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Multiple and particular solutions of a second order discrete boundary value problem with mixed periodic boundary conditions

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Abstract. In this paper, a second order discrete boundary value problem with a pair of mixed periodic boundary conditions is considered. Sufficient conditions on the existence of multiple solutions are obtained by using the critical point theory. Necessary conditions for a particular solution subject to pre-defined criteria are also investigated. Examples are given to illustrate the applications of the results as well.

Keywords: discrete boundary value problem, mixed periodic boundary conditions, variational methods, mountain pass lemma, Lagrange multiplier.

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1 Introduction

In this paper, we consider a boundary value problem (BVP) consisting of a second order difference equation

$$-\Delta(r(t-1)\Delta u(t-1)) = f(t, u(t)), \qquad t \in [2, N]_{\mathbb{Z}}, \tag{1.1}$$

and a pair of mixed periodic boundary conditions (BCs)

$$u(0) = u(N), \qquad r(0)\Delta u(0) = -r(N)\Delta u(N), \tag{1.2}$$

where

- $N \geq 2$ is an integer and $[a, b]_{\mathbb{Z}}$ denotes the discrete interval $\{a, \dots, b\}$ for any integers a and b with $a \leq b$;
- Δ is the forward difference operator defined by $\Delta u(t) = u(t+1) - u(t)$;
- $r(t) > 0$, $t \in [0, N]_{\mathbb{Z}}$; and

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- $f : [2, N]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ is odd and continuous with respect to the second variable, i.e. $f(t, -x) = -f(t, x)$ and $f(t, \cdot) \in C(\mathbb{R})$, $t \in [2, N]_{\mathbb{Z}}$.

By a solution of BVP (1.1), (1.2), we mean a function $u : [0, N+1]_{\mathbb{Z}} \to \mathbb{R}$ that satisfies (1.1) and (1.2).

BVPs with various BCs have been widely studied for decades due to both theoretic importance and extensive applications in science and engineering areas. Great effort has been made to study the existence, multiplicity, and uniqueness of solutions of BVPs, see for example [4–11, 13–18] and references therein for some recent advances in this area.

Recently, Kong and Wang [15] studied the existence and multiplicity of solutions of the mixed periodic BVP

$$-\Delta^2 u(t-1) = f(u(t)), \quad t \in [2, N]_{\mathbb{Z}}, \tag{1.3}$$

$$u(0) = -u(N), \quad \Delta u(0) = \Delta u(N), \tag{1.4}$$

by using the critical point theory. In that work, the asymmetry at the boundaries of the domain caused by the mixed periodic BC (1.4) was the major obstacle in the construction of a suitable functional for applying the variational technique. As the result, a particular Banach space and an associated functional were proposed to overcome the asymmetry of the mixed periodic BC (1.4). The reader is referred to [15, Lemma 2.3] for the details. We want to point out that there was a typo in Eq. (1.1) in [15] where the domain was mistakenly written as $t \in [1, N]_{\mathbb{Z}}$, which should be replaced by $t \in [2, N]_{\mathbb{Z}}$ as seen in Eq. (1.3) above. The reason why we propose $t \in [2, N]_{\mathbb{Z}}$ will be explained in Remark 2.5 below.

Clearly, Eq. (1.1) covers Eq. (1.3) as a special case and BC (1.2) and BC (1.4) are closely related to each other. So BVP (1.1), (1.2) is parallel to BVP (1.3), (1.4) but more general. Moreover, BC (1.2) leads to an asymmetry at the boundaries as well. This obstacle must be first eliminated to construct the functional. We will use an idea similar to [15] to overcome this difficulty and further apply the variational arguments and the critical point theory to study the existence of multiple solutions of BVP (1.1), (1.2). This will be the first contribution of this paper.

Once the multiplicity of solutions is proven, it is natural to raise a new question: *Which solution is the "right" one (in the sense that some pre-defined criteria are met)?* This question is practical in applications as there is a common need to identify a particular solution following certain pre-defined criteria, among all the solutions, due to constraints or demands of particular circumstances. In this paper, a framework to derive the necessary conditions for a particular solution of BVP (1.1), (1.2) following a set of pre-defined criteria, i.e. a *target solution*, will be presented. To the best of our knowledge, this type of questions have not been considered in the literature on BVPs. Our work will fill the void and be applicable to other problems with multiple solutions. This will be the second contribution of this paper.

The remainder of this paper is organized as follows. The Banach space, the functional, and the needed lemmas are given in Section 2; criteria on the existence of multiple solutions are proven in Section 3; the necessary conditions of the target solutions are derived in Section 4; and three examples are given in Section 5 to demonstrate the applications of our results.

