Electronic Journal of Qualitative Theory of Differential Equations
2020, No. 3, 1–16; https://doi.org/10.14232/ejqtde.2020.1.3 www.math.u-szeged.hu/ejqtde/

Optimal control problem for 3D micropolar fluid equations

Exequiel Mallea-Zepeda✉[1] and **Luis Medina**[2]

[1]Departamento de Matemática, Universidad de Tarapacá, Av. 18 de Septiembre 2222, Arica, Chile
[2]Departamento de Matemáticas, Universidad de Antofagasta, Av. Angamos 610, Antofagasta, Chile

Received 16 November 2019, appeared 12 January 2020
Communicated by Maria Alessandra Ragusa

Abstract. In this paper we study an optimal control problem related to strong solutions of 3D micropolar fluid equations. We deduce the existence of a global optimal solution with distributed control and, using a Lagrange multipliers theorem, we derive first-order optimality conditions for local optimal solutions.

Keywords: micropolar fluid equations, optimal control, optimality conditions.

2010 Mathematics Subject Classification: 49J20, 76D05, 76D55.

1 Introduction

The Navier–Stokes equations are a widely accepted model for the behavior of viscous incompressible fluids in the presence of convection. However, the classical Navier–Stokes theory is incapable of describing some physical phenomena for a class of fluids which exhibit certain microscopic effects arising from the local structure and micro-motions of the fluid elements. A subclass of these fluids is the micropolar fluids, which exhibit micro-rotational effects and micro-rotational inertia. Animal blood, liquid crystals, and certain polymeric fluids are a few examples of fluids which may be represented by the mathematical model of micropolar fluids, so that it is interesting to study the behavior of such fluids. The mathematical model that describes the movement of these fluids has been introduced by Eringen in [7] (see, also [6]). In this work we consider an optimal control problem restricted by the 3D micropolar fluid equations in which a distributed control acts on linear momentum as external source on the domain. Specifically, we consider $\Omega \subset \mathbb{R}^3$ be an open bounded domain with smooth boundary $\partial\Omega$ and $(0,T)$ a time interval, with $T > 0$. Then we study an optimal control problem related to the following system in the space-time domain $Q := \Omega \times (0,T)$

$$\begin{cases} \partial_t \mathbf{u} - (\nu+\nu_r)\Delta \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = 2\nu_r \mathrm{curl}\,\mathbf{w} + \mathbf{f}, \\ \partial_t \mathbf{w} - (c_a+c_d)\Delta \mathbf{w} + (\mathbf{u}\cdot\nabla)\mathbf{w} - (c_0+c_d-c_a)\nabla \mathrm{div}\,\mathbf{w} + 4\nu_r \mathbf{w} = 2\nu_r \mathrm{curl}\,\mathbf{u} + \mathbf{g}, \\ \mathrm{div}\,\mathbf{u} = 0, \end{cases} \tag{1.1}$$

✉Corresponding author. Email: emallea@uta.cl

where the unknowns are the linear velocity $\mathbf{u} = \mathbf{u}(x,t) \in \mathbb{R}^3$, the velocity of rotation of the particles $\mathbf{w} = \mathbf{w}(x,t) \in \mathbb{R}^3$ and the pressure $p = p(x,t) \in \mathbb{R}$. The functions $\mathbf{f}$ and $\mathbf{g}$ are given, and represent external sources of linear and angular momentum of particles, respectively. The positive real constant ν, ν_r, c_0, c_a and c_d characterize isotropic properties of the fluid; in particular, ν is the usual kinematic viscosity and ν_r, c_0, c_a and c_d are new viscosities related to the asymmetry of the stress tensor. These constants satisfy $c_0 + c_d > c_a$. For simplicity we denote $\nu_1 = \nu + \nu_r$, $\nu_2 = c_a + c_d$ and $\nu_3 = c_0 + c_d - c_a$. Without loss generality we can assume that density of the fluid is equal to one. The symbols Δ, ∇, curl and div denote the Laplacian, gradient, rotational and divergence operators, respectively; $\partial_t \mathbf{u}$ and $\partial_t \mathbf{w}$ stand for the time derivatives of $\mathbf{u}$ and $\mathbf{w}$, respectively. The i-th components of $(\mathbf{u} \cdot \nabla)\mathbf{u}$ and $(\mathbf{u} \cdot \nabla)\mathbf{w}$ are respectively given by

$$[(\mathbf{u} \cdot \nabla)\mathbf{u}]_i = \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} \quad \text{and} \quad [(\mathbf{u} \cdot \nabla)\mathbf{w}]_i = \sum_{j=1}^{3} u_j \frac{\partial w_i}{\partial x_j}.$$

When the microrotation viscous effects are not considered, that is, $\nu_r = 0$, or $\mathbf{w} = \mathbf{0}$, model (1.1) reduces to the well known incompressible Navier–Stokes system, which have been greatly studied (see, for instance, the classical text books [17], [18] and [31]).

We complete system (1.1) with initial conditions

$$\mathbf{u}(x,0) = \mathbf{u}_0(x), \quad \mathbf{w}(x,0) = \mathbf{w}_0(x) \quad \text{in } \Omega \tag{1.2}$$

and boundary conditions

$$\mathbf{u} = \mathbf{0}, \quad \mathbf{w} = \mathbf{0} \quad \text{on } \partial\Omega \times (0,T). \tag{1.3}$$

From the mathematical point of view, the initial-value problem (1.1)–(1.3) has been studied by several authors, and important results on existence of weak solutions and local strong solutions, large time asymptotic behavior, regularity of solutions, and general qualitative analysis, have been obtained (see [1,8–11,20,26,27,33], for instance).

There is an extensive literature devoted to the study of optimal control problems related with the classical Navier–Stokes equations (see, for instance, [3–5,14–16,25,32] and references therein). As far as known, the literature related to optimal control problems for micropolar fluids is scarce. In [29], an optimal control problem associated with themotion of a micropolar fluid, with applications in the control of the blood pressure, was studied. In [30], in a two-dimensional domain, the relation between the microrotation and vorticity of the fluid was analyzed. Also, a boundary control problem for the stationary case with mixed boundary conditions, including a Navier slip condition on a part of the boundary for the velocity field, was studied in [22,23]. In [22], for three-dimensional flows with constant density is considered, while in [23], the 2D case with variable density is studied.

For two-dimensional flows, an existence and uniqueness theorem for a weak solution of (1.1)–(1.3) has been known for a long time (see [20]). The study for 3D domains is more complicated. Here we can distinguish two types of solutions: weak and *strong* solutions. Under minimal assumptions in the initial data and external forces $\mathbf{f}$ and $\mathbf{g}$ the existence of weak solutions for (1.1)–(1.3) can be proved; however, the uniqueness is an open question (this is similar to what happens with the 3D Navier–Stokes equations). The existence of weak solutions is not sufficient to carry out the study of the optimal control problem, due to the lack of regularity of weak solutions. Indeed, we cannot obtain first-order necessary optimality conditions. To overcome this, following the ideas of Casas [3] and Casas et al. [4], we consider a convenient cost functional. Instead of setting the L^2-norm of $\mathbf{u} - \mathbf{u}_d$ in the objective functional as usual, we consider the functional

$$J(\mathbf{u}, \mathbf{w}, \mathbf{f}) := \frac{\alpha}{6} \int_0^T \|\mathbf{u}(t) - \mathbf{u}_d(t)\|_{L^6}^6 dt + \frac{\beta}{2} \int_0^T \|\mathbf{w}(t) - \mathbf{w}_d(t)\|^2 dt + \frac{\gamma}{2} \int_0^T \|\mathbf{f}(t)\|^2 dt, \qquad (1.4)$$

where $\alpha > 0$, $\beta, \gamma \geq 0$, and the functions $\mathbf{u}_d$ and $\mathbf{w}_d$ to be fixed more precisely later. The objective is to minimize $J(\mathbf{u}, \mathbf{w}, \mathbf{f})$ in a certain set, with $(\mathbf{u}, \mathbf{w}, \mathbf{f})$ satisfying system (1.1)–(1.3). From Loayza and Rojas-Medar [19] we deduce that, if $(\mathbf{u}, \mathbf{w})$ is a weak solution of (1.1)–(1.3) such that $J(\mathbf{u}, \mathbf{w}, \mathbf{f}) < +\infty$, then the pair $(\mathbf{u}, \mathbf{w})$ is a strong solution. With this formulation we can prove the existence of an optimal solution and obtain first-order optimality conditions.

The paper is organized as follow: in Section 2 we fix the notation, introduce the functional spaces to be used and give the definition of weak and strong solutions for system (1.1)–(1.3). In Section 3 we establish the optimal control problem, proving the existence of a global optimal solution and we derive the first-order optimality conditions using a Lagrange multipliers theorem in Banach spaces. Finally, we improve the regularity of Lagrange multipliers.

2 Preliminaries

Through this paper, we will use the Lebesgue space $L^p(\Omega)$, $1 \leq p \leq +\infty$, with norm denoted by $\|\cdot\|_{L^p}$. In particular, the L^2-norm and its inner product will be denoted by $\|\cdot\|$ and $(\cdot, \cdot)$, respectively. We consider the standard Sobolev spaces $W^{m,p}(\Omega) = \{u \in L^p(\Omega) : \|\partial^\alpha u\|_{L^p} < +\infty, \forall |\alpha| \leq m\}$, with norm denoted by $\|\cdot\|_{W^{m,p}}$. When $p = 2$, we write $H^m(\Omega) := W^{m,2}(\Omega)$ and we denote the respective norm by $\|\cdot\|_{H^m}$. Corresponding functional spaces of vector-valued functions will be denoted by bold letter; for instance $\mathbf{H}^1(\Omega)$, $\mathbf{L}^2(\Omega)$, and so on. We will use the Hilbert space $\mathbf{H}_0^1(\Omega) = \{\mathbf{u} \in \mathbf{H}^1(\Omega) : \mathbf{u} = \mathbf{0} \text{ on } \partial\Omega\}$, which is a Hilbert spaces with inner-product $(\mathbf{u}, \mathbf{v})_{\mathbf{H}_0^1} := (\nabla \mathbf{u}, \nabla \mathbf{v})$. Also, as usual we define $\mathcal{V} := \{\mathbf{u} \in \mathcal{C}_0^\infty(\Omega) : \operatorname{div} \mathbf{u} = 0\}$ and the spaces

$$\mathbf{H} := \text{ The closure of } \mathcal{V} \text{ in } \mathbf{L}^2(\Omega), \qquad \mathbf{V} := \text{ The closure of } \mathcal{V} \text{ in } \mathbf{H}^1(\Omega).$$

