4(r) = 0$ so we let $d_{1,3,0} = Y^1_{4,6}/4608\alpha^5 + d_{1,3,0}$, $d_{2,0,3} = -Y^1_{4,5}/4320\pi\alpha^6 + d_{2,0,3}$, $d_{3,0,2} = Y^1_{4,4}/15360\alpha^6 + d_{3,0,2}$, $d_{4,0,1} = -Y^1_{4,3}/5760\pi\alpha^6 + d_{4,0,1}$, $d_{5,0,0} = Y^1_{4,2}/23040\alpha^6 + d_{5,0,0}$, $c_{2,0,0} = 0$. Computing f_5 we obtain

$$r f_5(r) = -\frac{1}{5529600\alpha^{10}}\Big(Y^1_{5,6}r^6 + Y^1_{5,5}r^5 + Y^1_{5,4}r^4 + Y^1_{5,3}r^3 + Y^1_{5,2}r^2 + Y^1_{5,1}r + Y^1_{5,0}\Big),$$

where

$$Y^1_{5,6} = 115200\pi\alpha^4\Big((-2a_{1,1,2} - 2c_{1,1,2} - 3a_{1,3,0} - 3c_{1,3,0} + b_{1,2,1} + d_{1,2,1})\alpha^2 + 2(a_{1,0,2} + c_{1,0,2} + a_{1,2,0} + c_{1,2,0})\alpha - 2(a_{1,1,0} + c_{1,1,0})\Big),$$
$$Y^1_{5,0} = 86400\pi(a_{1,1,0} + b_{1,0,1})\Big(2(a_{3,0,0} + c_{3,0,0})\alpha^2 + b_{2,0,0}(a_{1,1,0} + c_{1,1,0})\Big) \cdot \Big(2(a_{3,0,0} - c_{3,0,0})\alpha^2 + b_{2,0,0}(a_{1,1,0} - c_{1,1,0})\Big).$$

We do not explicitly provide the expressions of $Y^1_{5,i}$ for $i = 1, 2, \ldots, 5$, since they are very long. Then $f_5(r)$ can have at most six positive zeros, we conclude that system (1.5) has at most six small limit cycles and this number can be reached.

CASE 2. $a_{2,0,0} = -c_{2,0,0}$, $a_{2,0,0} \neq c_{2,0,0}$ and $a_{1,1,0} \neq -b_{1,0,1}$.

In this case, computing f_4 we obtain

$$rf_4(r) = -\frac{1}{23040\alpha^8}\left(Y^2_{4,5}r^5 + Y^2_{4,4}r^4 + Y^2_{4,3}r^3 + Y^2_{4,2}r^2 + Y^2_{4,1}r + Y^2_{4,0}\right),$$

where

$$\begin{aligned} Y^2_{4,5} =& -1536\alpha^5\Big(8(a_{1,0,3} - c_{1,0,3}) + 2(a_{1,2,1} - c_{1,2,1}) - 2(b_{1,1,2} - d_{1,1,2}) - 3(b_{1,3,0} - d_{1,3,0})\Big),\\ Y^2_{4,0} =& -720\pi(a_{1,1,0} + b_{1,0,1})c_{2,0,0}\Big(4(a_{3,0,0} + c_{3,0,0})\alpha^2 + c_{2,0,0}(a_{1,0,1} - c_{1,0,1})\\ &+ 2b_{2,0,0}(a_{1,1,0} + c_{1,1,0}) - c_{2,0,0}(b_{1,1,0} - d_{1,1,0})\Big). \end{aligned}$$

We do not explicitly provide the expressions of $Y^2_{4,i}$ for $i = 1, 2, \ldots, 4$, since they are very long. Then $f_4(r)$ can have at most five positive simple zeros, we conclude that system (1.5) has at most five small limit cycles and this number can be reached.

To apply the fifth order averaging theorem, we need to have $f_4(r) = 0$ so we let $d_{1,3,0} = Y^2_{4,5}/4608\alpha^5 + d_{1,3,0}$, $d_{2,0,3} = -Y^2_{4,4}/4320\pi\alpha^6 + d_{2,0,3}$, $d_{3,0,2} = Y^2_{4,3}/15360\alpha^6 + d_{3,0,2}$, $d_{4,0,1} = -Y^2_{4,2}/5760\pi\alpha^6 + d_{4,0,1}$, $d_{5,0,0} = Y^2_{4,1}/23040\alpha^6 + d_{5,0,0}$. Note that in order to make $Y^2_{4,0} = 0$, we consider two subcases.

