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Explicit solution and dynamical properties of atmospheric Ekman flows with boundary conditions

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Abstract. In this paper, we study the classical problem of the wind in the steady atmospheric Ekman layer with the constant eddy viscosity. Different from the previous work, we modify the boundary conditions and derive the explicit solution by using the notation of matrix cosine and matrix sine. For the arbitrary height-dependent eddy viscosity, we get the solution of the classical problem with zero velocity and acceleration at the bottom of the layer. In addition, uniqueness is shown and dynamical properties of solution are characterized.

Keywords: Ekman layer, variable eddy viscosity, explicit solutions, existence, dynamical properties.

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1 Introduction

The Earth's atmosphere can be divided into several layers based on the behaviour of its temperature [11], these layers are, starting from ground level upwards, the troposphere, the stratosphere, the mesosphere and the thermosphere, A further region, beginning about 500 km above the ground level, is the exosphere, which fades away into the realm of interplanetary space. The troposphere contains more than 75% of all of the air in the atmosphere, and almost all of the water vapour (which forms clouds and rain). This is the region where the familiar weather phenomena occur. The lowest part-roughly the lower third-of the troposphere is called the atmospheric boundary layer, and it is here that friction plays an important role, while higher up, from the stratosphere upwards, the air flow is practically inviscid.

For a better understanding of the flow dynamics, it is useful to divide the atmospheric boundary layer into there parts [8, 11], i.e., the lamina sublayer, surface (Prandtl) layer and

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the Ekman layer(see Fig. 2.1), the lamina sublayer is only a few millimeters thick and is not relevant to the transfer of wind energy. Within the surface layer, confined to 20–100 meters of the atmosphere (above the lamina sublayer), the velocity profile is adjusted so that the horizontal frictional stress is nearly independent of height. In contrast to this, in the Ekman layer, located on top of the surface layer and extending to a height of about 1 km, on average, the flow is governed by a three-way balance among frictional effects, pressure gradient and the influence of the coriolis force [5,8,21]. Primarily the air flow is horizontal (the horizontal velocities are about 10^4 larger than the vertical velocity [20]).

The governing equations for mesoscale steady air flow at mid-latitudes in the Ekman layer are [8]

$$\begin{cases} f(v-v_g) = -\frac{\partial}{\partial z}(k\frac{\partial u}{\partial z}), \\ f(u-u_g) = \frac{\partial}{\partial z}(k\frac{\partial v}{\partial z}), \end{cases} \tag{1.1}$$

where (u,v) represents the horizontal wind velocity, with zonal (West-to-East, in the sense of the Earth's rotation) component $u = u(t,x,y,z)$ and meridional (positive meaning towards the North Pole) component $v = v(t,x,y,z)$, u_g and v_g are the corresponding geostrophic wind component, k denotes the eddy viscosity, $f = 2\Omega\sin\theta$ is the Coriolis parameter at the fixed latitude θ in the Northern Hemisphere and $\Omega \approx 7.29\times 10^{-5}s^{-1}$ is the angular speed of rotation of the Earth and $\theta \in (0,\pi/2]$ is the angle of latitude in right-handed rotating spherical coordinates ($\theta = 0$ corresponding to the Equator and $\theta = \pi/2$ to the North Pole).

The boundary conditions for the system (1.1) are

$$u = v = 0 \quad \text{at } z = 0, \tag{1.2}$$

and

$$u \to u_g, \qquad v \to v_g \quad \text{for } z \to \infty, \tag{1.3}$$

expressing the fact that, due to the frictional properties of the flow below the Ekman layer, a no-slip condition holds at the bottom $z = 0$ of the layer, while at the top of the Ekman layer the horizontal components of the wind must be in geostrophic balance: above the Ekman layer the flow is geostrophic (pressure-driven).

If k is a constant, then we can obtain the explicit formula of the solution to (1.1) with (1.2) and (1.3) by the classic Ekman theory, but this assumption is too restrictive. The dynamics of the atmospheric boundary-layer is very important in applications, for example, other than meteorology (weather prediction and climate studies), in the control and management of air pollution (since the dispersal of smog in urban environments depends strongly on meteorological conditions) and in agriculture (e.g. dewfall and frost formation). For this reason, it is important, both from the theoretical as well as from the practical point of view, to understand the flow dynamics of the atmospheric boundary-layer in the context of height-dependent eddy viscosities. The available explicit solutions for height-dependent eddy viscosities are very scare, being apparently restricted to special cases, for example, $k(z)$ denote linear and exponentially decaying functions [10,12] or $k(z)$ is a quadratic polynomial [15]. It is remarkable that Constantin and Johnson [2] studied the Atmospheric Ekman flows with variable eddy viscosity $k(z)$ which is a perturbation of the asymptotic and verify the existence of the solution by transforming the Ekman flows into a suitable integral equation and apply iterative technique to give an efficient approach to find the explicit solution, so that for other types of non-constant eddy viscosity we have to rely on case-by-case approximations and numerical simulations [4,6,9,13,14,16].

Remark 1.1. When $z = \pi\sqrt{\frac{2k}{f}}$, the wind (u,v) is parallel to and nearly equally to the geostrophic value (u_g, v_g), it is conventional to designate this level as the top of the Ekman layer [8], so we can change the condition (1.3) to

$$u = u_g, \qquad v = v_g, \quad \text{at } z = z_0, \tag{1.4}$$

where $z_0 > \pi\sqrt{\frac{2k}{f}}$.

For a constant eddy viscosity k, we can obtain the explicit formula of the solution to (1.1) with (1.2) and (1.3). Based on Remark 1.1, we consider (1.1) with (1.2) and (1.4). The first contribution of this paper is to apply the technique of second linear ODEs (using the notion of sin and cos matrix) to find the explicit solution of (1.1) with (1.2) and (1.4) and give a directly approach to compute the explicit solution.

If we assume the velocity and acceleration at the bottom of the layer are zero, then (1.2) is retained and (1.3) is changed into

$$u' = 0, \qquad v' = 0 \quad \text{at } z = 0, \tag{1.5}$$

so the second aim of this paper is to investigate the explicit solution of (1.1) with (1.2) and (1.5) for an arbitrary height-dependent eddy viscosity $k(z)$. We use the closed form of function series to give the representation of solutions. By using integral change and introducing Green function, a spectrum theorem of a corresponding anti-symmetric compact operator is used to deriving the uniqueness result. Finally, some dynamical properties of solution like asymptotic property, Lyapunov exponents, and stable manifold are characterized.