The spaces $\mathbf{H}$ and $\mathbf{V}$ are characterized by (see [31]):

$$\mathbf{H} = \{\mathbf{u} \in \mathbf{L}^2(\Omega) : \operatorname{div} \mathbf{u} = 0 \text{ and } \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\},$$
$$\mathbf{V} = \{\mathbf{u} \in \mathbf{H}_0^1(\Omega) : \operatorname{div} \mathbf{u} = 0\},$$

where $\mathbf{n}$ denotes the outward unit normal vector to $\partial\Omega$. If X is a Banach space, we denote by $L^p(0, T; X)$ the space of valued functions in X defined on the interval $[0, T]$ that are integrable in the Bochner sense, and its norm will denoted by $\|\cdot\|_{L^p(X)}$. For simplicity, we will denotes $L^p(Q) := L^p(0, T; L^p(\Omega))$ for $p \neq \infty$ and its norm by $\|\cdot\|_{L^p(Q)}$. In the case $p = +\infty$, $L^\infty(Q) := L^\infty(\Omega \times (0, T))$ and its respective norm will denoted by $\|\cdot\|_{L^\infty(Q)}$. Also, we denote by $C([0; T]; X)$ the space of continuous functions from $[0, T]$ into a Banach space X, and its norm by $\|\cdot\|_{C(X)}$. The topological dual space of a Banach space X will be denoted by X', and the duality for a pair X and X' by $\langle \cdot, \cdot \rangle_{X'}$ or simply by $\langle \cdot, \cdot \rangle$ unless this leads to ambiguity. In particular $\mathbf{V}'$ is the dual space of $\mathbf{V}$ and the space $\mathbf{H}^{-1}(\Omega)$ denotes the dual of $\mathbf{H}_0^1(\Omega)$. Moreover, the letters C, K, C_1, $K_1, \ldots$, are positive constants, independent of state $(\mathbf{u}, \mathbf{w})$ and control $\mathbf{f}$, but its value may change from line to line.

Now, we give the concept of weak solutions of system (1.1)–(1.3).

Definition 2.1 (Weak solutions)**.** Let $(\mathbf{f},\mathbf{g}) \in L^2(Q) \times L^2(Q)$ and $(\mathbf{u}_0,\mathbf{w}_0) \in \mathbf{H} \times \mathbf{L}^2(\Omega)$. A weak solution of (1.1)-(1.3) is a pair $(\mathbf{u},\mathbf{w})$ such that

$$\mathbf{u} \in L^\infty(0,T;\mathbf{H}) \cap L^2(0,T;\mathbf{V}), \quad \partial_t \mathbf{u} \in L^2(0,T;\mathbf{V}'), \tag{2.1}$$

$$\mathbf{w} \in L^\infty(0,T;\mathbf{L}^2(\Omega)) \cap L^2(0,T;\mathbf{H}_0^1(\Omega)), \quad \partial_t \mathbf{w} \in L^2(0,T;\mathbf{H}^{-1}(\Omega)), \tag{2.2}$$

and satisfies the following weak formulation

$$\int_0^T \langle \partial_t \mathbf{u}, \mathbf{v}\rangle + \nu_1 \int_0^T (\nabla \mathbf{u}, \nabla \mathbf{v}) + \int_0^T ((\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{v})$$

$$= 2\nu_r \int_0^T (\operatorname{curl}\mathbf{w},\mathbf{v}) + \int_0^T (\mathbf{f},\mathbf{v}) \qquad \forall \mathbf{v} \in L^2(0,T;\mathbf{V}), \tag{2.3}$$

$$\int_0^T \langle \partial_t \mathbf{w}, \mathbf{z}\rangle + \nu_2 \int_0^T (\nabla \mathbf{w}, \nabla \mathbf{z}) + \int_0^T ((\mathbf{u}\cdot\nabla)\mathbf{w},\mathbf{z}) + \nu_3 \int_0^T (\operatorname{div}\mathbf{w}, \operatorname{div}\mathbf{z}) + 4\nu_r \int_0^T (\mathbf{w},\mathbf{z})$$

$$= 2\nu_r \int_0^T (\operatorname{curl}\mathbf{u},\mathbf{z}) + \int_0^T (\mathbf{g},\mathbf{z}) \qquad \forall \mathbf{z} \in L^2(0,T;\mathbf{H}_0^1(\Omega)), \tag{2.4}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \mathbf{w}(0) = \mathbf{w}_0 \quad \text{in } \Omega, \tag{2.5}$$

$$\mathbf{u} = \mathbf{w} = \mathbf{0} \quad \text{on } \partial\Omega \times (0,T). \tag{2.6}$$

Remark 2.2. We consider the usual Stokes operator $A := -P\Delta$ with domain $D(A) = \mathbf{H}^2(\Omega) \cap \mathbf{V}$, where $P : \mathbf{L}^2(\Omega) \to \mathbf{H}$ is the Leray projector, and the strongly elliptic operator $L := -\nu_2\Delta - \nu_3\nabla\operatorname{div}$ with domain $D(L) = \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)$, then system (1.1)–(1.3) can be rewritten as follows

$$\begin{cases} \partial_t \mathbf{u} + \nu A\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = 2\nu_r \operatorname{curl}\mathbf{w} + P\mathbf{f} \text{ in } Q, \\ \partial_t \mathbf{w} + L\mathbf{w} + (\mathbf{u}\cdot\nabla)\mathbf{w} + 4\nu_r\mathbf{w} = 2\nu_r \operatorname{curl}\mathbf{u} + \mathbf{g} \text{ in } Q, \\ \operatorname{div}\mathbf{u} = 0 \text{ in } Q, \\ \mathbf{u}(x,0) = \mathbf{u}_0(x), \ \mathbf{w}(x,0) = \mathbf{w}_0(x) \text{ in } \Omega, \\ \mathbf{u} = \mathbf{0}, \ \mathbf{w} = \mathbf{0} \text{ on } \partial\Omega \times (0,T). \end{cases} \tag{2.7}$$

Thus, we have the following equivalent formulation of weak solutions of system (1.1)–(1.3).

Definition 2.3. Let $(\mathbf{f},\mathbf{g}) \in L^2(Q) \times L^2(Q)$ and $(\mathbf{u}_0,\mathbf{w}_0) \in \mathbf{H} \times \mathbf{L}^2(\Omega)$. Find a pair $(\mathbf{u},\mathbf{w})$ such that

$$\mathbf{u} \in L^\infty(0,T;\mathbf{H}) \cap L^2(0,T;\mathbf{V}), \quad \partial_t \mathbf{u} \in L^2(0,T;\mathbf{V}'), \tag{2.8}$$

$$\mathbf{w} \in L^\infty(0,T;\mathbf{L}^2(\Omega)) \cap L^2(0,T;\mathbf{H}_0^1(\Omega)), \quad \partial_t \mathbf{w} \in L^2(0,T;\mathbf{H}^{-1}(\Omega)), \tag{2.9}$$

and satisfies the system

$$\begin{cases} \partial_t \mathbf{u} + \nu A\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = 2\nu_r \operatorname{curl}\mathbf{w} + P\mathbf{f} \text{ in } D(A)', \\ \partial_t \mathbf{w} + L\mathbf{w} + (\mathbf{u}\cdot\nabla)\mathbf{w} + 4\nu_r\mathbf{w} = 2\nu_r \operatorname{curl}\mathbf{u} + \mathbf{g} \text{ in } D(L)', \\ \mathbf{u}(x,0) = \mathbf{u}_0(x) \in \mathbf{H}, \\ \mathbf{w}(x,0) = \mathbf{w}_0(x) \text{ in } \mathbf{L}^2(\Omega), \\ \mathbf{u} = \mathbf{0}, \ \mathbf{w} = \mathbf{0} \text{ on } \partial\Omega \times (0,T). \end{cases} \tag{2.10}$$

We are interested in studying an optimal control problem related the strong solutions of system (1.1)–(1.3), the following definition is given in this sense.

Definition 2.4 (Strong solutions)**.** Let $(\mathbf{f},\mathbf{g}) \in L^2(Q) \times L^2(Q)$ and $(\mathbf{u}_0,\mathbf{w}_0) \in \mathbf{V} \times \mathbf{H}^1_0(\Omega)$. We say that $(\mathbf{u},\mathbf{w})$ is a strong solution of system (1.1)–(1.3) in $(0,T)$ if

$$\mathbf{u} \in \mathbf{X}_{\mathbf{u}} := \{\mathbf{u} \in L^\infty(0,T;\mathbf{V}) \cap L^2(0,T;\mathbf{H}^2(\Omega)) \, : \, \partial_t \mathbf{u} \in L^2(Q)\}, \tag{2.11}$$
$$\mathbf{w} \in \mathbf{X}_{\mathbf{w}} := \{\mathbf{w} \in L^\infty(0,T;\mathbf{H}^1_0(\Omega)) \cap L^2(0,T;\mathbf{H}^2(\Omega)) \, : \, \partial_t \mathbf{w} \in L^2(Q)\}, \tag{2.12}$$

and satisfies

$$\begin{cases} \partial_t \mathbf{u} + \nu A\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = 2\nu_r \mathrm{curl}\,\mathbf{w} + \mathbf{f} \text{ in } L^2(Q), \\ \partial_t \mathbf{w} + L\mathbf{w} + (\mathbf{u}\cdot\nabla)\mathbf{w} + 4\nu_r\mathbf{w} = 2\nu_r \mathrm{curl}\,\mathbf{u} + \mathbf{g} \text{ in } L^2(Q), \\ \mathbf{u}(x,0) = \mathbf{u}_0(x) \in \mathbf{V}, \\ \mathbf{w}(x,0) = \mathbf{w}_0(x) \text{ in } \mathbf{H}^1_0(\Omega), \\ \mathbf{u} = \mathbf{0}, \;\; \mathbf{w} = \mathbf{0} \text{ on } \partial\Omega \times (0,T). \end{cases} \tag{2.13}$$

The following result is a criterion of regularity that allows us to obtain a strong solution of system (1.1)–(1.3), the proof can be consulted in [19].

Theorem 2.5. *Let $(\mathbf{u},\mathbf{w})$ be a weak solution of (1.1)–(1.3). If, in addition, the initial data $(\mathbf{u}_0,\mathbf{w}_0)$ belongs to $\mathbf{V} \times \mathbf{H}^1_0(\Omega)$ and*

$$\mathbf{u} \in L^4(0,T;\mathbf{L}^6(\Omega)), \tag{2.14}$$

then $(\mathbf{u},\mathbf{w})$ is a strong solution of (1.1)–(1.3).

Moreover, there exists a positive constant $K := K(\|\mathbf{u}_0\|_{\mathbf{V}}, \|\mathbf{w}_0\|_{\mathbf{H}^1_0}, \|\mathbf{f}\|_{L^2(Q)}, \|\mathbf{g}\|_{L^2(Q)})$ such that

$$\|(\mathbf{u},\mathbf{w})\|_{\mathbf{X}_{\mathbf{u}}\times\mathbf{X}_{\mathbf{w}}} \le K. \tag{2.15}$$

3 The optimal control problem

In this section we establish the statement of control problem. We formulate the control problem un such way a that any admissible state is a strong solution of (1.1)–(1.3). Due to the is no existence result of strong solutions of (1.1)–(1.3), we have to choose a suitable objective functional.