Subcase 1. $c_{2,0,0} = 0$ and $a_{3,0,0} \neq -\frac{1}{4\alpha^2}\big(c_{2,0,0}(a_{1,0,1} - c_{1,0,1}) + 2b_{2,0,0}(a_{1,1,0} + c_{1,1,0}) - c_{2,0,0}(b_{1,1,0} - d_{1,1,0})\big) - c_{3,0,0}$.

In this subcase, computing f_5 we obtain

$$rf_5(r) = -\frac{1}{5529600\alpha^{10}}\left(Y^{2,1}_{5,6}r^6 + Y^{2,1}_{5,5}r^5 + Y^{2,1}_{5,4}r^4 + Y^{2,1}_{5,3}r^3 + Y^{2,1}_{5,2}r^2 + Y^{2,1}_{5,1}r + Y^{2,1}_{5,0}\right),$$

where

$$\begin{aligned} Y^{2,1}_{5,6} &= 115200\pi\alpha^4\Big((-2a_{1,1,2} - 2c_{1,1,2} - 3a_{1,3,0} - 3c_{1,3,0} + b_{1,2,1} + d_{1,2,1})\alpha^2\\ &\quad + (2a_{1,0,2} + 2c_{1,0,2} + 2a_{1,2,0} + 2c_{1,2,0})\alpha - 2(a_{1,1,0} + c_{1,1,0})\Big),\\ Y^{2,1}_{5,0} &= 86400\pi(a_{1,1,0} + b_{1,0,1})\Big(2(a_{3,0,0} + c_{3,0,0})\alpha^2 + b_{2,0,0}(a_{1,1,0} + c_{1,1,0})\Big)\\ &\quad \cdot \Big(2(a_{3,0,0} - c_{3,0,0})\alpha^2 + b_{2,0,0}(a_{1,1,0} - c_{1,1,0})\Big). \end{aligned}$$

We do not explicitly provide the expressions of $Y^{2,1}_{5,i}$ for $i = 1, 2, \ldots, 5$, since they are very long. Then $f_5(r)$ can have at most six positive simple zeros, we conclude that system (1.5) has at most six small limit cycles and this number can be reached.

Subcase 2. $c_{2,0,0} \neq 0$ and $a_{3,0,0} = -\frac{1}{4\alpha^2}\big(c_{2,0,0}(a_{1,0,1} - c_{1,0,1}) + 2b_{2,0,0}(a_{1,1,0} + c_{1,1,0}) - c_{2,0,0}(b_{1,1,0} - d_{1,1,0})\big) - c_{3,0,0}$.

As in the **Subcase 1**, one can compute the expression of f_5 as follows:

$$rf_5(r) = -\frac{1}{5529600\alpha^{10}}\left(Y^{2,2}_{5,6}r^6 + Y^{2,2}_{5,5}r^5 + Y^{2,2}_{5,4}r^4 + Y^{2,2}_{5,3}r^3 + Y^{2,2}_{5,2}r^2 + Y^{2,2}_{5,1}r + Y^{2,2}_{5,0}\right),$$

where

$$
\begin{aligned}
Y^{2,2}_{5,6} &= Y^{2,1}_{5,6}, \\
Y^{2,2}_{5,0} &= 21600\pi(a_{1,1,0}+b_{1,0,1})c_{2,0,0}\Big[-32(a_{4,0,0}+c_{4,0,0})\alpha^4+8\Big(-2b_{2,0,0}(a_{2,1,0}+c_{2,1,0}) \\
&\quad +c_{3,0,0}(b_{1,1,0}-d_{1,1,0}-a_{1,0,1}+c_{1,0,1})+c_{2,0,0}(-a_{2,0,1}+c_{2,0,1}+b_{2,1,0}-d_{2,1,0}) \\
&\quad -2b_{3,0,0}(a_{1,1,0}+c_{1,1,0})\Big)\alpha^2+\Big(-4b_{2,0,0}(c_{1,1,0}(a_{1,0,1}+b_{1,1,0}-c_{1,0,1}+d_{1,1,0}) \\
&\quad +2a_{1,1,0}b_{1,1,0})+c_{2,0,0}(a^2_{1,0,1}+2a_{1,0,1}b_{1,1,0}+2a_{1,0,1}c_{1,0,1}-2a_{1,0,1}d_{1,1,0} \\
&\quad -4a_{1,1,0}b_{1,0,1}-4a_{1,1,0}c_{1,1,0}-4b_{1,0,1}c_{1,1,0}+b^2_{1,1,0}-2b_{1,1,0}c_{1,0,1} \\
&\quad +2b_{1,1,0}d_{1,1,0}-3c^2_{1,0,1}+2c_{1,0,1}d_{1,1,0}-4c^2_{1,1,0}-3d^2_{1,1,0})\Big)\Big].
\end{aligned}
$$

We do not explicitly provide the expressions of $Y^{2,2}_{5,i}$ for $i=1,2,\ldots,5$, since they are very long. Then $f_5(r)$ can have at most six positive simple zeros, we conclude that system (1.5) has at most six small limit cycles and this number can be reached.