2 Model description

Motivated by [8], we give the details to derive (1.1) by dividing into four steps.

Step 1. We set up the momentum equation in rotating coordinates.

We derive the relationship between the total derivative of a vector in an inertial reference frame and the corresponding total derivative in a rotating system. Let $\overrightarrow{A}$ be an arbitrary vector whose Cartesian components in an inertial frame given by

$$\overrightarrow{A} = \overrightarrow{i'} A'_x + \overrightarrow{j'} A'_y + \overrightarrow{k'} A'_z$$

and whose components in a frame rotating with the angular velocity $\overrightarrow{\Omega}$ are

$$\overrightarrow{A} = \overrightarrow{i} A_x + \overrightarrow{j} A_y + \overrightarrow{k} A_z,$$

here $\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}$ are unit vectors which are taken to be directed eastward, northward, and upward, respectively, $\overrightarrow{\Omega} = (0, \Omega\sin\phi, \Omega\cos\phi)$, ϕ is the latitude.

Letting $\frac{D_\alpha \overrightarrow{A}}{Dt}$ be the total derivative of $\overrightarrow{A}$ in the inertial frame, we can write

$$\begin{aligned}
\frac{D_\alpha \overrightarrow{A}}{Dt} &= \overrightarrow{i'}\frac{DA'_x}{Dt} + \overrightarrow{j'}\frac{DA'_y}{Dt} + \overrightarrow{k'}\frac{DA'_z}{Dt} \\
&= \overrightarrow{i}\frac{Du}{Dt} + \overrightarrow{j}\frac{Dv}{Dt} + \overrightarrow{k}\frac{Dw}{Dt} + \frac{D_\alpha \overrightarrow{i}}{Dt}u + \frac{D_\alpha \overrightarrow{j}}{Dt}v + \frac{D_\alpha \overrightarrow{k}}{Dt}w,
\end{aligned}$$

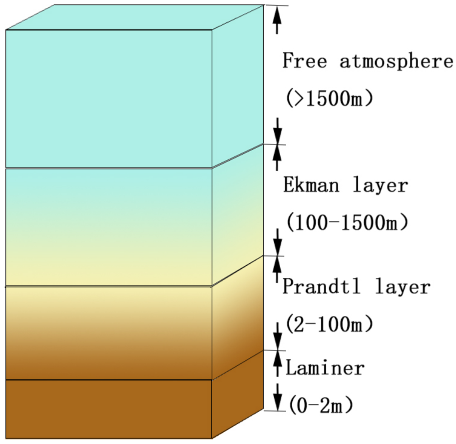

Figure 2.1: Ekman layer, surface layer and lamina sublayer are called the atmosphere boundary layer.

the first three terms on the left line above can be combined to give

$$\frac{D\overrightarrow{A}}{Dt} = \overrightarrow{i}\frac{DA_x}{Dt} + \overrightarrow{j}\frac{DA_y}{Dt} + \overrightarrow{k}\frac{DA_z}{Dt},$$

which is just the total derivative of $\overrightarrow{A}$ as viewed in the rotating coordinates. By direct calculation [8], we get

$$\frac{D_\alpha \overrightarrow{i}}{Dt} = \overrightarrow{\Omega} \times \overrightarrow{i}, \qquad \frac{D_\alpha \overrightarrow{j}}{Dt} = \overrightarrow{\Omega} \times \overrightarrow{j}, \qquad \frac{D_\alpha \overrightarrow{k}}{Dt} = \overrightarrow{\Omega} \times \overrightarrow{k},$$

there, the total derivative for $\overrightarrow{A}$ in an inertial frame is related to that in a rotating frame by

$$\frac{D_\alpha \overrightarrow{A}}{Dt} = \frac{D\overrightarrow{A}}{Dt} + \overrightarrow{\Omega} \times \overrightarrow{A}. \tag{2.1}$$

For a given air parcel the location (x, y, z) is a given function of t so that $x = x(t), y = y(t), z = z(t)$, let $\frac{Dx}{Dt} = u, \frac{Dy}{Dt} = v, \frac{Dz}{Dt} = w$, then u, v, w are the velocity components in the x, y, z directions, respectively, let $\overrightarrow{U}$ is the velocity vector , then $\overrightarrow{U} = \overrightarrow{i}u + \overrightarrow{j}v + \overrightarrow{k}w$.

In an inertial reference frame, Newton's second law of motion may be written as

$$\sum \overrightarrow{F} = \frac{D_\alpha \overrightarrow{U_\alpha}}{Dt},$$

here $\frac{D_\alpha \overrightarrow{U}_\alpha}{Dt}$ is the rate of change of the absolute velocity U_α. On the rotating Earth, if $\overrightarrow{r}$ is a position vector for an air parcel, from the (2.1), we get

$$\frac{D_\alpha \overrightarrow{r}}{Dt} = \frac{D\overrightarrow{r}}{Dt} + \overrightarrow{\Omega} \times \overrightarrow{r},$$

but $\frac{D_\alpha \vec{r}}{Dt} = \vec{U}_\alpha$, $\frac{D\vec{r}}{Dt} = \vec{U}$, so we obtain

$$\vec{U}_\alpha = \vec{U} + \vec{\Omega} \times \vec{r}. \tag{2.2}$$

We apply (2.1) to $\vec{U}_\alpha$ and obtain

$$\frac{D_\alpha \vec{U}_\alpha}{Dt} = \frac{D\vec{U}_\alpha}{Dt} + \vec{\Omega} \times \vec{U}_\alpha.$$

Using (2.2), we get

$$\begin{aligned} \frac{D_\alpha \vec{U}_\alpha}{Dt} &= \frac{D\vec{U}_\alpha}{Dt} + \vec{\Omega} \times \vec{U}_\alpha \\ &= \frac{D}{Dt}(\vec{U} + \vec{\Omega} \times \vec{r}) + \vec{\Omega} \times (\vec{U} + \vec{\Omega} \times \vec{r}) \\ &= \frac{D\vec{U}}{Dt} + 2\vec{\Omega} \times \vec{U} - \Omega^2 \vec{R}, \end{aligned}$$

here $\vec{R}$ is a vector with direction perpendicular to the axis of rotation, and the magnitude equal to the distance to the axis of rotation.