We suppose that $\mathcal{U} \subset L^2(Q)$ is a nonempty, closed and convex set and we consider the initial data $\mathbf{u}_0 \in \mathbf{V}$, $\mathbf{w}_0 \in \mathbf{H}^1_0(\Omega)$, and the function $\mathbf{f} \in \mathcal{U}$ describing the distributed control on the linear momentum equation.

Now, we define the following constrained extremal problem related to PDE system (1.1)–(1.3):

$$\begin{cases} \text{Find } (\mathbf{u},\mathbf{w},\mathbf{f}) \in \mathbf{X}_{\mathbf{u}} \times \mathbf{X}_{\mathbf{w}} \times \mathcal{U} \text{ such that the functional} \\ J(\mathbf{u},\mathbf{w},\mathbf{f}) := \dfrac{\alpha}{6}\displaystyle\int_0^T \|\mathbf{u}(t)-\mathbf{u}_d(t)\|^6_{L^6} dt + \dfrac{\beta}{2}\int_0^T \|\mathbf{w}(t)-\mathbf{w}_d(t)\|^2 dt + \dfrac{\gamma}{2}\int_0^T \|\mathbf{f}(t)\|^2 dt \\ \text{is minimized, subject to } (\mathbf{u},\mathbf{w},\mathbf{f}) \text{ be a strong solution of (1.1)–(1.3).} \end{cases} \tag{3.1}$$

Here $(\mathbf{u}_d,\mathbf{w}_d) \in L^{10}(Q) \times L^2(Q)$ represent the desires states (in the proof of Theorem 3.14 below is justified the fact that $\mathbf{u}_d \in L^{10}(Q)$) and the real numbers α, β and γ measure the cost of the states and control, respectively. These constants satisfy

$$\alpha > 0 \quad \text{and} \quad \beta,\gamma \ge 0.$$

The admissible set for the control problem (3.1) is defined by

$$\mathcal{S}_{ad} = \{\mathbf{s} = (\mathbf{u}, \mathbf{w}, \mathbf{f}) \in \mathbf{X}_{\mathbf{u}} \times \mathbf{X}_{\mathbf{w}} \times \mathcal{U} \, : \, \mathbf{s} \text{ is a strong solution of (1.1)–(1.3) in } (0, T)\}.$$

The functional J defined in (3.1) describes the deviation of the velocity of the fluid $\mathbf{u}$ and the microrotational velocity $\mathbf{w}$ from a desired velocity $\mathbf{u}_d$ and microrotational velocity $\mathbf{w}_d$ respectively, plus the control of the control measured in the L^2-norm.

Thus, we have the following definition.

Definition 3.1 (Optimal solution)**.** An element $\tilde{\mathbf{s}} = (\tilde{\mathbf{u}}, \tilde{\mathbf{w}}, \tilde{\mathbf{f}}) \in \mathcal{S}_{ad}$ will be called global optimal solution of problem (3.1) if

$$J(\tilde{\mathbf{u}}, \tilde{\mathbf{w}}, \tilde{\mathbf{f}}) = \min_{(\mathbf{u},\mathbf{w},\mathbf{f}) \in \mathcal{S}_{ad}} J(\mathbf{u}, \mathbf{w}, \mathbf{f}). \tag{3.2}$$

Remark 3.2. Notice that if $(\mathbf{u}, \mathbf{w})$ is a weak solution of (1.1)–(1.3) in $(0, T)$ such that $J(\mathbf{u}, \mathbf{w}, \mathbf{f}) < +\infty$, then, in particular $\mathbf{u} \in L^6(0, T; \mathbf{L}^4(\Omega))$; thus by Theorem 2.5 the pair $(\mathbf{u}, \mathbf{w})$ is a strong solution of (1.1)–(1.3) in $(0, T)$ (in sense of Definition 2.4). Due to there is no existence result of strong solutions, in what follows, we will assume that

$$\mathcal{S}_{ad} \neq \emptyset. \tag{3.3}$$

3.1 Existence of global optimal solution

In this subsection we will prove the existence of a global optimal solution of problem (3.1) in sense of Definition 3.1. Concretely, we will prove the following result.

Theorem 3.3. *Let* $(\mathbf{u}_0, \mathbf{w}_0) \in \mathbf{V} \times \mathbf{H}_0^1(\Omega)$*. We assume that either* $\gamma > 0$ *or* $\mathcal{U}$ *is bounded in* $L^2(Q)$ *and hypothesis* (3.3)*, then the optimal control problem* (3.1) *has at least one global optimal solution* $(\tilde{\mathbf{u}}, \tilde{\mathbf{w}}, \tilde{\mathbf{f}}) \in \mathcal{S}_{ad}$*.*

Proof. From (3.3) the admissible set $\mathcal{S}_{ad} \neq \emptyset$. Since functional J is nonnegative, then is bounded below. Hence there exists the infimum over all the admissible elements $\mathbf{s} := (\mathbf{u}, \mathbf{w}, \mathbf{f})$ belongs to $\mathcal{S}_{ad}$; that is,

$$0 \leq \inf_{\mathbf{s} \in \mathcal{S}_{ad}} J(\mathbf{s}) < +\infty.$$

Then, by definition of the infimum, there exists a minimizing sequence

$$\{\mathbf{s}_m\}_{m\geq 1} := \{(\mathbf{u}_m, \mathbf{w}_m, \mathbf{f}_m)\}_{m\geq 1}$$

such that

$$\lim_{m\to+\infty} J(\mathbf{s}_m) = \inf_{\mathbf{s} \in \mathcal{S}_{ad}} J(\mathbf{s}).$$

From definition of $\mathcal{S}_{ad}$, for each $m \in \mathbb{N}$, $\mathbf{s}_m$ is a strong solution of (1.1)–(1.3), then by definition of J and the assumption $\gamma > 0$ or $\mathcal{U}$ is bounded in $L^2(Q)$ we deduce that

$$\{(\mathbf{u}_m, \mathbf{f}_m)\}_{m\geq 1} \text{ is bounded in } L^6(Q) \times L^2(Q). \tag{3.4}$$

Also, from estimate (2.15) (given in Theorem 2.5) there exists a positive constant, independent of m such that

$$\|(\mathbf{u}_m, \mathbf{w}_m)\|_{\mathbf{X}_{\mathbf{u}} \times \mathbf{X}_{\mathbf{w}}} \leq K. \tag{3.5}$$

Thus, from (3.4), (3.5), and using the fact that $\mathcal{U} \subset L^2(Q)$ is a closed and convex (then is weakly closed in $L^2(Q)$), we conclude that there exists an element $\tilde{\mathbf{s}} = (\tilde{\mathbf{u}}, \tilde{\mathbf{w}}, \tilde{\mathbf{f}}) \in \mathbf{X}_{\mathbf{u}} \times \mathbf{X}_{\mathbf{w}} \times \mathcal{U}$ such that, for some subsequence of $\{\mathbf{s}_n\}_{m\geq 1}$; which, for simplicity, still will denoted by $\{\mathbf{s}_m\}_{m\geq 1}$, the following convergences hold (as $m \to +\infty$):

$$\mathbf{u}_m \to \tilde{\mathbf{u}} \quad \text{weak in } L^2(0,T;\mathbf{H}^2(\Omega)) \text{ and weak* in } L^\infty(0,T;\mathbf{V}), \tag{3.6}$$

$$\mathbf{w}_m \to \tilde{\mathbf{w}} \quad \text{weak in } L^2(0,T;\mathbf{H}^2(\Omega)) \text{ and weak* in } L^\infty(0,T;\mathbf{H}_0^1(\Omega)), \tag{3.7}$$

$$\partial_t \mathbf{u}_m \to \partial_t \tilde{\mathbf{u}} \quad \text{weak in } L^2(Q), \tag{3.8}$$

$$\partial_t \mathbf{w}_m \to \partial_t \tilde{\mathbf{w}} \quad \text{weak in } L^2(Q), \tag{3.9}$$

$$\mathbf{f}_m \to \tilde{\mathbf{f}} \quad \text{weak in } L^2(Q). \tag{3.10}$$

Furthermore, from (3.6)–(3.9), the Aubin–Lions lemma (see [18, Théorème 5.1, p. 58]) and [28, Corollary 4], we deduce the strong convergences

$$\mathbf{u}_m \to \tilde{\mathbf{u}} \quad \text{in } L^2(0,T;\mathbf{H}^1(\Omega)) \cap C([0,T];\mathbf{L}^2(\Omega)), \tag{3.11}$$

$$\mathbf{w}_m \to \tilde{\mathbf{w}} \quad \text{in } L^2(0,T;\mathbf{H}^1(\Omega)) \cap C([0,T];\mathbf{L}^2(\Omega)). \tag{3.12}$$

From (3.11) and (3.12) we have that the pair $(\mathbf{u}_m(0), \mathbf{w}_m(0))$ converges to $(\tilde{\mathbf{u}}, \tilde{\mathbf{w}})$ in $\mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega)$, and since $\mathbf{u}_m(0) = \mathbf{u}_0$ and $\mathbf{w}_m(0) = \mathbf{w}_0$ we conclude that $(\tilde{\mathbf{u}}(0), \tilde{\mathbf{w}}(0)) = (\mathbf{u}_0, \mathbf{w}_0)$. Thus, the limit element $\tilde{\mathbf{s}}$ satisfies the initial conditions given in (1.2). The convergences (3.6)–(3.12), and a standard argument allow us to pass to the limit in system (2.3)–(2.6) written by $(\mathbf{u}_m, \mathbf{w}_m, \mathbf{f}_m)$, as m goes to $+\infty$; consequently we have that $\tilde{\mathbf{s}} = (\tilde{\mathbf{u}}, \tilde{\mathbf{w}}, \tilde{\mathbf{f}})$ is a strong solution of (1.1)–(1.3), that is, $\tilde{\mathbf{s}}$ belongs to admissible set $\mathcal{S}_{ad}$. Therefore

$$\lim_{m\to+\infty} J(\mathbf{s}_m) = \inf_{\mathbf{s}\in\mathcal{S}_{ad}} J(\mathbf{s}) \leq J(\tilde{\mathbf{s}}). \tag{3.13}$$

Finally, taking into account that the functional J is weakly lower semicontinuous on $\mathcal{S}_{ad}$, we have

$$J(\tilde{\mathbf{s}}) \leq \liminf_{m\to+\infty} J(\mathbf{s}_m). \tag{3.14}$$

Therefore, from (3.13) and (3.14) we deduce (3.2), which implies that optimal control problem (3.1) has at least global optimal solution. □