2 Preliminary

We first introduce a few definition and lemmas needed to prove our existence results.

Definition 2.1. Assume H is a real Banach space. We say that a functional $J \in C^1(H, \mathbb{R})$ satisfies the Palais–Smale (PS) condition if every sequence $\{u_n\} \subset H$, such that $J(u_n)$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$, has a convergent subsequence. The sequence $\{u_n\}$ is called a PS sequence.

The following version of Clark's Theorem is taken from [19] and will play a key role in proving our existence theorem.

Lemma 2.2 ([19, Theorem 9.1]). *Let H be a real Banach space with $\mathbf{0}$ the zero of H, S^{n-1} be the $(n-1)$-dimensional unit sphere, and $J \in C^1(H, \mathbb{R})$ with J even, bounded from below and satisfying the PS condition. Suppose $J(\mathbf{0}) = 0$, and there is a set $K \subset H$ such that K is homeomorphic to S^{n-1} by an odd map, and $\sup_K J < 0$. Then J possesses at least n distinct pairs of critical points.*

In the sequel, we let H be defined by

$$H = \{u : [0, N+1]_{\mathbb{Z}} \to \mathbb{R} \mid u(0) = u(N),\ u(1) = 0,\ r(0)\Delta u(0) = -r(N)\Delta u(N)\}. \tag{2.1}$$

Remark 2.3. By (2.1), we see that any $u \in H$ must satisfy

$$u(0) = u(N), \quad u(1) = 0, \quad u(N+1) = \frac{r(0) + r(N)}{r(N)} u(N). \tag{2.2}$$

So H is isomorphic to $\mathbb{R}^{N-1}$. Then, equipped with the norm $\|u\| = \left(\sum_{t=1}^{N} u^2(t)\right)^{\frac{1}{2}}$, H is an $N-1$ dimensional Banach space. When we write the vector $u = (0, u(2), \dots, u(N)) \in \mathbb{R}^N$, we always imply that u can be extended as a vector in H so that (2.2) holds, i.e., u can be extended to the vector

$$\left(u(N), 0, u(2), \dots, u(N), \frac{r(0) + r(N)}{r(N)} u(N) \right).$$

Moreover, for any $u \in H$, when we write $u = (0, u(2), \dots, u(N)) \in \mathbb{R}^N$, we mean that u have been extended in the above sense.

Let $\tilde{f} : [1, N]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ and $\tilde{F} : [1, N]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$\tilde{f}(t, x) = \begin{cases} 0, & t = 1, \\ f(t, x), & t \in [2, N-1]_{\mathbb{Z}}, \\ f(N, x) + 2r(0)x, & t = N, \end{cases} \tag{2.3}$$

and

$$\tilde{F}(t, x) = \int_0^x \tilde{f}(t, s) ds, \qquad t \in [1, N]_{\mathbb{Z}}, \tag{2.4}$$

resectively. It is clear that $\tilde{f}(t, x)$ and $\tilde{F}(t, x)$ are continuous in x and $\tilde{f}(t, x)$ is odd in x if $f(t, x)$ is odd in x.

Define $J : H \to \mathbb{R}$ by

$$J(u) = -\frac{1}{2} \sum_{t=1}^{N} r(t-1)(\Delta u(t-1))^2 + \sum_{t=1}^{N} \tilde{F}(t, u(t)). \tag{2.5}$$

Lemma 2.4. *If $u \in H$ is a critical point of J, then u is a solution of BVP (1.1), (1.2).*

Proof. By (2.3)–(2.5), for any $u \in H$,

$$J(u) = -\frac{1}{2}\sum_{t=1}^{N} r(t-1)(\Delta u(t-1))^2 + \sum_{t=2}^{N}\int_0^{u(t)} f(t,s)ds + 2\int_0^{u(N)} r(0)sds.$$

Then J is continuously differentiable and its derivative $J'(u)$ at $u \in H$ is given by

$$\langle J'(u), v\rangle = -\sum_{t=1}^{N} r(t-1)\Delta u(t-1)\Delta v(t-1) + \sum_{t=2}^{N} f(t,u(t))v(t) + 2r(0)u(N)v(N) \tag{2.6}$$

for any $v \in H$.