If we assume that the only real forces acting on the atmosphere are the pressure gradient force $\vec{F_p}$, gravitation force $\vec{F_g}$ and friction force $\vec{F_r}$, then we have

$$\frac{D\vec{U}}{Dt} = \vec{F_g} + \vec{F_p} + \vec{F_r},$$

so we get

$$\frac{D\vec{U}}{Dt} = -2\vec{\Omega} \times \vec{U} + \Omega^2 \vec{R} + \vec{F_g} + \vec{F_p} + \vec{F_r}. \tag{2.3}$$

Step 2. We set up the component equations in spherical coordinates.

Let (λ, ϕ, z) be the spherical coordinates, λ is longitude, ϕ is latitude, and z is the vertical distance above the surface of the Earth, using the formula for the transformation of local rectangular coordinate system and spherical coordinate system, we can get the following relationships,

$$dx = a\cos\phi d\lambda, \qquad dy = ad\phi, \qquad dz = dr,$$

where a is the radius of the Earth, r is the distance to the center of the Earth, which is related to z by $r = a + z$.

The direction of the $\vec{i}$, $\vec{j}$, $\vec{k}$ unit vectors are not constant, they are the functions of position on the spherical Earth, thus we write

$$\frac{D\vec{U}}{Dt} = \vec{i}\frac{Du}{Dt} + \vec{j}\frac{Dv}{Dt} + \vec{k}\frac{Dw}{Dt} + u\frac{D\vec{i}}{Dt} + v\frac{D\vec{j}}{Dt} + w\frac{D\vec{k}}{Dt}, \tag{2.4}$$

from [8], we get

$$\frac{D\vec{i}}{Dt} = \frac{u}{a\cos\phi}\left(\vec{j}\sin\phi - \vec{k}\cos\phi\right), \qquad \frac{D\vec{j}}{Dt} = -\frac{u\tan\phi}{a}\vec{i} - \frac{v}{a}\vec{k}, \tag{2.5}$$

and

$$\frac{D\overrightarrow{k}}{Dt}=\frac{u}{a}\overrightarrow{i}+\frac{v}{a}\overrightarrow{j}, \tag{2.6}$$

substituting (2.5) and (2.6) into (2.4) and rearranging the terms, we obtain

$$\begin{aligned}\frac{D\overrightarrow{U}}{Dt}=&\left(\frac{Du}{Dt}-\frac{uv\tan\phi}{a}+\frac{uw}{a}\right)\overrightarrow{i}+\left(\frac{Dv}{Dt}+\frac{u^2\tan\phi}{a}+\frac{vw}{a}\right)\overrightarrow{j}\\&+\left(\frac{Dw}{Dt}-\frac{u^2+v^2}{a}\right)\overrightarrow{k}.\end{aligned} \tag{2.7}$$

We know that

$$\Omega^2\overrightarrow{R}+\overrightarrow{F_g}=\overrightarrow{g}, \tag{2.8}$$

and

$$\begin{aligned}-2\overrightarrow{\Omega}\times\overrightarrow{U}&=-2\Omega\begin{bmatrix}\overrightarrow{i}&\overrightarrow{j}&\overrightarrow{k}\\0&\cos\phi&\sin\phi\\u&v&w\end{bmatrix}\\&=-(2\Omega w\cos\phi-2\Omega v\sin\phi)\overrightarrow{i}-2\Omega u\sin\phi\overrightarrow{j}+2\Omega u\cos\phi\overrightarrow{k}.\end{aligned} \tag{2.9}$$

We consider an infinitesimal volume element of air, $\delta V=\delta x\delta y\delta z$, center at the point (x_0,y_0,z_0) (see Fig. 2.2), so we can easily get the total pressure gradient force per unit mass is

$$\overrightarrow{F_p}=\frac{1}{\rho}\nabla\overrightarrow{p}=\overrightarrow{i}\frac{1}{\rho}\frac{\partial p}{\partial x}+\overrightarrow{j}\frac{1}{\rho}\frac{\partial P}{\partial y}+\overrightarrow{k}\frac{1}{\rho}\frac{\partial P}{\partial z}, \tag{2.10}$$

we know that

$$\overrightarrow{g}=-\overrightarrow{k}g, \tag{2.11}$$

and

$$\overrightarrow{F_r}=\overrightarrow{i}F_{rx}+\overrightarrow{j}F_{ry}+\overrightarrow{k}F_{rz}, \tag{2.12}$$

where

$$\begin{cases}F_{rx}=v[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}],\\F_{ry}=v[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}],\\F_{rz}=v[\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}],\end{cases}$$

$v=\frac{\mu}{\rho}$ is the kinematic viscosity coefficient [8].

From (2.3) and using (2.7), (2.8), (2.9), (2.10), (2.11), (2.12), we get the following equations

$$\begin{cases}\frac{Du}{Dt}=-\frac{1}{\rho}\frac{1}{\cos\phi}\frac{\partial P}{\partial\lambda}+2\Omega v\sin\phi-2\Omega w\cos\phi+\frac{uv\tan\phi}{a}-\frac{uw}{a}+F_{rx},\\\frac{Dv}{Dt}=-\frac{1}{\rho}\frac{1}{a}\frac{\partial P}{\partial\phi}-2\Omega u\sin\phi-\frac{u^2\tan\phi}{a}-\frac{vw}{a}+F_{ry},\\\frac{Dw}{Dt}=-\frac{1}{\rho}\frac{\partial P}{\partial z}-g-2\Omega u\cos\phi+\frac{u^2+v^2}{a}-\frac{uw}{a}+F_{rz}.\end{cases} \tag{2.13}$$

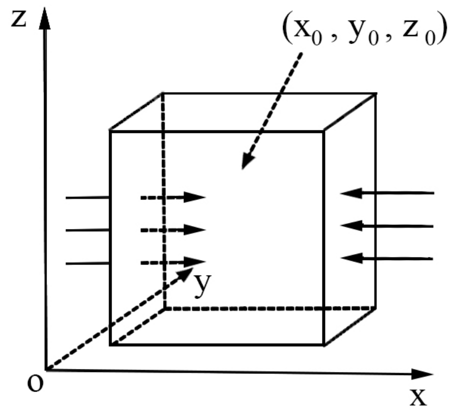

Figure 2.2: The x component of the pressure gradient forcer.