3.2 Optimality system

In this subsection we will derive the first-order necessary optimality conditions for a local optimal solution $\tilde{\mathbf{s}} = (\tilde{\mathbf{u}}, \tilde{\mathbf{w}}, \tilde{\mathbf{f}})$ of problem (3.1), using a Lagrange multiplier theorem in Banach spaces. We will base on a generic result given by Zowe et al. [34] (see, also [32, Chapter 6]). This method has been used by Guillén-González et al. [12,13] in the context of chemo-repulsion systems and in [21] for other models. In order to introduce the concepts and results given in [34] we consider the following extremal problem:

$$\min_{\mathbf{x}\in\mathcal{M}} J(\mathbf{x}) \text{ subject to } R(\mathbf{x}) = \mathbf{0}, \tag{3.15}$$

where $J : \mathbf{X} \to \mathbb{R}$ is a functional, $R : \mathbf{X} \to \mathbf{Y}$ is an operator, $\mathbf{X}$ and $\mathbf{Y}$ are Banach spaces, and $\mathcal{M} \subset \mathbf{X}$ is a nonempty, closed and convex set. The admissible set for problem (3.15) is given by

$$\mathcal{S} = \{\mathbf{x} \in \mathcal{M} \,:\, R(\mathbf{x}) = \mathbf{0}\}.$$

The so-called *Lagrangian functional* $\mathcal{L} : \mathbf{X} \times \mathbf{Y}' \to \mathbb{R}$ related to problem (3.15) is given by

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) := J(\mathbf{x}) - \langle \boldsymbol{\lambda}, R(\mathbf{x}) \rangle_{\mathbf{Y}'}. \tag{3.16}$$

Definition 3.4 (Lagrange multiplier)**.** Let $\tilde{\mathbf{x}} \in \mathcal{S}$ be a local optimal solution of (3.15). Suppose that J and R are Fréchet differentiable in $\tilde{\mathbf{x}}$, with derivatives denoted by $J'(\tilde{\mathbf{x}})$ and $R'(\tilde{\mathbf{x}})$, respectively. Then, $\boldsymbol{\lambda} \in \mathbf{Y}'$ is called Lagrange multiplier for problem (3.15) at the point $\tilde{\mathbf{x}}$ if

$$\begin{cases} \langle \boldsymbol{\lambda}, R(\tilde{\mathbf{x}})\rangle_{\mathbf{Y}'} = 0, \\ \mathcal{L}'(\tilde{\mathbf{x}}, \boldsymbol{\lambda})[\mathbf{s}] := J'(\tilde{\mathbf{x}})[\mathbf{s}] - \langle \boldsymbol{\lambda}, R'(\tilde{\mathbf{x}})[\mathbf{s}]\rangle_{\mathbf{Y}'} \geq 0 \quad \forall \mathbf{s} \in \mathcal{C}(\tilde{\mathbf{x}}), \end{cases} \tag{3.17}$$

where $\mathcal{C}(\tilde{\mathbf{x}})$ is the conical hull of $\tilde{\mathbf{x}}$ in $\mathcal{M}$, that is, $\mathcal{C}(\tilde{\mathbf{x}}) = \{\theta(\mathbf{x} - \tilde{\mathbf{x}}) \,:\, \mathbf{x} \in \mathcal{M},\ \theta \geq 0\}$.

Definition 3.5. Let $\tilde{\mathbf{x}} \in \mathcal{S}$ be a local optimal solution of problem (3.15). We say that $\tilde{\mathbf{x}}$ is a regular point if

$$R'(\tilde{\mathbf{x}})[\mathcal{C}(\tilde{\mathbf{x}})] = \mathbf{Y}. \tag{3.18}$$

The following result guarantees the existence of Lagrange multiplier for problem (3.15); the proof can be found in [34, Theorem 3.1] and [32, Theorem 6.3, p. 330].

Theorem 3.6. *Let $\tilde{\mathbf{x}} \in \mathcal{S}$ be a local optimal solution of problem* (3.15)*. Suppose that J is Fréchet differentiable in $\tilde{\mathbf{x}}$ and R is continuously Fréchet differentiable in $\tilde{\mathbf{x}}$. If $\tilde{\mathbf{x}}$ is a regular point, then the set of Lagrange multipliers for* (3.15) *at $\tilde{\mathbf{x}}$ is nonempty.*

Now, we will reformulate the optimal control problem (3.1) in the abstract setting (3.15). We consider the Banach spaces

$$\mathbf{X} := \widehat{\mathbf{X}}_{\mathbf{u}} \times \widehat{\mathbf{X}}_{\mathbf{w}} \times L^2(Q), \qquad \mathbf{Y} := L^2(Q) \times L^2(Q) \times \mathbf{V} \times \mathbf{H}_0^1(\Omega),$$

where

$$\widehat{\mathbf{X}}_{\mathbf{u}} := \{\mathbf{u} \in \mathbf{X}_{\mathbf{u}} \,:\, \mathbf{u} = \mathbf{0} \text{ on } \partial\Omega \times (0,T)\}, \tag{3.19}$$

$$\widehat{\mathbf{X}}_{\mathbf{w}} := \{\mathbf{u} \in \mathbf{X}_{\mathbf{w}} \,:\, \mathbf{w} = \mathbf{0} \text{ on } \partial\Omega \times (0,T)\}, \tag{3.20}$$

and the operator $R = (R_1, R_2, R_3, R_4) : \mathbf{X} \to \mathbf{Y}$, where

$$R_1 : \mathbf{X} \to L^2(Q), \quad R_2(\mathbf{X}) \to L^2(Q), \quad R_3 : \mathbf{X} \to \mathbf{V},\ R_4 : \mathbf{X} \to \mathbf{H}_0^1(\Omega)$$

are defined at each point $\mathbf{s} = (\mathbf{u}, \mathbf{w}, \mathbf{f}) \in \mathbf{X}$ by

$$\begin{cases} R_1(\mathbf{s}) = \partial_t \mathbf{u} + \nu A\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - 2\nu_r \mathrm{curl}\, \mathbf{w} - P\mathbf{f}, \\ R_2(\mathbf{s}) = \partial_t \mathbf{w} + L\mathbf{w} + (\mathbf{u} \cdot \nabla)\mathbf{w} + 4\nu_r \mathbf{w} - 2\nu_r \mathrm{curl}\, \mathbf{u} - \mathbf{g}, \\ R_3(\mathbf{s}) = \mathbf{u}(0) - \mathbf{u}_0, \\ R_4(\mathbf{s}) = \mathbf{w}(0) - \mathbf{w}_0. \end{cases} \tag{3.21}$$

Hence, the control problem (3.1) is reformulated as follows

$$\min_{\mathbf{s} \in \mathbf{M}} J(\mathbf{s}) \text{ subject to } R(\mathbf{s}) = \mathbf{0}. \tag{3.22}$$

Notice that $\mathbf{M} := \widehat{\mathbf{X}}_{\mathbf{u}} \times \widehat{\mathbf{X}}_{\mathbf{w}} \times \mathcal{U}$ is a closed convex subset of $\mathbf{X}$ and the admissible set is rewritten as follows

$$\mathcal{S}_{ad} = \{\mathbf{s} = (\mathbf{u}, \mathbf{w}, \mathbf{f}) \in \mathbf{M} \,:\, R(\mathbf{s}) = \mathbf{0}\}. \tag{3.23}$$

Concerning to differentiability of the functional J and constraint operator R we have the following lemmas.

Lemma 3.7. *The functional J is Fréchet differentiable and the Fréchet derivative of J in $\tilde{\mathbf{s}} = (\tilde{\mathbf{u}}, \tilde{\mathbf{w}}, \tilde{\mathbf{f}}) \in \mathbf{X}$ in the direction $\mathbf{t} = (\mathbf{U}, \mathbf{W}, \mathbf{F}) \in \mathbf{X}$ is given by*

$$J'(\tilde{\mathbf{s}})[\mathbf{t}] = \alpha \int_0^T \int_\Omega |\tilde{\mathbf{u}} - \mathbf{u}_d|^4 (\tilde{\mathbf{u}} - \mathbf{u}_d) \cdot \mathbf{U} + \beta \int_0^T \int_\Omega (\tilde{\mathbf{w}} - \mathbf{w}_d) \cdot \mathbf{W} + \gamma \int_0^T \int_\Omega \tilde{\mathbf{f}} \cdot \mathbf{F}. \tag{3.24}$$

Lemma 3.8. *The operator R is continuously-Fréchet differentiable and the Fréchet derivative of R in $\tilde{\mathbf{s}} = (\tilde{\mathbf{u}}, \tilde{\mathbf{w}}, \tilde{\mathbf{f}}) \in \mathbf{X}$, in the direction $\mathbf{t} = (\mathbf{U}, \mathbf{W}, \mathbf{F}) \in \mathbf{X}$, is the linear and bounded operator $R'(\tilde{\mathbf{s}})[\mathbf{t}] = (R_1'(\tilde{\mathbf{s}})[\mathbf{t}], R_2'(\tilde{\mathbf{s}})[\mathbf{t}], R_3'(\tilde{\mathbf{s}})[\mathbf{t}], R_4'(\tilde{\mathbf{s}})[\mathbf{t}])$ defined by*

$$\begin{cases} R_1'(\tilde{\mathbf{s}})[\mathbf{t}] = \partial_t \mathbf{U} + \nu A\mathbf{U} + (\mathbf{U} \cdot \nabla)\tilde{\mathbf{u}} + (\tilde{\mathbf{u}} \cdot \nabla)\mathbf{U} - 2\nu_r \operatorname{curl} \mathbf{W} - P\mathbf{F}, \\ R_2(\tilde{\mathbf{s}})[\mathbf{t}] = \partial_t \mathbf{W} + L\mathbf{W} + (\tilde{\mathbf{u}} \cdot \nabla)\mathbf{W} + (\mathbf{U} \cdot \nabla)\tilde{\mathbf{w}} + 4\nu_r \mathbf{W} - 2\nu_r \operatorname{curl} \mathbf{U}, \\ R_3'(\tilde{\mathbf{s}})[\mathbf{t}] = \mathbf{U}(0), \\ R_4'(\tilde{\mathbf{s}})[\mathbf{t}] = \mathbf{W}(0). \end{cases} \tag{3.25}$$

Remark 3.9. From Definition 3.5 we conclude that $\tilde{\mathbf{s}} = (\tilde{\mathbf{u}}, \tilde{\mathbf{w}}, \tilde{\mathbf{f}}) \in \mathcal{S}_{ad}$ is a regular point if given $(\mathbf{g_u}, \mathbf{g_w}, \mathbf{U}_0, \mathbf{W}_0) \in \mathbf{Y}$ there exists $\mathbf{t} = (\mathbf{U}, \mathbf{W}, \mathbf{F}) \in \widehat{\mathbf{X}}_\mathbf{u} \times \widehat{\mathbf{X}}_\mathbf{w} \times \mathcal{C}(\tilde{\mathbf{f}})$ such that

$$R'(\tilde{\mathbf{s}})[\mathbf{t}] = (\mathbf{g_u}, \mathbf{g_w}, \mathbf{U}_0, \mathbf{W}_0), \tag{3.26}$$

where $\mathcal{C}(\tilde{\mathbf{f}}) := \{\theta(\mathbf{f} - \tilde{\mathbf{f}}) : \theta \geq 0, \mathbf{f} \in \mathcal{U}\}$ is the conical hull of $\tilde{\mathbf{f}}$ in $\mathcal{U}$.