By the summation by parts formula and (2.1),

$$\begin{aligned}\sum_{t=1}^{N} r(t-1)\Delta u(t-1)\Delta v(t-1) &= r(N)\Delta u(N)v(N) - r(0)\Delta u(0)v(0) \\ &\quad - \sum_{t=1}^{N} \Delta(r(t-1)\Delta u(t-1))v(t) \\ &= -2r(0)\Delta u(0)v(0) - \sum_{t=1}^{N} \Delta(r(t-1)\Delta u(t-1))v(t) \\ &= 2r(0)u(N)v(N) - \sum_{t=2}^{N} \Delta(r(t-1)\Delta u(t-1))v(t). \end{aligned} \tag{2.7}$$

Then by (2.6) and (2.7), we have $\langle J'(u), v\rangle = \sum_{t=2}^{N} [\Delta(r(t-1)\Delta u(t-1)) + f(t,u(t))]\, v(t)$. This completes the proof of the lemma. □

Remark 2.5. Below, we provide some justification why we introduce the space H and the functional J as given above and why Eq. (1.1) is defined on $[2, N]_{\mathbb{Z}}$ instead of $[1, N]_{\mathbb{Z}}$. To see this, assume Eq. (1.1) is defined on $[1, N]_{\mathbb{Z}}$, and as in the traditional way, let

$$\tilde{H} = \{u : [0, N+1]_{\mathbb{Z}} \to \mathbb{R} \mid u \text{ satisfies the BCs (1.2)}\}$$

and

$$\tilde{J}(u) = -\frac{1}{2}\sum_{t=1}^{N} r(t-1)(\Delta u(t-1))^2 + \sum_{t=1}^{N}\int_0^{u(t)} f(t,s)ds.$$

Then, if $u \in \tilde{H}$ is a critical point of $\tilde{J}(u)$, by summation by parts formula and (1.2), we have

$$\begin{aligned}\langle \tilde{J}'(u), v\rangle &= -\sum_{t=1}^{N} r(t-1)\Delta u(t-1)\Delta v(t-1) + \sum_{t=1}^{N} f(t,u(t))v(t) \\ &= -2r(N)\Delta u(N)v(N) + \sum_{t=1}^{N}[\Delta(r(t-1)\Delta u(t-1)) + f(t,u(t))]v(t)\end{aligned}$$

for any $v \in H$. Since $v \in \tilde{H}$ is arbitrary, u satisfies (1.1) at $t \in [1, N-1]_{\mathbb{Z}}$. However, u satisfies Eq. (1.1) at $t = N$ only if $\Delta u(N) = 0$. Then the BCs (1.2) now become

$$u(0) = u(N), \quad \Delta u(0) = \Delta u(N) = 0,$$

which is very restrictive and is a special case of the periodic BCs studied in the literature, for example, in [12, 16]. We do not have an interest in such a simple case. In this work, in order to make u satisfy Eq. (1.1) at $t = N$ without introducing the extra assumption $\Delta u(N) = 0$, unlike the traditional way, we introduce a modification, $\tilde{f}$, of the function f, as given in (2.3), and the corresponding functional J in (2.5). In addition to the BCs, we also impose an extra condition $u(1) = 0$ in our working space H defined by (2.1). Then, as seen in Lemma 2.4, any critical point $u \in H$ of J satisfies Eq. (1.1) for all $t \in [2, N-1]_{\mathbb{Z}}$ and the BCs

$$u(0) = u(N), \quad u(1) = 0, \quad r(0)\Delta u(0) = -r(N)\Delta u(N).$$

That is, u is a solution of BVP (1.1), (1.2) with the property that $u(1) = 0$. This type of problems are new and are worthy of our studies. The above explanations also explain why we only require Eq. (1.1) to be defined on $[2, N]_{\mathbb{Z}}$. We propose Eq. (1.3) in [15] due to a similar reason.

Remark 2.6. Lemma 2.5 offers a general setting to study the BVPs with mixed periodic BCs. With the functional defined by (2.5), other variational techniques may be applied as well, see, for example, [1,3].

Next, let us consider an equivalent form of J. Let

$$A = \begin{bmatrix} r(0)+r(1) & -r(1) & 0 & \dots & 0 & -r(0) \\ -r(1) & r(1)+r(2) & -r(2) & \dots & 0 & 0 \\ 0 & -r(2) & r(2)+r(3) & \dots & 0 & 0 \\ & \cdots & & \cdots & & \\ -r(0) & 0 & 0 & \dots & -r(N-1) & r(N-1)+r(0) \end{bmatrix}_{N\times N}. \tag{2.8}$$

Then it can be verified by direct computation that for any $u \in H$,

$$J(u) = -\frac{1}{2}uAu^T + \sum_{t=1}^{N} \tilde{F}(t, u(t)), \tag{2.9}$$

where $(\cdot)^T$ denotes the transpose.