Step 3. We simplify (2.13) in local rectangular coordinates system.

The table 2.1 in [8] shows the terms proportional to $\frac{1}{a}$ on the above equations are unimportant for midlatitude synoptic scale motions, so we omit this terms and get

$$\begin{cases} \frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + 2\Omega v \sin\phi - 2\Omega w \cos\phi + F_{rx}, \\ \frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial y} - 2\Omega u \sin\phi + F_{ry}, \\ \frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial z} - g - 2\Omega u \cos\phi + F_{rz}. \end{cases}$$

As $u = u(t, x, y, z)$, and $\frac{Dx}{Dt} = u$, $\frac{Dy}{Dt} = v$, $\frac{Dz}{Dt} = w$, we get

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z},$$

$\frac{Dv}{Dt}$ and $\frac{Dw}{Dt}$ are similar.

For a wide range of air movements, $w \ll u, v$ [21], so we assume $w = 0$, for the atmosphere below 100km, kinematic viscosity coefficient is negligible except in a thin layer within a few centimeters of the Earth's surface where the vertical shear is very large [8], so $F_{rx} = 0$, $F_{ry} = 0$ in Ekman layer, as shown in chapter 3 in [8], the magnitude of w can be deduced from knowledge of the horizontal velocity u, v, so we omit the last equation of the system and get

$$\begin{cases} \frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + 2\Omega v \sin\phi = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv, \\ \frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial y} - 2\Omega u \sin\phi = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu. \end{cases}$$

Step 4. We set up the mean equations.

In a turbulent fluid, a field variable such as velocity measured at a point generally fluctuates rapidly in time as eddies of various scales pass the point, so we assume that the field variables can be separated into slowing varying turbulent components, for example, $u = \overline{u} + u'$, the corresponding means are indicated by overbars and the fluctuating component by primes. With the aid of the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

and the chain rule of the differentiation, we get

$$\begin{aligned} \frac{Du}{Dt} &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + u(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) \\ &= \frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z}. \end{aligned} \tag{2.14}$$

Separating each dependant variable into mean and fluctuating parts, substituting into (2.14), and averaging then yields

$$\overline{\frac{Du}{Dt}} = \frac{\partial \overline{u}}{\partial t} + \frac{\partial}{\partial x}(\overline{u}\,\overline{u} + \overline{u'u'}) + \frac{\partial}{\partial y}(\overline{u}\,\overline{v} + \overline{u'v'}) + \frac{\partial}{\partial z}(\overline{u}\,\overline{w} + \overline{u'w'}).$$

Noting that the mean velocity fields satisfy the continuity equation, we get

$$\overline{\frac{Du}{Dt}} = \frac{\overline{D}\,\overline{u}}{Dt} + \frac{\partial}{\partial x}(\overline{u'u'}) + \frac{\partial}{\partial y}(\overline{u'v'}) + \frac{\partial}{\partial z}(\overline{u'w'}),$$

where

$$\frac{\overline{D}}{Dt} = \frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x} + \overline{v}\frac{\partial}{\partial y} + \overline{w}\frac{\partial}{\partial z}$$

is the rate of change following the mean motion, the mean equations thus have the following form,

$$\begin{cases} \frac{\overline{D}\overline{u}}{Dt} = -\frac{1}{\rho}\frac{\partial \overline{P}}{\partial x} + f\overline{v} - [\frac{\partial \overline{u'u'}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y} + \frac{\partial \overline{u'w'}}{\partial z}], \\ \frac{\overline{D}\overline{v}}{Dt} = -\frac{1}{\rho}\frac{\partial \overline{P}}{\partial y} - f\overline{u} - [\frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{v'v'}}{\partial y} + \frac{\partial \overline{v'w'}}{\partial z}]. \end{cases}$$

Away from region with horizontal inhomogeneities (e.g., shorelines terms, forest edges), we can assume turbulent fluxes are horizontally homogeneous because they are too small in comparison to the term involving vertical differentiation [8], so we assume $\frac{\partial \overline{u'u'}}{\partial x} = \frac{\partial \overline{u'v'}}{\partial y} = \frac{\partial \overline{u'v'}}{\partial x} = \frac{\partial \overline{v'v'}}{\partial y} = 0$.

Outside the boundary layer, the resulting approximation was geostrophic balance, i.e.,

$$\begin{cases} \frac{1}{\rho}\frac{\partial \overline{P}}{\partial x} = f\overline{v_g}, \\ \frac{1}{\rho}\frac{\partial \overline{P}}{\partial y} = -f\overline{u_g}. \end{cases}$$

For midlatitude synoptic-scale motions, the inertial acceleration terms (the terms on the left of above equations) can be neglected compared to the Cariolis force and pressure gradient force terms [8], so we get

$$\begin{cases} f(\overline{v} - \overline{v_g}) - \frac{\partial \overline{u'w'}}{\partial z} = 0, \\ -f(\overline{u} - \overline{u_g}) - \frac{\partial \overline{v'w'}}{\partial z} = 0. \end{cases}$$

By the Flux-Gradient theory, we get

$$\begin{cases} \overline{u'w'} = -k(\frac{\partial \overline{u}}{\partial z}), \\ \overline{v'w'} = -k(\frac{\partial \overline{v}}{\partial z}), \end{cases}$$

where $k(m^2 s^{-1})$ is the eddy viscosity coefficient, then we have

$$\begin{cases} f(\overline{v} - \overline{v_g}) = -\frac{\partial}{\partial z}(k\frac{\partial \overline{u}}{\partial z}), \\ f(\overline{u} - \overline{u_g}) = \frac{\partial}{\partial z}(k\frac{\partial \overline{v}}{\partial z}). \end{cases}$$

Finally, we omit the overbars for simplicity to obtain (1.1).

3 Main results

3.1 Existence of explicit solution

Note that if k reduces to a constant, then (1.1) reduces to

$$\begin{cases} \frac{d^2 v}{dz^2} = \frac{f}{k}(u - u_g), \\ \frac{d^2 u}{dz^2} = -\frac{f}{k}(v - v_g). \end{cases} \tag{3.1}$$

Based on Remark 1.1, we change the condition (1.3) to (1.4) in the following theorems, and we try to find explicit solution of (3.1) with (1.2) and (1.4) by using the notion of sin and cos matrices.