Lemma 3.10. *Let $\tilde{\mathbf{s}} = (\tilde{\mathbf{u}}, \tilde{\mathbf{w}}, \tilde{\mathbf{f}}) \in \mathcal{S}_{ad}$, then $\tilde{\mathbf{s}}$ is a regular point.*

Proof. Due to $\mathbf{0}$ belongs to $\mathcal{C}(\tilde{\mathbf{f}})$; then, given $(\mathbf{g_u}, \mathbf{g_w}, \mathbf{U}_0, \mathbf{W}_0) \in \mathbf{Y}$, it is sufficient to show the existence of $(\mathbf{U}, \mathbf{W}) \in \widehat{\mathbf{X}}_\mathbf{u} \times \widehat{\mathbf{X}}_\mathbf{w}$ such that

$$\begin{cases} \partial_t \mathbf{U} + \nu A\mathbf{U} + (\mathbf{U} \cdot \nabla)\tilde{\mathbf{u}} + (\tilde{\mathbf{u}} \cdot \nabla)\mathbf{U} - 2\nu_r \operatorname{curl} \mathbf{W} = \mathbf{g_u} & \text{in } Q, \\ \partial_t \mathbf{W} + L\mathbf{W} + (\tilde{\mathbf{u}} \cdot \nabla)\mathbf{W} + (\mathbf{U} \cdot \nabla)\tilde{\mathbf{w}} + 4\nu_r \mathbf{W} - 2\nu_r \operatorname{curl} \mathbf{U} = \mathbf{g_w} & \text{in } Q, \\ \mathbf{U}(0) = \mathbf{U}_0 & \text{in } \Omega, \\ \mathbf{W}(0) = \mathbf{W}_0 & \text{in } \Omega. \end{cases} \tag{3.27}$$

Since system (3.27) is a linear, we argue in a formal manner, proving that any regular enough solution is bounded in $\widehat{\mathbf{X}}_\mathbf{u} \times \widehat{\mathbf{X}}_\mathbf{w}$.

Testing in $(3.27)_1$ by $A\mathbf{U}$ we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla \mathbf{U}\|^2 + \nu_1 \|A\mathbf{U}\|^2 = & -((\mathbf{U} \cdot \nabla)\tilde{\mathbf{u}}, A\mathbf{U}) - ((\tilde{\mathbf{u}} \cdot \nabla)\mathbf{U}, A\mathbf{U}) \\ & + 2\nu_r (\operatorname{curl} \mathbf{W}, A\mathbf{U}) + (\mathbf{g_u}, A\mathbf{U}). \end{aligned} \tag{3.28}$$

Now, we will bound the terms of right-side of (3.28). Using the Hölder, Poincaré and Young inequalities, and taking into account the continuous injection $\mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^q(\Omega)$ ($q \in [1, 6]$) we have

$$\begin{aligned} ((\mathbf{U} \cdot \nabla)\tilde{\mathbf{u}}, A\mathbf{U}) &\leq \|\mathbf{U}\|_{\mathbf{L}^3} \|\nabla \tilde{\mathbf{u}}\|_{\mathbf{L}^6} \|A\mathbf{U}\| \leq C\|\mathbf{U}\|_{\mathbf{H}^1} \|\nabla \tilde{\mathbf{u}}\|_{\mathbf{L}^6} \|A\mathbf{U} \leq C\|\nabla \mathbf{U}\| \|\nabla \tilde{\mathbf{u}}\|_{\mathbf{L}^6} \|A\mathbf{U}\| \\ &\leq \varepsilon \|A\mathbf{U}\|^2 + C_\varepsilon \|\nabla \tilde{\mathbf{u}}\|_{\mathbf{L}^6}^2 \|\nabla \mathbf{U}\|^2. \end{aligned} \tag{3.29}$$

From the equivalence $\frac{1}{2\nu\sqrt{3}}\|A\mathbf{u}\| \leq \|\mathbf{u}\|_{\mathbf{H}^2} \leq C\|A\mathbf{u}\|$ (see [24, Lemma 3.1]) and the known interpolation inequality in 3D domains $\|\mathbf{u}\|_{\mathbf{L}^3} \leq C\|\mathbf{u}\|^{1/2}\|\nabla \mathbf{u}\|^{1/2}$, we obtain

$$\begin{aligned} |((\tilde{\mathbf{u}} \cdot \nabla)\mathbf{U}, A\mathbf{U})| &\leq \|\tilde{\mathbf{u}}\|_{\mathbf{L}^6} \|\nabla \mathbf{U}\|_{\mathbf{L}^3} \|A\mathbf{U}\| \leq C\|\tilde{\mathbf{u}}\|_{\mathbf{L}^6} \|\nabla \mathbf{U}\|^{1/2} \|\nabla \mathbf{U}\|_{\mathbf{H}^1}^{1/2} \|A\mathbf{U}\| \\ &\leq C\|\tilde{\mathbf{u}}\|_{\mathbf{L}^6} \|\nabla \mathbf{U}\|^{1/2} \|\mathbf{U}\|_{\mathbf{H}^2}^{1/2} \|A\mathbf{U}\| \leq C\|\tilde{\mathbf{u}}\|_{\mathbf{L}^6} \|\nabla \mathbf{U}\|^{1/2} \|A\mathbf{U}\|^{3/2} \\ &\leq \varepsilon \|A\mathbf{U}\|^2 + C_\varepsilon \|\tilde{\mathbf{u}}\|_{\mathbf{L}^6}^4 \|\nabla \mathbf{U}\|. \end{aligned} \tag{3.30}$$

Again using the Hölder and Young inequalities, we have

$$\begin{aligned} 2\nu_r|(\operatorname{curl}\mathbf{W}, A\mathbf{U})| &\leq 2\nu_r\|\operatorname{curl}\mathbf{W}\|\|A\mathbf{U}\| \leq \varepsilon\|A\mathbf{U}\|^2 + C_\varepsilon\|\operatorname{curl}\mathbf{W}\|^2 \\ &\leq \varepsilon\|A\mathbf{U}\|^2 + C_\varepsilon\|\nabla\mathbf{W}\|^2, \quad (3.31) \\ |(\mathbf{g_u}, A\mathbf{U})| &\leq \|\mathbf{g_u}\|\|A\mathbf{U}\| \leq \varepsilon\|A\mathbf{U}\|^2 + C_\varepsilon\|\mathbf{g_u}\|^2. \quad (3.32) \end{aligned}$$

Thus, replacing (3.29)–(3.32) in (3.28) and choosing ε suitably, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{U}\|^2 + C\|A\mathbf{U}\|^2 \leq C\|\nabla\tilde{\mathbf{u}}\|^2_{\mathbf{L}^6}\|\nabla\mathbf{U}\|^2 + C\|\tilde{\mathbf{u}}\|^4_{\mathbf{L}^6}\|\nabla\mathbf{U}\|^2 + C\|\mathbf{g_u}\|^2 + C\|\nabla\mathbf{W}\|^2. \quad (3.33)$$

Now, testing in $(3.27)_2$ by $-\Delta\mathbf{W}$ we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{W}\|^2 + \nu_2\|\Delta\mathbf{W}\|^2 + \nu_3(\nabla\operatorname{div}\mathbf{W}, \Delta\mathbf{W}) + 4\nu_r\|\nabla\mathbf{W}\|^2 \\ &\quad\leq |((\tilde{\mathbf{u}}\cdot\nabla)\mathbf{W}, \Delta\mathbf{W})| + |((\mathbf{U}\cdot\nabla)\tilde{\mathbf{w}}, \Delta\mathbf{W})| + 2\nu_r|(\operatorname{curl}\mathbf{U}, \Delta\mathbf{W})| + |(\mathbf{g_w}, \Delta\mathbf{W})|. \quad (3.34) \end{aligned}$$

Applying the Hölder and Young inequalities, we deduce

$$\begin{aligned} |((\tilde{\mathbf{u}}\cdot\nabla\mathbf{W}), \Delta\mathbf{W})| &\leq \|\tilde{\mathbf{u}}\|_{\mathbf{L}^6}\|\nabla\mathbf{W}\|_{\mathbf{L}^3}\|\Delta\mathbf{W}\| \leq C\|\tilde{\mathbf{u}}\|_{\mathbf{L}^6}\|\nabla\mathbf{W}\|^{1/2}\|\Delta\mathbf{W}\|^{3/2} \\ &\leq \varepsilon\|\Delta\mathbf{W}\|^2 + C_\varepsilon\|\tilde{\mathbf{u}}\|^4_{\mathbf{L}^6}\|\nabla\mathbf{W}\|^2, \quad (3.35) \\ |((\mathbf{U}\cdot\nabla)\tilde{\mathbf{w}}, \Delta\mathbf{W})| &\leq \|\mathbf{U}\|_{\mathbf{L}^3}\|\nabla\tilde{\mathbf{w}}\|_{\mathbf{L}^6}\|\Delta\mathbf{W}\| \\ &\leq \varepsilon\|\Delta\mathbf{W}\|^2 + C_\varepsilon\|\nabla\tilde{\mathbf{w}}\|^2_{\mathbf{L}^6}\|\nabla\mathbf{U}\|^2, \quad (3.36) \\ 2\nu_r|(\operatorname{curl}\mathbf{U}, \Delta\mathbf{W})| &\leq 2\nu_r\|\nabla\mathbf{U}\|\|\Delta\mathbf{W}\| \leq \varepsilon\|\Delta\mathbf{W}\|^2 + C_\varepsilon\|\nabla\mathbf{U}\|^2, \quad (3.37) \\ |(\mathbf{g_w}, \Delta\mathbf{W})| &\leq \varepsilon\|\Delta\mathbf{W}\|^2 + C_\varepsilon\|\mathbf{g_w}\|^2. \quad (3.38) \end{aligned}$$

Then, carrying (3.35)–(3.38) to (3.34) and choosing ε suitably, we can obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{W}\|^2 + C\|\Delta\mathbf{W}\|^2 + \nu_3(\nabla\operatorname{div}\mathbf{W}, \Delta\mathbf{W}) + 4\nu_r\|\nabla\mathbf{W}\|^2 \\ &\quad\leq C\|\tilde{\mathbf{u}}\|^4_{\mathbf{L}^6}\|\nabla\mathbf{W}\|^2 + C(\|\nabla\tilde{\mathbf{w}}\|^2_{\mathbf{L}^6} + 1)\|\nabla\mathbf{U}\|^2 + C\|\mathbf{g_w}\|^2. \quad (3.39) \end{aligned}$$