Matrix A has been studied in [16]. Some needed conclusions are summarized in the following lemma. The reader is referred to [16] for the details.

Lemma 2.7. *Let A be defined by (2.8) with $r(t) > 0$, $t \in [0, N-1]_{\mathbb{Z}}$. Then*

(a) A is positively semi-definite with $\mathrm{Rank}(A) = N - 1$.

(b) A has N nonnegative eigenvalues $0 = \lambda_0 < \lambda_1 \leq \dots \leq \lambda_{N-1}$ with the associated orthonormal eigenvectors $\{\eta_0, \dots, \eta_{N-1}\}$, where $\eta_0 = (\frac{\sqrt{N}}{N}, \frac{\sqrt{N}}{N}, \dots, \frac{\sqrt{N}}{N})$.

(c) Let $\|\cdot\|$ denote the standard Euclidean norm of $\mathbb{R}^N$. For any $u \in \mathbb{R}^N$, $uAu^T \leq \lambda_{N-1}\|u\|^2$; for any $u \in \mathrm{span}\{\eta_2, \dots, \eta_{N-1}\}$, $uAu^T \geq \lambda_1 \|u\|^2$.

Similary to [16, Lemma 3.1], we can prove the following lemma.

Lemma 2.8. *Assume there exists a constant $\beta > \lambda_{N-1}$ such that*

$$\lim_{x\to\infty} \frac{f(t,x)}{x} \geq \beta, \qquad t \in [2, N]_{\mathbb{Z}}. \tag{2.10}$$

Then J satisfies the PS condition.

Proof. Let $\{u_n\}_{n=1}^{\infty} \subset H$ be any sequence with $\{J(u_n)\}$ bounded and $J'(u_n) \to 0$ as $n \to \infty$. For any u_n, by (2.6), (2.5), and (2.9),

$$\begin{aligned}\langle J'(u_n), u_n\rangle &= -\sum_{t=1}^{N} r(t-1)(\Delta u_n(t-1))^2 + \sum_{t=2}^{N} f(t, u_n(t))u_n(t) + 2r(0)(u_n(N))^2 \\ &= -u_n A u_n^T + \sum_{t=2}^{N} f(t, u_n(t))u_n(t) + 2r(0)(u_n(N))^2.\end{aligned}$$

Then by Lemma 2.7,

$$\begin{aligned}\sum_{t=2}^{N} f(t, u_n(t))u_n(t) + 2r(0)(u_n(N))^2 &= \langle J'(u_n), u_n\rangle + u_n A u_n^T \\ &\le \langle J'(u_n), u_n\rangle + \lambda_{N-1}\|u_n\|^2. \end{aligned} \tag{2.11}$$

On the other hand, by the oddity of f and (2.10), there exists constant $C > 0$ such that

$$f(t, u_n(t))u_n(t) \ge \left(\frac{\beta + \lambda_{N-1}}{2}\right)(u_n(t))^2 - C, \qquad t \in [2, N]_{\mathbb{Z}}.$$

Hence

$$\sum_{t=2}^{N} f(t, u_n(t))u_n(t) + 2r(0)(u_n(N))^2 \ge \left(\frac{\beta + \lambda_{N-1}}{2}\right)\|u_n\|^2 - NC. \tag{2.12}$$

By (2.11) and (2.12),

$$\left(\frac{\beta - \lambda_{N-1}}{2}\right)\|u_n\|^2 \le \langle J'(u_n), u_n\rangle + NC \le \|J'(u_n)\|\|u_n\| + NC.$$

Since $(\beta - \lambda_{N-1})/2 > 0$ and $J'(u_n) \to 0$ as $n \to \infty$, $\{u_n\}$ is bounded. Therefore, the PS condition holds. □

3 Existence of solutions

In this section, we consider the existence of multiple solutions of BVP (1.1), (1.2).

Theorem 3.1. *Let $0 = \lambda_0 < \lambda_1 \le \cdots \le \lambda_{N-1}$ be the eigenvalues of A defined by (2.8) respectively. Assume that $f(t, x)$ is continuous and odd in its second variable x, and satisfies (2.10) for some $\beta > \lambda_{N-1}$. If in addition there exists a constant $\mu < \lambda_m$, $m \in [1, N-1]_{\mathbb{Z}}$, such that*

$$\lim_{x\to 0} \frac{f(t,x)}{x} \le \mu, \quad t \in [2, N-1]_{\mathbb{Z}}, \quad \text{and} \quad \lim_{x\to 0} \frac{f(N,x)}{x} + 2r(0) \le \mu. \tag{3.1}$$

Then BVP (1.1), (1.2) has at least $2N - 2m$ distinct solutions.