Definition 3.1 ((see [7]). It is well known that

$$\sin \Omega z = \Omega \frac{z}{1!} - \Omega^3 \frac{z^3}{3!} + \cdots + (-1)^k \Omega^{2k+1} \frac{z^{2k+1}}{(2k+1)!} + \cdots,$$
$$\cos \Omega z = I - \Omega^2 \frac{z^2}{2!} + \cdots + (-1)^k \Omega^{2k} \frac{z^{2k}}{(2k)!} + \cdots.$$

Theorem 3.2. *The solution of* (3.1) *with* (1.2) *and* (1.4) *can be expressed by the following formula*

$$\begin{bmatrix} v \\ u \end{bmatrix} = \cos \Omega z \begin{bmatrix} -v_g \\ -u_g \end{bmatrix} + \sin \Omega z \begin{bmatrix} C_{21} \\ C_{22} \end{bmatrix} + \begin{bmatrix} v_g \\ u_g \end{bmatrix}, \tag{3.2}$$

where

$$\Omega = \begin{bmatrix} \sqrt{\frac{f}{2k}} & -\sqrt{\frac{f}{2k}} \\ \sqrt{\frac{f}{2k}} & \sqrt{\frac{f}{2k}} \end{bmatrix},$$

and

$$\begin{bmatrix} C_{21} \\ C_{22} \end{bmatrix} = (\sin \Omega z_0)^{-1} \cos \Omega z_0 \begin{bmatrix} v_g \\ u_g \end{bmatrix}.$$

Remark 3.3. Note that $(\sin\Omega z_0)^{-1}$ does exist because z_0 is a positive number, so using *Wolfram Mathematica*, $(\sin\Omega z_0)^{-1}\cos\Omega z_0$ can be solved by the following computations:

$$\sin\Omega z_0 = \begin{pmatrix} \sin\sqrt{\frac{f}{2k}}z_0\cosh\sqrt{\frac{f}{2k}}z_0 & -\cos\sqrt{\frac{f}{2k}}z_0\sinh\sqrt{\frac{f}{2k}}z_0 \\ \cos\sqrt{\frac{f}{2k}}z_0\sinh\sqrt{\frac{f}{2k}}z_0 & \sin\sqrt{\frac{f}{2k}}z_0\cosh\sqrt{\frac{f}{2k}}z_0 \end{pmatrix},$$

$$\det\sin\Omega z_0 = \frac{1}{2}\left(\cosh\left(2\sqrt{\frac{f}{2k}}z_0\right) - \cos\left(2\sqrt{\frac{f}{2k}}z_0\right)\right) > 0,$$

$$(\sin\Omega z_0)^{-1} = \begin{pmatrix} \frac{2\sin\sqrt{\frac{f}{2k}}z_0\cosh\sqrt{\frac{f}{2k}}z_0}{\cosh(2\sqrt{\frac{f}{2k}}z_0)-\cos(2\sqrt{\frac{f}{2k}}z_0)} & \frac{2\cos(\sqrt{\frac{f}{2k}}z_0)\sinh\sqrt{\frac{f}{2k}}z_0}{\cosh(2\sqrt{\frac{f}{2k}}z_0)-\cos(2\sqrt{\frac{f}{2k}}z_0)} \\ \frac{2\cos\sqrt{\frac{f}{2k}}z_0\sinh\sqrt{\frac{f}{2k}}z_0}{\cos(2\sqrt{\frac{f}{2k}}z_0)-\cosh(2\sqrt{\frac{f}{2k}}z_0)} & \frac{2\sin\sqrt{\frac{f}{2k}}z_0\cosh\sqrt{\frac{f}{2k}}z_0}{\cosh(2\sqrt{\frac{f}{2k}}z_0)-\cos(2\sqrt{\frac{f}{2k}}z_0)} \end{pmatrix},$$

and

$$\cos\Omega z_0 = \begin{pmatrix} \cos\sqrt{\frac{f}{2k}}z_0\cosh\sqrt{\frac{f}{2k}}z_0 & \sin\sqrt{\frac{f}{2k}}z_0\sinh\sqrt{\frac{f}{2k}}z_0 \\ -\sin\sqrt{\frac{f}{2k}}z_0\sinh\sqrt{\frac{f}{2k}}z_0 & \cos\sqrt{\frac{f}{2k}}z_0\cosh\sqrt{\frac{f}{2k}}z_0 \end{pmatrix},$$

$$\det\cos\Omega z_0 = \frac{1}{2}\left(\cosh\left(2\sqrt{\frac{f}{2k}}z_0\right) + \cos\left(2\sqrt{\frac{f}{2k}}z_0\right)\right) > 0,$$

$$(\sin\Omega z_0)^{-1}\cos\Omega z_0 = \begin{pmatrix} -\frac{\sin(2\sqrt{\frac{f}{2k}}z_0)}{\cos(2\sqrt{\frac{f}{2k}}z_0)-\cosh(2\sqrt{\frac{f}{2k}}z_0)} & -\frac{\sinh(2\sqrt{\frac{f}{2k}}z_0)}{\cos(2\sqrt{\frac{f}{2k}}z_0)-\cosh(2\sqrt{\frac{f}{2k}}z_0)} \\ \frac{\sinh(2\sqrt{\frac{f}{2k}}z_0)}{\cos(2k)-\cosh(2\sqrt{\frac{f}{2k}}z_0)} & -\frac{\sin(2\sqrt{\frac{f}{2k}}z_0)}{\cos(2\sqrt{\frac{f}{2k}}z_0)-\cosh(2\sqrt{\frac{f}{2k}}z_0)} \end{pmatrix},$$

and

$$\det\left((\sin\Omega z_0)^{-1}\cos\Omega z_0\right) = \frac{\cos\left(2\sqrt{\frac{f}{2k}}z_0\right) + \cosh\left(2\sqrt{\frac{f}{2k}}z_0\right)}{\cosh\left(2\sqrt{\frac{f}{2k}}z_0\right) - \cos\left(2\sqrt{\frac{f}{2k}}z_0\right)} > 0.$$