Moreover, since operator $L = -\nu_2\Delta - \nu_3\nabla\operatorname{div}$ is strongly elliptic, we have

$$(L\mathbf{W}, -\Delta\mathbf{W}) \geq C_1\|\Delta\mathbf{W}\|^2 - C_2\|\nabla\mathbf{W}\|^2, \quad (3.40)$$

where C_1 and C_2 are positive constant which depend only on ν_2, ν_3 and $\partial\Omega$ (see [19], for more details). Then, estimates (3.39) and (3.40) implies

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{W}\|^2 + C\|\Delta\mathbf{W}\|^2 + 4\nu_r\|\nabla\mathbf{W}\|^2 &\leq C(\|\tilde{\mathbf{u}}\|^4_{\mathbf{L}^6} + 1)\|\nabla\mathbf{W}\|^2 \\ &\quad + C(\|\nabla\tilde{\mathbf{w}}\|^2_{\mathbf{L}^6} + 1)\|\nabla\mathbf{U}\|^2 + C\|\mathbf{g_w}\|^2. \quad (3.41) \end{aligned}$$

Therefore, from (3.33) and (3.41) we deduce

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\nabla\mathbf{U}\|^2 + \|\nabla\mathbf{W}\|^2) + (\|A\mathbf{U}\|^2 + \|\Delta\mathbf{W}\|^2) + 4\nu_r\|\nabla\mathbf{W}\|^2 \\ &\quad\leq (\|\nabla\tilde{\mathbf{u}}\|^2_{\mathbf{L}^6} + \|\tilde{\mathbf{u}}\|^4_{\mathbf{L}^6} + \|\nabla\tilde{\mathbf{w}}\|^2_{\mathbf{L}^6} + 1)\|\nabla\mathbf{U}\|^2 + C(\|\tilde{\mathbf{u}}\|^4_{\mathbf{L}^6} + 1)\|\nabla\mathbf{W}\|^2 \\ &\qquad + C(\|\mathbf{g_u}\|^2 + \|\mathbf{g_w}\|^2). \quad (3.42) \end{aligned}$$

Then, from (3.42) and Gronwall lemma, we can deduce that $(\mathbf{U}, \mathbf{W}) \in \widehat{\mathbf{X}}_{\mathbf{u}} \times \widehat{\mathbf{X}}_{\mathbf{w}}$. □

Now we are able to prove the existence of Lagrange multipliers.

Theorem 3.11. *Let $\tilde{\mathbf{s}} = (\tilde{\mathbf{u}}, \tilde{\mathbf{w}}, \tilde{\mathbf{f}}) \in \mathcal{S}_{ad}$ be a local optimal solution for the control problem* (3.22)*. Then, there exist Lagrange multipliers $(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3, \boldsymbol{\lambda}_4) \in L^2(Q) \times L^2(Q) \times \mathbf{V}' \times \mathbf{H}^{-1}(\Omega)$ such that*

$$\begin{aligned}
\alpha \int_0^T \int_\Omega |\tilde{\mathbf{u}} - \mathbf{u}_d|^4 (\tilde{\mathbf{u}} - \mathbf{u}_d) \cdot \mathbf{U} + \beta \int_0^T \int_\Omega (\tilde{\mathbf{w}} - \mathbf{w}_d) \cdot \mathbf{W} + \gamma \int_0^T \int_\Omega \tilde{\mathbf{f}} \cdot \mathbf{F} \\
- \int_0^T \int_\Omega (\partial_t \mathbf{U} + \nu A \mathbf{U} + (\mathbf{U} \cdot \nabla)\tilde{\mathbf{u}} + (\tilde{\mathbf{u}} \cdot \nabla)\mathbf{U} - 2\nu_r \mathrm{curl}\, \mathbf{W} - P\mathbf{F}) \cdot \boldsymbol{\lambda}_1 \\
- \int_0^T \int_\Omega (\partial_t \mathbf{W} + L\mathbf{W} + (\tilde{\mathbf{u}} \cdot \nabla)\mathbf{W} + (\mathbf{U} \cdot \nabla)\tilde{\mathbf{w}} + 4\nu_r \mathbf{W} - 2\nu_r \mathrm{curl}\, \mathbf{U}) \cdot \boldsymbol{\lambda}_2 \\
- \int_\Omega \mathbf{U}(0) \cdot \boldsymbol{\lambda}_3 - \int_\Omega \mathbf{W}(0) \cdot \boldsymbol{\lambda}_4 \geq 0, \qquad \forall \in (\mathbf{U}, \mathbf{W}, \mathbf{F}) \in \widehat{\mathbf{X}}_{\mathbf{u}} \times \widehat{\mathbf{X}}_{\mathbf{w}} \times \mathcal{C}(\tilde{\mathbf{f}}).
\end{aligned} \tag{3.43}$$

Proof. From Lemma 3.10 we have that $\tilde{\mathbf{s}} = (\tilde{\mathbf{u}}, \tilde{\mathbf{w}}, \tilde{\mathbf{f}})$ is a regular point. Therefore, from Theorem 3.6 we deduce that there exist Lagrange multipliers satisfying (3.43). □

Theorem 3.11 allows us derive an optimality system for problem (3.22), for this purpose we consider the following spaces

$$\widehat{\mathbf{X}}_{\mathbf{u}_0} = \{\mathbf{u} \in \widehat{\mathbf{X}}_{\mathbf{u}} \,:\, \mathbf{u}(0) = \mathbf{0}\}, \qquad \widehat{\mathbf{X}}_{\mathbf{w}_0} = \{\mathbf{u} \in \widehat{\mathbf{X}}_{\mathbf{w}} \,:\, \mathbf{u}(0) = \mathbf{0}\}. \tag{3.44}$$

Corollary 3.12. *Let $\tilde{\mathbf{s}} = (\tilde{\mathbf{u}}, \tilde{\mathbf{w}}, \tilde{\mathbf{f}}) \in \mathcal{S}_{ad}$ be a local optimal solution of control problem* (3.22)*. Then the Lagrange multipliers $(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) \in L^2(Q) \times L^2(Q)$ satisfy the system*

$$\begin{aligned}
\int_0^T \int_\Omega (\partial_t \mathbf{U} + \nu A \mathbf{U} + (\mathbf{U} \cdot \nabla)\tilde{\mathbf{u}} + (\tilde{\mathbf{u}} \cdot \nabla)\mathbf{U} - 2\nu_r \mathrm{curl}\, \mathbf{W}) \cdot \boldsymbol{\lambda}_1 \\
= \alpha \int_0^T \int_\Omega |\tilde{\mathbf{u}} - \mathbf{u}_d|^4 (\tilde{\mathbf{u}} - \mathbf{u}_d) \cdot \mathbf{U},
\end{aligned} \tag{3.45}$$

$$\begin{aligned}
\int_0^T \int_\Omega (\partial_t \mathbf{W} + L\mathbf{W} + (\tilde{\mathbf{u}} \cdot \nabla)\mathbf{W} + 4\nu_r \mathbf{W} - 2\nu_r \mathrm{curl}\, \mathbf{U}) \cdot \boldsymbol{\lambda}_2 \\
= \beta \int_0^T \int_\Omega (\tilde{\mathbf{w}} - \mathbf{w}_d) \cdot \mathbf{W},
\end{aligned} \tag{3.46}$$

for all $(\mathbf{U}, \mathbf{W}) \in \widehat{\mathbf{X}}_{\mathbf{u}_0} \times \widehat{\mathbf{W}}_{\mathbf{w}_0}$, and the optimality condition

$$\gamma \int_0^T \int_\Omega (\tilde{\mathbf{f}} + \boldsymbol{\lambda}_1) \cdot (\mathbf{f} - \tilde{\mathbf{f}}) \geq 0 \qquad \forall \mathbf{f} \in \mathcal{U}. \tag{3.47}$$

Proof. Notice that $\widehat{\mathbf{W}}_{\mathbf{u}_0} \times \widehat{\mathbf{W}}_{\mathbf{w}_0}$ is a vector space; then, from (3.43), taking $(\mathbf{U}, \mathbf{F}) = (\mathbf{0}, \mathbf{0})$ we have (3.45). Analogously, taking $(\mathbf{W}, \mathbf{F}) = (\mathbf{0}, \mathbf{0})$ in (3.43), we deduce (3.46). Finally, taking $(\mathbf{U}, \mathbf{W}) = (\mathbf{0}, \mathbf{0})$ in (3.43) we obtain

$$\gamma \int_0^T \int_\Omega \tilde{\mathbf{f}} \cdot \mathbf{F} + \int_0^T \int_\Omega \mathbf{F} \cdot \boldsymbol{\lambda}_1 \geq 0 \qquad \forall \mathbf{F} \in \mathcal{C}(\tilde{\mathbf{f}}). \tag{3.48}$$

Thus, choosing $F = \mathbf{f} - \tilde{\mathbf{f}} \in \mathcal{C}(\tilde{\mathbf{f}})$ in (3.48) we have (3.47). □

Remark 3.13. Problem (3.45)–(3.46) corresponds to the concept of the very weak solution of the parabolic linear problem

$$
\begin{aligned}
&-\partial_t\boldsymbol{\lambda}_1-\nu_1\Delta\boldsymbol{\lambda}_1-\tilde{\mathbf{u}}\cdot\nabla\boldsymbol{\lambda}_1+(\nabla\boldsymbol{\lambda}_1)^T\cdot\tilde{\mathbf{u}}+(\nabla\boldsymbol{\lambda}_2)^T\cdot\tilde{\mathbf{w}}+\nabla q\\
&\qquad=2\nu_r\mathrm{curl}\,\boldsymbol{\lambda}_2-\alpha|\tilde{\mathbf{u}}-\mathbf{u}_d|^4(\tilde{\mathbf{u}}-\mathbf{u}_d)\quad\text{in } Q, &(3.49)\\
&-\partial_t\boldsymbol{\lambda}_2-\nu_2\Delta\boldsymbol{\lambda}_2-\nu_3\nabla\mathrm{div}\,\boldsymbol{\lambda}_2-\tilde{\mathbf{u}}\cdot\nabla\boldsymbol{\lambda}_2+4\nu_r\boldsymbol{\lambda}_2\\
&\qquad=2\nu_r\mathrm{curl}\,\boldsymbol{\lambda}_1-\beta(\tilde{\mathbf{w}}-\mathbf{w}_d)\quad\text{in } Q, &(3.50)\\
&\mathrm{div}\,\boldsymbol{\lambda}_1=0\quad\text{in } Q, &(3.51)\\
&\boldsymbol{\lambda}_1(T)=\mathbf{0},\ \boldsymbol{\lambda}_2(T)=\mathbf{0}\quad\text{in }\Omega, &(3.52)\\
&\boldsymbol{\lambda}_1=\mathbf{0},\ \boldsymbol{\lambda}_2=\mathbf{0}\quad\text{on }\partial\Omega\times(0,T). &(3.53)
\end{aligned}
$$

Now, we will obtain some extra regularity for the Lagrange multipliers $(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)$ provided by Theorem 3.11.