Remark 3.2. In (3.1), when $N = 2$, we have $[2, N-1]_{\mathbb{Z}} = \emptyset$. Then, the first limit disappears.

Proof. By Lemma 2.8, J satisfies the PS condition. Since f is odd in x, by (2.3) and (2.4), $\tilde{F}(t, x)$ is even in x.

Let $\{\eta_0, \ldots, \eta_{N-1}\}$ be the orthonormal eigenvectors of A defined in Lemma 2.7, $X = \operatorname{span}\{\eta_1, \ldots, \eta_{N-1}\}$, and $Y = \operatorname{span}\{\eta_0\}$. Then it is easy to see that $\mathbb{R}^N = X \oplus Y$. By (2.1),

$H \cap Y = \mathbf{0}$, so $H = X$. For any $u \in H$, there exist $b_1, \dots, b_{N-1} \in \mathbb{R}$ such that $u = \sum_{i=1}^{N-1} b_i \eta_i$ and $\|u\|^2 = \sum_{i=1}^{N-1} b_i^2$. By (2.9) and Lemma 2.7, for any $u \in H$,

$$\begin{aligned} J(u) &= -\frac{1}{2} u A u^T + \sum_{t=1}^{N} \tilde{F}(t, u(t)) = -\frac{1}{2} \sum_{i=1}^{N-1} \lambda_i b_i^2 + \sum_{t=1}^{N} \tilde{F}(t, u(t)) \\ &\geq -\frac{1}{2} \lambda_{N-1} \sum_{i=1}^{N-1} b_i^2 + \sum_{t=1}^{N} \tilde{F}(t, u(t)) = -\frac{1}{2} \lambda_{N-1} \|u\|^2 + \sum_{t=1}^{N} \tilde{F}(t, u(t)). \end{aligned}$$

Similar to the proof of Lemma 2.8, there exists $\tilde{C} > 0$ such that

$$\sum_{t=1}^{N} \tilde{F}(t, u(t)) \geq \left(\frac{\beta + \lambda_{N-1}}{4} \right) \|u\|^2 - N\tilde{C}, \qquad u \in H.$$

Therefore, $\inf_{u \in H} J(u) > -\infty$, i.e. J is bounded below.

By (3.1), there exist $\rho > 0$ and $0 < D < \lambda_m$ such that for any $x \in [-\rho, \rho]$,

$$\int_0^x f(t,s) ds \leq \frac{D}{2} x^2, \quad t \in [2, N-1] \quad \text{and} \quad \int_0^x f(N,s) ds + r(0) x^2 \leq \frac{D}{2} x^2. \tag{3.2}$$

Let $K = \{u \in \operatorname{span}\{\eta_m, \dots, \eta_{N-1}\} \subset H \mid \|u\| = \rho\}$. It is clear that K is homeomorphic to S^{N-m-1} by an odd map $\Gamma : K \to X$ defined by $\Gamma u = \frac{u}{\rho}$. By (2.9), (2.3), (2.4), (3.2), and Lemma 2.7, for any $u \in K$,

$$\begin{aligned} J(u) &= -\frac{1}{2} u A u^T + \sum_{t=1}^{N} \tilde{F}(t, u(t)) = -\frac{1}{2} \sum_{i=m}^{N-1} \lambda_i b_i^2 + \sum_{t=1}^{N} \tilde{F}(t, u(t)) \\ &\leq -\frac{1}{2} \lambda_m \sum_{i=1}^{N-1} b_i^2 + \sum_{t=1}^{N} \tilde{F}(t, u(t)) = -\frac{1}{2} \lambda_m \|u\|^2 + \sum_{t=1}^{N} \tilde{F}(t, u(t)) \leq \frac{D - \lambda_m}{2} \rho^2 < 0. \end{aligned}$$

Therefore, $\sup_K J < 0$. By Lemma 2.2, J possesses at least $N - m$ distinct pairs of critical points. Hence BVP (1.1), (1.2) has at least $2N - 2m$ solutions by Lemma 2.4. □

The following corollary is an immediate conclusion of Theorem 3.1.