Proof. Let $U = u - u_g$, $V = v - v_g$ and $\bar{k} = \frac{f}{k}$. Then (3.1) becomes

$$\begin{cases} \frac{d^2V}{dz^2} = \bar{k}U, \\ \frac{d^2U}{dz^2} = -\bar{k}V, \end{cases} \tag{3.3}$$

and the conditions (1.2), (1.4) are transformed into the equivalent forms

$$U = -u_g, \quad V = -v_g \quad \text{at } z = 0, \tag{3.4}$$

$$U = 0, \quad V = 0 \quad \text{at } z = z_0. \tag{3.5}$$

From the (3.3), we get

$$\begin{bmatrix} V \\ U \end{bmatrix}'' + \begin{bmatrix} 0 & -\bar{k} \\ \bar{k} & 0 \end{bmatrix}\begin{bmatrix} V \\ U \end{bmatrix} = 0.$$

By using the matrix Ω, we obtain

$$\begin{bmatrix} V \\ U \end{bmatrix}'' + \Omega^2\begin{bmatrix} V \\ U \end{bmatrix} = 0. \tag{3.6}$$

So we get the solution of the (3.6) as following form,

$$\begin{bmatrix} V \\ U \end{bmatrix} = \cos \Omega z \begin{bmatrix} C_{11} \\ C_{12} \end{bmatrix} + \sin \Omega z \begin{bmatrix} C_{21} \\ C_{22} \end{bmatrix}.$$

We determine the constants such that the initial conditions (3.4) and (3.5) are satisfied. Considering the condition (3.4), we get

$$C_{11} = -v_g, \qquad C_{12} = -u_g.$$

Considering the condition (3.5), we obtain

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \cos \Omega z_0 \begin{bmatrix} -v_g \\ -u_g \end{bmatrix} + \sin \Omega z_0 \begin{bmatrix} C_{21} \\ C_{22} \end{bmatrix}.$$

Because the matrix $\sin \Omega z_0$ is nonsingular, so we get

$$\begin{bmatrix} C_{21} \\ C_{22} \end{bmatrix} = (\sin \Omega z_0)^{-1} \cos \Omega z_0 \begin{bmatrix} v_g \\ u_g \end{bmatrix}.$$

As $U = u - u_g$, $V = v - v_g$, so we obtain (3.2). □

We recall the following result.

Lemma 3.4 (see [1,18]). *For the matrix equation*

$$\Phi'(t, t_0) = A(t)\Phi(t, t_0), \qquad t \in [t_0, t_\alpha]$$

with the initial boundary condition $\Phi(t_0, t_0) = I$*, where the matrix* $\Phi(t, t_0)$ *and* $A(t)$ *are* $n \times n$ *matrices,* $t_\alpha > t_0 \geq 0$*, the solution* $\Phi(t, t_0)$ *is given by*

$$\Phi(t, t_0) = I + \int_{t_0}^{t} A(\tau)d\tau + \int_{t_0}^{t} A(\tau_1) \left[\int_{t_0}^{\tau_1} A(\tau_2)d\tau_2 \right] d\tau_1 + \int_{t_0}^{t} A(\tau_1) \int_{t_0}^{\tau_1} A(\tau_2) \int_{t_0}^{\tau_2} A(\tau_3)d\tau_3 d\tau_2 d\tau_1 + \cdots$$

For (1.1), we assume the $k = k(z) \neq 0$, then we will get

$$\begin{cases} \frac{d^2 v}{dz^2} + \frac{k'(z)}{k(z)} \frac{dv}{dz} = \frac{f}{k(z)}(u - u_g), \\ \frac{d^2 u}{dz^2} + \frac{k'(z)}{k(z)} \frac{du}{dz} = -\frac{f}{k(z)}(v - v_g). \end{cases}$$

Let $u - u_g = U$, $v - v_g = V$, then we will get

$$\begin{cases} \frac{d^2 V}{dz^2} + \alpha(z)\frac{dV}{dz} = \beta(z)U, \\ \frac{d^2 U}{dz^2} + \alpha(z)\frac{dU}{dz} = -\beta(z)V, \end{cases}$$

where $\alpha(z) = \frac{k'(z)}{k(z)}$, $\beta(z) = \frac{f}{k(z)}$, and the conditions (1.2) and (1.5) will become

$$U(0) = -u_g, \qquad V(0) = -v_g, \tag{3.7}$$

and

$$U'(0) = 0, \qquad V'(0) = 0. \tag{3.8}$$

Let $V'(z) = w_1,\ U'(z) = w_2$, then we obtain

$$X'(z) = A(z)X(z), \qquad X(0) = X_0, \tag{3.9}$$

where

$$X = \begin{bmatrix} V \\ U \\ W_1 \\ W_2 \end{bmatrix}, \qquad X_0 = \begin{bmatrix} -v_g \\ -u_g \\ 0 \\ 0 \end{bmatrix},$$

and

$$A(z) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \beta(z) & -\alpha(z) & 0 \\ -\beta(z) & 0 & 0 & -\alpha(z) \end{bmatrix}.$$

We get the solution of (3.9) by using Lemma 3.4, that is

$$X(z) = \Phi(z, z_0)X_0,$$

where

$$\Phi(z, z_0) = I + \int_0^z A(\tau)d\tau + \int_0^z A(\tau_1)\left[\int_0^{\tau_1} A(\tau_2)d\tau_2\right] d\tau_1 + \cdots,$$

as $u - u_g = U, v - v_g = V$, so we get the solution of (1.1) with the conditions (1.2) and (1.5).

If $k(z)$ is constant, then we will solve (3.9) with the conditions (1.2) and (1.5).