Theorem 3.14. *Let $(\tilde{\mathbf{u}},\tilde{\mathbf{w}},\tilde{\mathbf{f}})\in\mathcal{S}_{ad}$ be a local optimal solution of problem (3.22). Then, the Lagrange multipliers $(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)$, provided by Theorem 3.11, satisfy*

$$
\boldsymbol{\lambda}_1\in L^\infty(0,T;\mathbf{V})\cap L^2(0,T;\mathbf{H}^2(\Omega)),\quad \partial_t\boldsymbol{\lambda}_1\in L^2(Q), \tag{3.54}
$$

$$
\boldsymbol{\lambda}_2\in L^\infty(0,T;\mathbf{H}_0^1(\Omega))\cap L^2(0,T;\mathbf{H}^2(\Omega)),\quad \partial_t\boldsymbol{\lambda}_2\in L^2(Q). \tag{3.55}
$$

Proof. First we will show that the solution of system (3.49)–(3.53) has regularity (3.54)–(3.55). In fact, let $\tau:=T-t$, with $t\in(0,T)$, and $\boldsymbol{\eta}_1(\tau):=\boldsymbol{\lambda}_1(t)$, $\boldsymbol{\eta}_2(\tau):=\boldsymbol{\lambda}_2(t)$. Then, system (3.49)–(3.53) is equivalent to

$$
\left\{
\begin{aligned}
&\partial_\tau\boldsymbol{\eta}_1-\nu_1\Delta\boldsymbol{\eta}_1-\tilde{\mathbf{u}}\cdot\nabla\boldsymbol{\eta}_1+(\nabla\boldsymbol{\eta}_1)^T\cdot\tilde{\mathbf{u}}+(\nabla\boldsymbol{\eta}_2)^T\cdot\tilde{\mathbf{w}}+\nabla q\\
&\qquad=2\nu_r\mathrm{curl}\,\boldsymbol{\eta}_2-\alpha|\tilde{\mathbf{u}}-\mathbf{u}_d|^4(\tilde{\mathbf{u}}-\mathbf{u}_d)\quad\text{in } Q,\\
&\partial_\tau\boldsymbol{\eta}_2-\nu_2\Delta\boldsymbol{\eta}_2-\nu_3\nabla\mathrm{div}\,\boldsymbol{\eta}_2-\tilde{\mathbf{u}}\cdot\nabla\boldsymbol{\eta}_2+4\nu_r\boldsymbol{\eta}_2\\
&\qquad=2\nu_r\mathrm{curl}\,\boldsymbol{\eta}_1-\beta(\tilde{\mathbf{w}}-\mathbf{w}_d)\quad\text{in } Q,\\
&\mathrm{div}\,\boldsymbol{\eta}_1=0\quad\text{in } Q,\\
&\boldsymbol{\eta}_1(T)=\mathbf{0},\ \boldsymbol{\eta}_2(T)=\mathbf{0}\quad\text{in }\Omega,\\
&\boldsymbol{\eta}_1=\mathbf{0},\ \boldsymbol{\eta}_2=\mathbf{0}\quad\text{on }\partial\Omega\times(0,T).
\end{aligned}
\right. \tag{3.56}
$$

Following similar arguments that in the proof of Lemma 3.10 we can obtain that the unique solution $(\boldsymbol{\eta}_1,\boldsymbol{\eta}_2)$ of problem (3.56) satisfies

$$
\begin{aligned}
&\boldsymbol{\eta}_1\in L^\infty(0,T;\mathbf{V})\cap L^2(0,T;\mathbf{H}^2(\Omega)),\quad \partial_t\boldsymbol{\eta}_1\in L^2(Q),\\
&\boldsymbol{\eta}_2\in L^\infty(0,T;\mathbf{H}_0^1(\Omega))\cap L^2(0,T;\mathbf{H}^2(\Omega)),\quad \partial_t\boldsymbol{\eta}_2\in L^2(Q).
\end{aligned}
$$

Consequently, the unique solution of system (3.49)–(3.53) satisfies the regularity (3.54)–(3.55). Now, let $(\overline{\boldsymbol{\lambda}}_1,\overline{\boldsymbol{\lambda}}_2)$ the unique solution of (3.49)–(3.53); then, it suffices to identify $(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)$ with $(\overline{\boldsymbol{\lambda}}_1,\overline{\boldsymbol{\lambda}}_2)$. For this, we consider the unique solution $(\mathbf{U},\mathbf{W})\in\widehat{\mathbf{X}}_{\mathbf{u}}\times\widehat{\mathbf{X}}_{\mathbf{w}}$ of problem (3.27) (see the proof of Lemma 3.10 above) for $\mathbf{g}_{\mathbf{u}}:=(\boldsymbol{\lambda}_1-\overline{\boldsymbol{\lambda}}_1)\in L^2(Q)$ and $\mathbf{g}_{\mathbf{w}}:=(\boldsymbol{\lambda}_2-\overline{\boldsymbol{\lambda}}_2)\in L^2(Q)$. Then, written (3.49)-(3.52) for $(\overline{\boldsymbol{\lambda}}_1,\overline{\boldsymbol{\lambda}}_2)$ instead of $(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)$, and testing the first equation by $\mathbf{U}$

and the second equation by $\mathbf{W}$, we can obtain

$$\int_0^T \int_\Omega (\partial_t \mathbf{U} + \nu A\mathbf{U} + (\mathbf{U}\cdot\nabla)\tilde{\mathbf{u}} + (\tilde{\mathbf{u}}\cdot\nabla)\mathbf{U} - 2\nu_r \mathrm{curl}\,\mathbf{W})\cdot\overline{\boldsymbol{\lambda}}_1 = \alpha\int_0^T\int_\Omega |\tilde{\mathbf{u}}-\mathbf{u}_d|^4(\tilde{\mathbf{u}}-\mathbf{u}_d)\cdot\mathbf{U}, \quad (3.57)$$

$$\int_0^T \int_\Omega (\partial_t \mathbf{W} + L\mathbf{W} + (\tilde{\mathbf{u}}\cdot\nabla)\mathbf{W} + 4\nu_r\mathbf{W} - 2\nu_r \mathrm{curl}\,\mathbf{U})\cdot\overline{\boldsymbol{\lambda}}_2 = \beta\int_0^T\int_\Omega (\tilde{\mathbf{w}}-\mathbf{w}_d)\cdot\mathbf{W}. \quad (3.58)$$

Making the difference between (3.45) for and (3.57), and between (3.46) and (3.58), and then adding the respective equations, we can deduce

$$\int_0^T \int_\Omega (\partial_t \mathbf{U} + \nu A\mathbf{U} + (\mathbf{U}\cdot\nabla)\tilde{\mathbf{u}} + (\tilde{\mathbf{u}}\cdot\nabla)\mathbf{U} - 2\nu_r \mathrm{curl}\,\mathbf{W})\cdot(\boldsymbol{\lambda}_1-\overline{\boldsymbol{\lambda}}_1) + \int_0^T \int_\Omega (\partial_t \mathbf{W} + L\mathbf{W} + (\tilde{\mathbf{u}}\cdot\nabla)\mathbf{W} + 4\nu_r\mathbf{W} - 2\nu_r \mathrm{curl}\,\mathbf{U})\cdot(\boldsymbol{\lambda}_2-\overline{\boldsymbol{\lambda}}_2) = 0. \quad (3.59)$$

Therefore, taking into account that $(\mathbf{U},\mathbf{W})$ is the unique solution of (3.27) for $(\boldsymbol{\lambda}_1-\overline{\boldsymbol{\lambda}}_1)$ and $(\boldsymbol{\lambda}_2-\overline{\boldsymbol{\lambda}}_2)$, from (3.59) we obtain

$$\|\boldsymbol{\lambda}_1-\overline{\boldsymbol{\lambda}}_1\|^2_{L^2(Q)} + \|\boldsymbol{\lambda}_2-\overline{\boldsymbol{\lambda}}_2\|^2_{L^2(Q)} = 0,$$

which implies that $(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2) = (\overline{\boldsymbol{\lambda}}_1,\overline{\boldsymbol{\lambda}}_2)$ in $L^2(Q)\times L^2(Q)$. Consequently, the regularity of $(\overline{\boldsymbol{\lambda}}_1,\overline{\boldsymbol{\lambda}}_2)$ imply that

$$\boldsymbol{\lambda}_1 \in L^\infty(0,T;\mathbf{V})\cap L^2(0,T;\mathbf{H}^2(\Omega)), \quad \partial_t\boldsymbol{\lambda}_1\in L^2(Q),$$
$$\boldsymbol{\lambda}_2 \in L^\infty(0,T;\mathbf{H}^1_0(\Omega))\cap L^2(0,T;\mathbf{H}^2(\Omega)), \quad \partial_t\boldsymbol{\lambda}_2\in L^2(Q). \qquad \square$$

Finally, we deduce the optimality system of control problem (3.22).