Corollary 3.3. *Assume that $f(t,x)$ is continuous and odd in its second variable x, and satisfies*

$$\liminf_{x \to \infty} \min_{t \in [2,N]_{\mathbb{Z}}} \frac{f(t,x)}{x} = \infty$$

and

$$\max \left\{ \limsup_{x \to 0} \max_{t \in [2,N-1]_{\mathbb{Z}}} \frac{f(t,x)}{x}, \limsup_{x \to 0} \frac{f(N,x)}{x} + 2r(0) \right\} < \lambda_m, \tag{3.3}$$

where λ_m is the mth positive eigenvalue of A following the increasing order. Then BVP (1.1), (1.2) has at least $2N - 2m$ distinct solutions.

A note similar to Remark 3.2 applies to Eq. (3.3) in Corollary 3.3.

4 Necessary conditions of the target solution

In this section, we investigate how to identify a target solution among multiple solutions following a set of pre-defined criteria. The main idea is to find the target solution by solving an optimization problem (OP) with constraints.

Let I be a subset of $[0, N+1]_{\mathbb{Z}}$ and $u^* : I \to \mathbb{R}$ be a function defined on I. Assume the pre-defined criteria is given as a performance index, or objective function, $L : \mathbb{R}^{N+2} \to \mathbb{R}$ defined by

$$L(u) = \sum_{t \in I} (u(t) - u^*(t))^2. \tag{4.1}$$

We need to find a particular solution of BVP (1.1), (1.2) that minimizes the objective function L. In other words, BVP (1.1), (1.2) is the constraints of the OP.

We first introduce some auxiliary functions to simplify the notations. Define $G : \mathbb{R}^{N+2} \times [2, N]_{\mathbb{Z}} \to \mathbb{R}$, $B_0 : \mathbb{R}^{N+2} \to \mathbb{R}$, $B_1 : \mathbb{R}^{N+2} \to \mathbb{R}$, and $B_2 : \mathbb{R}^{N+2} \to \mathbb{R}$ by

$$\begin{aligned} G(u,t) &= r(t)u(t+1) - (r(t) + r(t-1))u(t) + r(t-1)u(t-1) + f(t, u(t)), \\ B_0(u) &= u(1), \ B_1(u) = u(0) - u(N), \text{ and} \\ B_2(u) &= r(0)u(1) - r(0)u(0) + r(N)u(N+1) - r(N)u(N). \end{aligned}$$

It is easy to verify that BVP (1.1), (1.2) is equivalent to the following system consisting of $N+2$ equations

$$G(u,t) = 0, \ t \in [2, N]_{\mathbb{Z}}, \tag{4.2}$$
$$B_0(u) = 0, \tag{4.3}$$
$$B_1(u) = 0, \tag{4.4}$$
$$B_2(u) = 0. \tag{4.5}$$

In the sequel, we use Eq. (4.2)–(4.5) as the constraints and solve the OP (4.1), (4.2)–(4.5) by the Lagrange multiplier method, see for example [2]. Clearly, $N+2$ Lagrange multipliers are needed. Let $\theta : [0, N+1] \to \mathbb{R}$ be the Lagrange multipliers and $\Phi : \mathbb{R}^{N+2} \times \mathbb{R}^{N+2} \to \mathbb{R}$ be defined by

$$\Phi(u, \theta) = \zeta L(u) + \sum_{t=2}^{N} \theta(t+1) G(u,t) + \theta(0) B_0(u) + \theta(1) B_1(u) + \theta(2) B_2(u), \tag{4.6}$$

where $\zeta > 0$ is a parameter. Then by the Lagrange multiplier method, we obtain the following necessary conditions for the target solution.

Theorem 4.1. *A target solution of BVP (1.1), (1.2) subject to L must satisfy Eq. (4.2)–(4.5) and*

$$\frac{\partial \Phi(u, \theta)}{\partial u(t)} = 0, \qquad t \in [0, N+1]_{\mathbb{Z}}.$$

Remark 4.2. The value of ζ in (4.6) does not impact the theoretic result in Theorem 4.1. However, numerical experiments reveal that the value of ζ impacts the performance of numerical optimization algorithms. This is the main reason to introduce the parameter ζ.

5 Examples

In this section, we will demonstrate the applications of our results by considering the BVP

$$-\Delta^2 u(t-1) = (u(t))^3,\ t \in [2,10]_{\mathbb{Z}}, \tag{5.1}$$
$$u(0) = u(10),\ \Delta u(0) = -\Delta u(10). \tag{5.2}$$

Let $r(t) \equiv 1$ on $[0,N]_Z$ and $f(t,x) \equiv x^3$. It is easy to verify that

$$A = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 & -1 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & \dots & -1 & 2 \end{bmatrix}_{10\times 10},$$

$$\lim_{x\to\infty} \frac{f(x)}{x} = \infty, \quad \text{and} \quad \lim_{x\to 0} \frac{f(x)}{x} = 0.$$

Computing the eigenvalues of A with Matlab, we have $\lambda_4 < 2 < \lambda_5$. Hence all the conditions of Corollary 3.3 are satisfied. Therefore, BVP (5.1), (5.2) has at least 10 solutions.