Remark 3.5. If $k(z)$ is a constant k, then $\alpha(z) = 0$, $\beta(z) = \frac{f}{k}$, and (1.1) will become the following form,

$$X'(z) = AX(z), \tag{3.10}$$

the corresponding initial conditions are

$$X(0) = X_0,$$

where

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \beta & 0 & 0 \\ -\beta & 0 & 0 & 0 \end{bmatrix}. \tag{3.11}$$

The characteristic equation of (3.11) is

$$\lambda^4 + \beta^2 = 0,$$

so we get the four eigenvalues:

$$\lambda_1 = \sqrt{\frac{f}{2k}} + \sqrt{\frac{f}{2k}}i, \quad \lambda_2 = -\sqrt{\frac{f}{2k}} - \sqrt{\frac{f}{2k}}i, \quad \lambda_3 = \sqrt{\frac{f}{2k}} - \sqrt{\frac{f}{2k}}i, \quad \lambda_4 = -\sqrt{\frac{f}{2k}} + \sqrt{\frac{f}{2k}}i.$$

Let $\lambda = \lambda_1$, then we have

$$(A - \lambda_1 I) = \begin{bmatrix} -\lambda_1 & 0 & 1 & 0 \\ 0 & -\lambda_1 & 0 & 1 \\ 0 & \beta & -\lambda_1 & 0 \\ -\beta & 0 & 0 & -\lambda_1 \end{bmatrix},$$

so the corresponding eigenvector is

$$\xi_1 = \begin{bmatrix} 1 \\ i \\ \sqrt{\frac{f}{2k}} + \sqrt{\frac{f}{2k}} i \\ -\sqrt{\frac{f}{2k}} + \sqrt{\frac{f}{2k}} i \end{bmatrix},$$

thus we obtain

$$e^{\lambda_1 z}\xi_1 = e^{\sqrt{\frac{f}{2k}}z} \begin{bmatrix} \cos\sqrt{\frac{f}{2k}}z + i\sin\sqrt{\frac{f}{2k}}z \\ -\sin\sqrt{\frac{f}{2k}}z + i\cos\sqrt{\frac{f}{2k}}z \\ \sqrt{\frac{f}{2k}}\left(\cos\sqrt{\frac{f}{2k}}z - \sin\sqrt{\frac{f}{2k}}z\right) + \sqrt{\frac{f}{2k}}\left(\cos\sqrt{\frac{f}{2k}}z + \sin\sqrt{\frac{f}{2k}}z\right)i \\ -\sqrt{\frac{f}{2k}}\left(\cos\sqrt{\frac{f}{2k}}z + \sin\sqrt{\frac{f}{2k}}z\right) + \sqrt{\frac{f}{2k}}\left(\cos\sqrt{\frac{f}{2k}}z - \sin\sqrt{\frac{f}{2k}}z\right)i \end{bmatrix}.$$

The two linear independent solutions are obtained:

$$X_1(z) = e^{\sqrt{\frac{f}{2k}}z} \begin{bmatrix} \cos\sqrt{\frac{f}{2k}}z \\ -\sin\sqrt{\frac{f}{2k}}z \\ \sqrt{\frac{f}{2k}}\left(\cos\sqrt{\frac{f}{2k}}z - \sin\sqrt{\frac{f}{2k}}z\right) \\ -\sqrt{\frac{f}{2k}}\left(\cos\sqrt{\frac{f}{2k}}z + \sin\sqrt{\frac{f}{2k}}z\right) \end{bmatrix},$$

and

$$X_2(z) = e^{\sqrt{\frac{f}{2k}}z} \begin{bmatrix} \sin\sqrt{\frac{f}{2k}}z \\ \cos\sqrt{\frac{f}{2k}}z \\ \sqrt{\frac{f}{2k}}\left(\cos\sqrt{\frac{f}{2k}}z + \sin\sqrt{\frac{f}{2k}}z\right) \\ \sqrt{\frac{f}{2k}}\left(\cos\sqrt{\frac{f}{2k}}z - \sin\sqrt{\frac{f}{2k}}z\right) \end{bmatrix}.$$

Similarly, let $\lambda_3 = -\sqrt{\frac{f}{2k}} + \sqrt{\frac{f}{2k}}i$, we will get the eigenvector

$$\xi_2 = \begin{bmatrix} 1 \\ -i \\ -\sqrt{\frac{f}{2k}} + \sqrt{\frac{f}{2k}} i \\ \sqrt{\frac{f}{2k}} + \sqrt{\frac{f}{2k}} i \end{bmatrix},$$

therefore we have

$$e^{\lambda_2 z}\xi_2 = e^{-\sqrt{\frac{f}{2k}}z}\begin{bmatrix} \cos\sqrt{\frac{f}{2k}}z + i\sin\sqrt{\frac{f}{2k}}z \\ \sin\sqrt{\frac{f}{2k}}z - i\cos\sqrt{\frac{f}{2k}}z \\ -\sqrt{\frac{f}{2k}}\left(\cos\sqrt{\frac{f}{2k}}z + \sin\sqrt{\frac{f}{2k}}z\right) + \sqrt{\frac{f}{2k}}\left(\cos\sqrt{\frac{f}{2k}}z - \sin\sqrt{\frac{f}{2k}}z\right)i \\ \sqrt{\frac{f}{2k}}\left(\cos\sqrt{\frac{f}{2k}}z - \sin\sqrt{\frac{f}{2k}}z\right) + \sqrt{\frac{f}{2k}}\left(\cos\sqrt{\frac{f}{2k}}z + \sin\sqrt{\frac{f}{2k}}z\right)i \end{bmatrix}.$$

The two linear independent solutions can be stated as follows,

$$X_3(z) = e^{-\sqrt{\frac{f}{2k}}z}\begin{bmatrix} \cos\sqrt{\frac{f}{2k}}z \\ \sin\sqrt{\frac{f}{2k}}z \\ -\sqrt{\frac{f}{2k}}\left(\cos\sqrt{\frac{f}{2k}}z + \sin\sqrt{\frac{f}{2k}}z\right) \\ \sqrt{\frac{f}{2k}}\left(\cos\sqrt{\frac{f}{2k}}z - \sin\sqrt{\frac{f}{2k}}z\right) \end{bmatrix},$$

and

$$X_4(z) = e^{-\sqrt{\frac{f}{2k}}z}\begin{bmatrix} \sin\sqrt{\frac{f}{2k}}z \\ -\cos\sqrt{\frac{f}{2k}}z \\ \sqrt{\frac{f}{2k}}\left(\cos\sqrt{\frac{f}{2k}}z - \sin\sqrt{\frac{f}{2k}}z\right) \\ \sqrt{\frac{f}{2k}}\left(\cos\sqrt{\frac{f}{2k}}z + \sin\sqrt{\frac{f}{2k}}z\right) \end{bmatrix}.$$