CASE 3. $a_{1,1,0}=-b_{1,0,1}$ and $a^2_{2,0,0}-c^2_{2,0,0}\neq 0$.

Since the calculations and arguments are quite similar to those used in the **CASE 1**, we just provide the expressions of f_4 and f_5 as follows:

$$
\begin{aligned}
rf_4(r) &= -\frac{1}{5760\alpha^8}\Big(Y^3_{4,5}r^5+Y^3_{4,4}r^4+Y^3_{4,3}r^3+Y^3_{4,2}r^2+Y^3_{4,1}r+Y^3_{4,0}\Big), \\
rf_5(r) &= -\frac{1}{23040\alpha^{10}}\Big(Y^3_{5,6}r^6+Y^3_{5,5}r^5+Y^3_{5,4}r^4+Y^3_{5,3}r^3+Y^3_{5,2}r^2+Y^3_{5,1}r+Y^3_{5,0}\Big),
\end{aligned}
$$

where

$$
\begin{aligned}
Y^3_{4,5} &= -384\alpha^5\Big(8(a_{1,0,3}-c_{1,0,3})+2(a_{1,2,1}-c_{1,2,1})-2(b_{1,1,2}-d_{1,1,2})-3(b_{1,3,0}-d_{1,3,0})\Big), \\
Y^3_{4,0} &= -360\alpha\pi(a^2_{2,0,0}-c^2_{2,0,0})(-\alpha(a_{2,1,0}+b_{2,1,0})+4a_{2,0,0}), \\
Y^3_{5,6} &= 480\pi\alpha^4\Big((-2a_{1,1,2}-2c_{1,1,2}-3a_{1,3,0}-3c_{1,3,0}+b_{1,2,1}+d_{1,2,1})\alpha^2 \\
&\quad +2(a_{1,0,2}+c_{1,0,2}+a_{1,2,0}+c_{1,2,0})\alpha+2(b_{1,0,1}-c_{1,1,0})\Big), \\
Y^3_{5,0} &= 720\pi\alpha(a^2_{2,0,0}-c^2_{2,0,2})\Big(2(a_{3,1,0}+b_{3,0,1})\alpha^3-8a_{3,0,0}\alpha^2+(-a_{1,1,1}a_{2,0,0}+2a_{1,2,0}b_{2,0,0} \\
&\quad -2a_{2,0,0}b_{1,0,2}+b_{1,1,1}b_{2,0,0})\alpha+4(a_{1,0,1}a_{2,0,0}+b_{1,0,1}b_{2,0,0})\Big).
\end{aligned}
$$

We do not explicitly provide the expressions of $Y^3_{4,i}$ for $i=1,2,\ldots,4$ and $Y^3_{5,j}$ for $j=1,2,\ldots,5$, since they are very long. Then $f_5(r)$ can have at most six positive simple zeros, we conclude that system (1.5) has at most six small limit cycles and this number can be reached.

Using the results of Sections 5.1 and 5.2, we complete the proof of Theorem 1.4.

In summary, we give a remark for the averaging method that we are using in Section 5. We know that if the averaged functions $f_j=0$ for $j=1,\ldots,k-1$ and $f_k\neq 0$, and $\bar{r}$ is a simple zero of f_k, then by Theorem 2.1 there is a limit cycle $r(\theta,\varepsilon)$ of the differential system (5.3) such that $r(0,\varepsilon)=\bar{r}+\mathcal{O}(\varepsilon)$. Then, going back through the changes of variables ($x=\varepsilon r\cos\theta$, $y=\varepsilon r\sin\theta$) we have for the discontinuous piecewise differential system (1.5) the limit cycle $(x(t,\varepsilon),y(t,\varepsilon))=\varepsilon\big(\bar{r}\cos\theta,\bar{r}\sin\theta\big)+\mathcal{O}(\varepsilon^2)$, which tends to the origin of system (1.5) when the parameter $\varepsilon\to 0$. In other words, this limit cycle is a small limit cycle bifurcating from the origin, i.e., is a limit cycle coming by a Hopf bifurcation.