Next, we choose different objective functions to demonstrate the applications of Theorem 4.1.

Example 5.1. We first consider a solution of BVP (5.1), (5.2) that minimizes the objective function

$$L_1(u) = \sum_{t=4}^{6} (u(t)-1)^2.$$

Let $\zeta = 2$. By Theorem 4.1, the target solution u must satisfy the following system

$$u(t-1) - 2u(t) + u(t+1) + (u(t))^3 = 0, \quad t \in [2,10]_{\mathbb{Z}}, \tag{5.3}$$
$$u(1) = 0 \tag{5.4}$$
$$u(0) - u(10) = 0, \tag{5.5}$$
$$-u(0) + u(1) - u(10) + u(11) = 0, \tag{5.6}$$
$$\tilde{\Theta}(u,\theta,t) + \tilde{\Phi}(u,\theta,t) = 0, \quad t \in [0,11]_{\mathbb{Z}}, \tag{5.7}$$

where

$$\tilde{\Theta}(u,\theta,0) := \theta(1) - r(0)\theta(2), \tag{5.8}$$
$$\tilde{\Theta}(u,\theta,1) := \theta(0) + r(0)\theta(2) + r(1)\theta(3), \tag{5.9}$$
$$\tilde{\Theta}(u,\theta,2) := (3(u(2))^2 - (r(1)+r(2)))\theta(3) + r(2)\theta(4), \tag{5.10}$$
$$\begin{aligned}\tilde{\Theta}(u,\theta,t) := {}& r(t-1)\theta(t) + (3(u(t))^2 - (r(t-1)+r(t)))\theta(t+1) \\ & + r(t)\theta(t+2), \quad t = [3,9]_{\mathbb{Z}},\end{aligned} \tag{5.11}$$
$$\tilde{\Theta}(u,\theta,10) := -\theta(1) - r(10)\theta(2) + r(9)\theta(10) + (3(u(10))^2 - (r(9)+r(10)))\theta(11), \tag{5.12}$$
$$\tilde{\Theta}(u,\theta,11) := r(10)\theta(2) + r(10)\theta(11), \tag{5.13}$$

and

$$\tilde{\Phi}(u,\theta,t) := 4(u(t)-1), \quad t \in [4,6]_{\mathbb{Z}}, \tag{5.14}$$
$$\tilde{\Phi}(u,\theta,t) := 0, \quad \text{otherwise}. \tag{5.15}$$

Note that by (4.2)–(4.5), Eq. (5.3)–(5.6) are equivalent to BVP (5.1), (5.2); $\tilde{\Theta}$ defined by (5.8)–(5.13) are the partial derivatives of

$$\sum_{t=2}^{N} \theta(t+1)G(u,t) + \theta(0)B_0(u) + \theta(1)B_1(u) + \theta(2)B_2(u)$$

in (4.6) with respect to $u(t)$, $t \in [0,11]_{\mathbb{Z}}$; and $\tilde{\Phi}$ defined by (5.14) and (5.15) are the partial derivatives of $\zeta L(u)$ in (4.6) with respect to $u(t)$, $t \in [0,11]_{\mathbb{Z}}$.

System (5.3)–(5.7) is solved with Matlab. The graph of the numerical solution u_1 subject to L_1 is given in Figure 5.1. Clearly, the behavior of u_1 is consistent with our expectation.

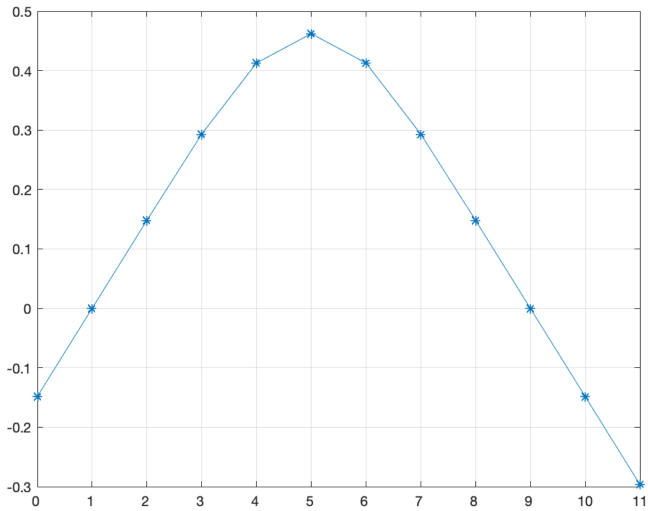

Figure 5.1: Numerical solution u_1 subject to L_1.