Corollary 3.15. *Let* $(\tilde{\mathbf{u}},\tilde{\mathbf{w}},\tilde{\mathbf{f}})\in\mathcal{S}_{ad}$ *be a local optimal solution of problem* (3.22)*. Then, the Lagrange multipliers* $(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)$*, with*

$$\boldsymbol{\lambda}_1 \in L^\infty(0,T;\mathbf{V})\cap L^2(0,T;\mathbf{H}^2(\Omega)), \quad \partial_t\boldsymbol{\lambda}_1\in L^2(Q),$$
$$\boldsymbol{\lambda}_2 \in L^\infty(0,T;\mathbf{H}^1_0(\Omega))\cap L^2(0,T;\mathbf{H}^2(\Omega)), \quad \partial_t\boldsymbol{\lambda}_2\in L^2(Q).$$

satisfiy the following optimality system

$$\begin{cases} -\partial_t\boldsymbol{\lambda}_1 - \nu_1\Delta\boldsymbol{\lambda}_1 - \tilde{\mathbf{u}}\cdot\nabla\boldsymbol{\lambda}_1 + (\nabla\boldsymbol{\lambda}_1)^T\cdot\tilde{\mathbf{u}} + (\nabla\boldsymbol{\lambda}_2)^T\cdot\tilde{\mathbf{w}} + \nabla q \\ \qquad = 2\nu_r\mathrm{curl}\,\boldsymbol{\lambda}_2 - \alpha|\tilde{\mathbf{u}}-\mathbf{u}_d|^4(\tilde{\mathbf{u}}-\mathbf{u}_d) \quad \text{in } Q, \\ -\partial_t\boldsymbol{\lambda}_2 - \nu_2\Delta\boldsymbol{\lambda}_2 - \nu_3\nabla\mathrm{div}\,\boldsymbol{\lambda}_2 - \tilde{\mathbf{u}}\cdot\nabla\boldsymbol{\lambda}_2 + 4\nu_r\boldsymbol{\lambda}_2 \\ \qquad = 2\nu_r\mathrm{curl}\,\boldsymbol{\lambda}_1 - \beta(\tilde{\mathbf{w}}-\mathbf{w}_d) \quad \text{in } Q, \\ \mathrm{div}\,\boldsymbol{\lambda}_1 = 0 \quad \text{in } Q, \\ \boldsymbol{\lambda}_1(T)=\mathbf{0},\ \boldsymbol{\lambda}_2(T)=\mathbf{0} \quad \text{in } \Omega, \\ \boldsymbol{\lambda}_1=\mathbf{0},\ \boldsymbol{\lambda}_2=\mathbf{0} \quad \text{on } \partial\Omega\times(0,T), \\ \gamma\int_0^T\int_\Omega(\tilde{\mathbf{f}}+\boldsymbol{\lambda}_1)\cdot(\mathbf{f}-\tilde{\mathbf{f}})\geq 0 \qquad \forall\mathbf{f}\in\mathcal{U}. \end{cases} \quad (3.60)$$

Remark 3.16. If $\gamma > 0$. Then, from (3.60)$_6$, the fact that the control set $\mathcal{U}$ is closed and convex, and [2, Theorem 5.2, p. 132], we can characterizes the optimal control $\tilde{\mathbf{f}}$ as the projection of $-\frac{\lambda_1}{\gamma}$ onto $\mathcal{U}$; that is,

$$\tilde{\mathbf{f}} = \operatorname*{Proj}_{\mathcal{U}}\left(-\frac{\boldsymbol{\lambda}_1}{\gamma}\right).$$

Acknowledgments

This work was partially supported by MINEDUC-UA project, code ANT1855 (research stay in December 2018), and Coloquio de Matemática CR-4486 of Universidad de Antofagasta (research stay in September 2019). Moreover, the first author wishes to dedicate this work to his grandfather Exequiel Mallea Pino, rest in peace!

Conflict of interest

The authors declare that they have no conflict of interest.

References

[1] A. M. Alghamdi, S. Gala, M. A. Ragusa, On the blow-up criterion for incompressible Stokes-MHD equations, *Results Math.* **73**(2018), No. 3, Article No. 110, 6 pp. https://doi.org/10.1007/s00025-018-0874-x; MR3836182

[2] H. Brézis, *Functional analysis, Sobolev spaces and partial differential equations*, Springer, New York, 2011. https://doi.org/10.1007/978-0-387-70914-7; MR2759829

[3] E. Casas, An optimal control problem governed by the evolution Navier–Stokes equations, in: S. S. Sritharan (ed.), *Optimal control of viscous flow*, SIAM, Philadelphia, 1998, pp. 79–95. https://doi.org/10.1137/1.9781611971415.ch4; MR1632422

[4] E. Casas, K. Chrysafinos, Analysis of the velocity tracking control problem for the 3D evolutionary Navier–Stokes equations, *SIAM J. Control Optim.* **54**(2016), No. 1, 99–128. https://doi.org/10.1137/140978107; MR3448340

[5] J.C. De los Reyes, K. Kunisch, A semi-smooth Newton method for control constrained boundary optimal control of the Navier–Stokes equations, *Nonlinear Anal.* **62**(2005), No. 7, 1289–1316. https://doi.org/10.1016/j.na.2005.04.035; MR2154110

[6] A. C. Eringen, Simple microfluids, *Internat. J. Engrg. Sci.* **2**(1964), 205–217. https://doi.org/10.1016/0020-7225(64)90005-9; MR0169468

[7] A. C. Eringen, Theory of micropolar fluids, *J. Math. Mech.* **16**(1966), 1–16. https://doi.org/10.1512/iumj.1967.16.16001; MR0204005

[8] L. C. F. Ferreira, E. J. Villamizar-Roa, Micropolar fluid system in a space of distributions and large time behavior, *J. Math. Anal. Appl.* **332**(2007), No. 2, 1425–1445. https://doi.org/10.1016/j.jmaa.2006.11.018; MR2324348

[9] S. Gala, M. A. Ragusa, A new regularity criterion for the Navier–Stokes equations in terms of the two components of the velocity, *Electron. J. Qual. Theory Differ. Equ.* **2016**, No. 26, 1–9. https://doi.org/10.14232/ejqtde.2016.1.26; MR3498744

[10] S. Gala, M. A. Ragusa, A new regularity criterion for the 3D incompressible MHD equations via partial derivatives, *J. Math. Anal. Appl.* **481**(2020), Article No. 123497, 7 pp. https://doi.org/10.1016/j.jmaa.2019.123497; MR4007203

[11] S. Gala, M. A. Ragusa, Z. Zhang, A regularity criterion in terms of pressure for the 3D viscous MHD equations, *Bull. Malays. Math. Sci. Soc.* **40**(2017), No. 4, 1677–1690. https://doi.org/10.1007/s40840-015-0160-y; MR3712578

[12] F. Guillén-González, E. Mallea-Zepeda, M. A. Rodríguez-Bellido, Optimal bilinear control problem related to a chemo-repulsion system in 2D domains, *ESAIM Control Optim. Calc. Var.*, published online, 2019. https://doi.org/10.1051/cocv/2019012

[13] F. Guillén-González, E. Mallea-Zepeda, M. A. Rodríguez-Bellido, A regularity criterion for a 3D chemo-repulsion system and its application to a bilinear optimal control problem. Submitted for publication (2018). arXiv:1808.09294.

[14] M. D. Gunzburger, S. Manservisi, The velocity tracking problem for Navier–Stokes flows with bounded distributed controls, *SIAM J. Control Optim.* **37**(1999), No. 6, 1913–1945. https://doi.org/10.1137/S0363012998337400; MR1720145

[15] M. D. Gunzburger, S. Manservisi, Analysis and approximation of the velocity tracking problem for Navier–Stokes flows with distributed control, *SIAM J. Numer. Anal.* **37**(2000), No. 5, 1481–1512. https://doi.org/10.1137/S0036142997329414; MR1759904

[16] B. T. Kien, A. Rösch, D. Wachsmuth, Pontryagin's principle for optimal control problem governed by 3D Navier–Stokes equations, *J. Optim. Theory Appl.* **173**(2017), No. 1, 30–55. https://doi.org/10.1007/s10957-017-1081-8; MR3626636

[17] O. A. Ladyzhenskaya, *The mathematical theory of viscous incompressible flow*, Mathematics and its Applications, Vol. 2, Gordon and Breach, Science Publishers, New York–London–Paris, 1969. MR0254401

[18] J. L. Lions, *Quelques métodes de résolution de probèmes aux limite non linéares* (in French), Dunod. Paris, 1969. MR0259693

[19] M. Loayza, M. A. Rojas-Medar, A weak-L^p Prodi–Serrin type regularity criterion for the micropolar fluid equations, *J. Math. Phys.* **57**(2016), 021512, 6 pp. https://doi.org/10.1063/1.4942047; MR3462971

[20] G. Lukaszewicz, *Micropolar fluids. Theory and applications*, Birkhaüser. Boston, 1999. https://doi.org/10.1007/978-1-4612-0641-5; MR1711268

[21] E. Mallea-Zepeda, E. Ortega-Torres, E. J. Villamizar-Roa, An optimal control problem for the Navier–Stokes-α system. Submitted for its publication (2019). arXiv:1905.01415

[22] E. Mallea-Zepeda, E. Ortega-Torres, E. J. Villamizar-Roa, A boundary control problem for micropolar fluids, *J. Optim. Theory Appl.* **169**(2016), No. 2, 349–369. https://doi.org/10.1007/s10957-016-0925-y; MR3489810

[23] E. Mallea-Zepeda, E. Ortega-Torres, E. J. Villamizar-Roa, An optimal control problem for the steady nonhomogeneous ssymmetric fluids, *Appl. Math. Optim.* **80**(2019), No. 2, 299–329. https://doi.org/10.1007/s00245-017-9466-5; MR4008667

[24] G. Mulone, F. Salemi, On the existence of hydrodynamic motion in a domain with free boundary type conditions, *Meccanica* **18**(1983), No. 3, 136–144. https://doi.org/10.1007/BF02128580

[25] A. Nowakowski, First order sufficient optimality conditions for Navier–Stokes flow. Dual feedback controls, *SIAM J. Control Optim.* **55**(2017), No. 4, 2734–2747. https://doi.org/10.1137/16M1082998; MR3691214

[26] E. Ortega-Torres, M. A. Rojas-Medar, On the regularity for solutions of the micropolar fluid equations, *Rend. Semin. Mat. Univ. Padova* **122**(2009), 27–37. https://doi.org/10.4171/RSMUP/122-3; MR2582828

[27] E. Ortega-Torres, E. J. Villamizar-Roa, M. A. Rojas-Medar, Micropolar fluids with vanishing viscosity, *Abstr. Appl. Anal.* **2010**, Article ID 843692, 18 pp. https://doi.org/10.1155/2010/843692; MR2644033

[28] J. Simon, Compact sets in the space $L^p(0;T;B)$, *Ann. Mat. Pura Appl.* **146**(1987), 65–96. https://doi.org/10.1007/BF01762360; MR916688

[29] R. Stavre, The control of the pressure for a micropolar fluid, *Z. Angew. Math. Phys.* **53**(2002), No. 2, 912–922. https://doi.org/10.1007/PL00012619; MR1963543

[30] R. Stavre, Optimization and numerical approximation for micropolar fluids, *Numer. Funct. Anal. Optim.* **24**(2003), No. 3-4, 223–241. https://doi.org/10.1081/NFA-120022919; MR1990647

[31] R. Temam, *Navier–Stokes equations. Theory and numerical analysis*, North-Holland. Amsterdam, 1979. MR0603444

[32] F. Tröltzsch, *Optimal control of partial differential equations. Theory, methods and applications*, AMS, Rhode Island. Providence, 2010. https://doi.org/10.1090/gsm/112; MR2583281

[33] E.J. Villamizar-Roa, M. A. Rodríguez-Bellido, Global existence and exponential stability for the micropolar fluid system, *Z. Angew. Math. Phys.* **59**(2008), No. 5, 790–809. https://doi.org/10.1007/s00033-007-6090-2; MR2442951

[34] J. Zowe, S. Kurcyusz, Regularity and stability for the mathematical programming problem in Banach spaces. *Appl. Math. Optim.* **5**(1979), No. 1, 49–62. https://doi.org/10.1007/BF01442543; MR0526427