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A Fifth order averaging formulae

$$f_i(z) = \frac{y_i^+(\gamma, z) - y_i^-(\gamma - 2\pi, z)}{i!}, \quad \text{for } i = 1, \ldots, 5,$$

where

$$\begin{aligned}
y_1^\pm(\theta, z) = &\int_0^\theta F_1^\pm(\varphi, z) d\varphi, \\
y_2^\pm(\theta, z) = &\int_0^\theta \Big(2F_2^\pm(\varphi, z) + 2\partial F_1^\pm(\varphi, z) y_1^\pm(\varphi, z)\Big) d\varphi, \\
y_3^\pm(\theta, z) = &\int_0^\theta \Big(6F_3^\pm(\varphi, z) + 6\partial F_2^\pm(\varphi, z) y_1^\pm(\varphi, z) \\
&+ 3\partial^2 F_1^\pm(\varphi, z) y_1^\pm(\varphi, z)^2 + 3\partial F_1^\pm(\varphi, z) y_2^\pm(\varphi, z)\Big) d\varphi, \\
y_4^\pm(\theta, z) = &\int_0^\theta \Big(24F_4^\pm(\varphi, z) + 24\partial F_3^\pm(\varphi, z) y_1^\pm(\varphi, z) + 12\partial^2 F_2^\pm(\varphi, z) y_1^\pm(\varphi, z)^2 \\
&+ 12\partial F_2^\pm(\varphi, z) y_2^\pm(\varphi, z) + 12\partial^2 F_1^\pm(\varphi, z) y_1^\pm(\varphi, z) y_2^\pm(\varphi, z) \\
&+ 4\partial^3 F_1^\pm(\varphi, z) y_1^\pm(\varphi, z)^3 + 4\partial F_1^\pm(\varphi, z) y_3^\pm(\varphi, z)\Big) d\varphi, \\
y_5^\pm(\theta, z) = &\int_0^\theta \Big(120F_5^\pm(\varphi, z) + 120\partial F_4^\pm(\varphi, z) y_1^\pm(\varphi, z) + 60\partial^2 F_3^\pm(\varphi, z) y_1^\pm(\varphi, z)^2 \\
&+ 60\partial F_3^\pm(\varphi, z) y_2^\pm(\varphi, z) + 60\partial^2 F_2^\pm(\varphi, z) y_1^\pm(\varphi, z) y_2^\pm(\varphi, z) \\
&+ 20\partial^3 F_2^\pm(\varphi, z) y_1^\pm(\varphi, z)^3 + 20\partial F_2^\pm(\varphi, z) y_3^\pm(\varphi, z) \\
&+ 20\partial^2 F_1^\pm(\varphi, z) y_1^\pm(\varphi, z) y_3^\pm(\varphi, z) + 15\partial^2 F_1^\pm(\varphi, z) y_2^\pm(\varphi, z)^2 \\
&+ 30\partial^3 F_1^\pm(\varphi, z) y_1^\pm(\varphi, z)^2 y_2^\pm(\varphi, z) + 5\partial^4 F_1^\pm(\varphi, z) y_1^\pm(\varphi, z)^4 \\
&+ 5\partial F_1^\pm(\varphi, z) y_4^\pm(\varphi, z)\Big) d\varphi.
\end{aligned}$$

B Algorithm for generating ω_0

Algorithm 1

Input: a function $G = \sum_{i=0}^{7} a_i f_i(\omega) + k f_8(\omega)$

Output: a zero ω_0 of G with multiplicity 8

with(RandomTools);

Ω:=Generate(list(rational(range=0..1,denominator=10001),500));

for ω_0 **in** Ω **do**

  $G_1 := \mathrm{subs}(\omega - \omega_0 = s, \mathrm{convert}(\mathrm{series}(G, \omega = \omega_0, 9), \mathrm{polynom}))$;

  $e_0 := \mathrm{tcoeff}(G_1, s)$;

  for i **from** 1 **to** 8 **do**

    $e_i := \mathrm{coeff}(G_1, s^i)$;

  $S_0 := \mathrm{solve}(\{\mathrm{seq}(e_j = 0, j = 0..7)\}, \{\mathrm{seq}(a_j, j = 0..7)\})$;

  $A := \mathrm{normal}(\mathrm{subs}(S_0, e_8)/k)$;

  $G_2 := \mathrm{convert}(\mathrm{series}(\mathrm{subs}(S_0, G), \omega = 0, 2), \mathrm{polynom})$;

  $B := \mathrm{normal}(\mathrm{coeff}(G_2, \omega)/k)$;

  if $\mathrm{signum}(\mathrm{evalf}(A)) - \mathrm{signum}(\mathrm{limit}(\mathrm{subs}(S_0, G), \omega = 1, \mathrm{left})/\mathrm{signum}(k)) = 0$ **and** $\mathrm{signum}(\mathrm{evalf}(AB)) < 0$ **then**

return ω_0;

The following result is one output of Algorithm 1:

$$\frac{781}{10001}, \quad \frac{834}{10001}, \quad \frac{515}{10001}, \quad \frac{878}{10001}, \quad \frac{622}{10001}, \quad \frac{740}{10001}.$$

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