Example 5.2. For the comparison purpose, we also consider the solution of BVP (5.1), (5.2) that minimizes the objective function

$$L_2(u) = \sum_{t=4}^{6} (u(t)+1)^2.$$

Let $\zeta = 2$. By Theorem 4.1, the target solution must satisfy Eq. (5.3)–(5.7) with

$$\tilde{\Phi}(u,\theta,t) := 4(u(t)+1), \quad t \in [4,6]_{\mathbb{Z}},$$
$$\tilde{\Phi}(u,\theta,t) := 0, \quad \text{otherwise}.$$

The graph of the numerical solution u_2 subject to L_2 is given in Figure 5.2.

Example 5.3. In this example, we seek a solution of BVP (5.1), (5.2) that minimizes the objective function

$$L_3(u) = \sum_{t=7}^{9} (u(3)-10)^2.$$

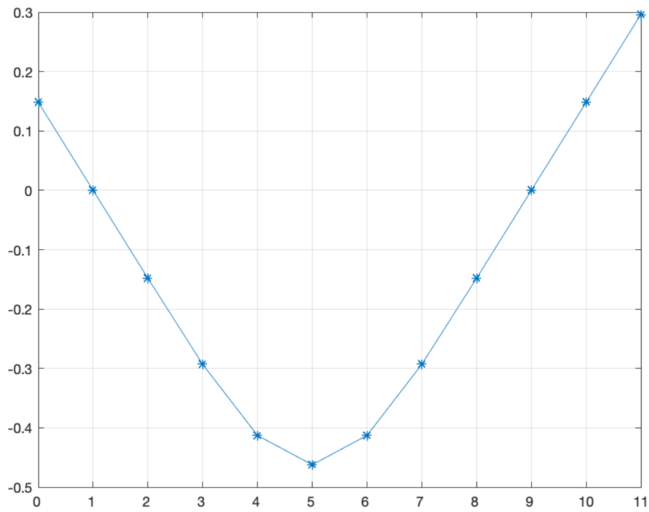

Figure 5.2: Numerical solution u_2 subject to L_2.

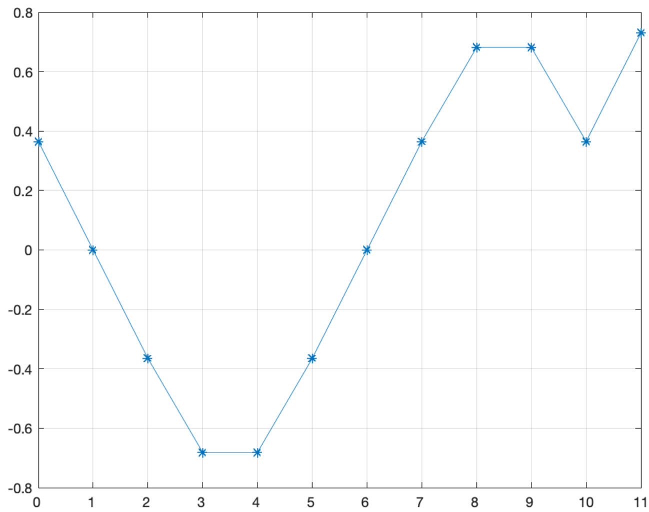

Figure 5.3: Numerical solution u_3 subject to L_3.

Let $\zeta = 1$. By Theorem 4.1, the target solution must satisfy Eq. (5.3)–(5.7) with

$$\tilde{\Phi}(u, \theta, t) := 2(u(t) - 10), \quad t \in [7, 9]_{\mathbb{Z}},$$
$$\tilde{\Phi}(u, \theta, t) := 0, \quad \text{otherwise}.$$

The graph of the numerical solution u_3 subject to L_3 is given in Figure 5.3.

Remark 5.4. Examples 5.1, 5.2, and 5.3 found three different solutions from the same BVP following different criteria. These examples demonstrated the effectiveness of Theorem 4.1. This idea can also be extended to the objective functions of other forms as well as other BVPs with multiple solutions.

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