So the general solution of (3.9) is

$$X(z) = c_1X_1(z) + c_2X_2(z) + c_3X_3(z) + c_4X_4(z),$$

then

$$V = c_1e^{\sqrt{\frac{f}{2k}}z}\cos\sqrt{\frac{f}{2k}}z + c_2e^{\sqrt{\frac{f}{2k}}z}\sin\sqrt{\frac{f}{2k}}z + c_3e^{-\sqrt{\frac{f}{2k}}z}\cos\sqrt{\frac{f}{2k}}z + c_4e^{-\sqrt{\frac{f}{2k}}z}\sin\sqrt{\frac{f}{2k}}z, \tag{3.12}$$

and

$$\begin{aligned} U = &- c_1e^{\sqrt{\frac{f}{2k}}z}\sin\sqrt{\frac{f}{2k}}z + c_2e^{\sqrt{\frac{f}{2k}}z}\cos\sqrt{\frac{f}{2k}}z \\ &+ c_3e^{-\sqrt{\frac{f}{2k}}z}\sin\sqrt{\frac{f}{2k}}z - c_4e^{-\sqrt{\frac{f}{2k}}z}\cos\sqrt{\frac{f}{2k}}z. \end{aligned} \tag{3.13}$$

By using the conditions (3.7), (3.8), we get $c_1 = c_3 = -\frac{1}{2}v_g$, $c_2 = -\frac{1}{2}u_g$, $c_4 = \frac{1}{2}u_g$, so the solution of (3.10) with the conditions (3.7), (3.8) is obtained.

Remark 3.6. From the above example, we know that the general solution of (3.10) is (3.12), (3.13), so if we use the traditional boundary conditions (1.2), (1.3), then we will get

$$c_1 = c_2 = 0, \qquad c_3 = -v_g, \qquad c_4 = u_g,$$

then we have

$$\begin{cases} V = e^{-\sqrt{\frac{f}{2k}}z}\left(u_g \sin\sqrt{\frac{f}{2k}}z\right) - v_g \cos\sqrt{\frac{f}{2k}}z, \\ U = e^{-\sqrt{\frac{f}{2k}}z}\left(-v_g \sin\sqrt{\frac{f}{2k}}z\right) - u_g \cos\sqrt{\frac{f}{2k}}z, \end{cases}$$

so the solution is

$$\begin{cases} v = v_g + e^{-\sqrt{\frac{f}{2k}}z}\left(u_g \sin\sqrt{\frac{f}{2k}}z\right) - v_g \cos\sqrt{\frac{f}{2k}}z, \\ u = u_g + e^{-\sqrt{\frac{f}{2k}}z}\left(-v_g \sin\sqrt{\frac{f}{2k}}z\right) - u_g \cos\sqrt{\frac{f}{2k}}z, \end{cases}$$

this coincides with the result in [8].

3.2 Uniqueness

For the constant k, the explicit solution of (1.1) with (1.2) and (1.4) is obtained by Theorem 3.2, in the following theorem, we try to find the uniqueness for $k(z)$.

Theorem 3.7. *Assume $f \neq 0$, then there is a unique solution of (1.1) with conditions (1.2) and (1.4).*

Proof. Let $\hat{u}$ and $\hat{v}$ be the solutions of (1.1) for $f = 0$ with (1.2) and (1.4). Then

$$\hat{u}(z) = \frac{l(z)}{l(z_0)}u_g, \qquad \hat{v}(z) = \frac{l(z)}{l(z_0)}v_g$$

for $l(z) = \int_0^z \frac{ds}{k(s)}$. Thus using in (1.1) the exchange

$$u \leftrightarrow u + \hat{u}, \qquad v \leftrightarrow v + \hat{v},$$

we get

$$\begin{cases} f(v + \hat{v} - v_g) = -\frac{\partial}{\partial z}(k(z)\frac{\partial u}{\partial z}), \\ f(u + \hat{u} - u_g) = \frac{\partial}{\partial z}(k(z)\frac{\partial v}{\partial z}), \\ u = v = 0 \quad \text{at} \quad z = 0, z_0. \end{cases} \tag{3.14}$$

Introducing the corresponding Green function

$$G(z,s) = \begin{cases} l(s)\left(\frac{l(z)}{l(z_0)} - 1\right) & \text{for } 0 \le s \le z \le z_0, \\ l(z)\left(\frac{l(s)}{l(z_0)} - 1\right) & \text{for } 0 \le z \le s \le z_0, \end{cases}$$

(3.14) is rewritten as

$$\begin{cases} \hat{f}u(z) = -\int_0^{z_0} G(z,s)(v(s) + \hat{v}(s) - v_g)ds, \\ \hat{f}v(z) = \int_0^{z_0} G(z,s)(u(s) + \hat{u}(s) - u_g)ds \end{cases} \tag{3.15}$$

for $\hat{f} = f^{-1}$. Now we consider a Hilbert space $H = L^2(0,z_0)^2$ with an inner product

$$((u_1,v_1),(u_2,v_2)) = \int_0^{z_0}(u_1(z)v_1(z) + u_2(z)v_2(z))dz.$$

Next introducing a linear operator $A : H \to H$ by

$$A(u,v)(z) = \left(\int_0^{z_0} G(z,s)v(s)ds, -\int_0^{z_0} G(z,s)u(s)ds\right)$$

and functions

$$\tilde{u}(z) = -\int_0^{z_0} G(z,s)(\hat{v}(s) - v_g)ds,$$
$$\tilde{v}(z) = \int_0^{z_0} G(z,s)(\hat{u}(s) - u_g)ds,$$

(3.15) is equivalent to

$$\hat{f}(u,v) + A(u,v) = (\tilde{u},\tilde{v}).$$

Since $G(z,s) = G(s,z)$, it is easy to see that A is anti-symmetric $A^* = -A$. It is also well-known that A is compact [1,19]. Thus a spectrum of A consists from isolated pure imaginary eigenvectors with a limit at the zero and the corresponding eigenvalues form an orthogonal bases of H. Consequently, for any $0 \neq f \in \mathbb{R}$, there is a unique solution of (3.14), and thus also for (1.1). Some approximations methods can be used for general $k(z)$ in order to construct these solutions. If $k(z)$ is constant then a method presented above is applied. □

3.3 Dynamical properties

Conditions (1.2) and (1.5) are Cauchy initial value conditions for (1.1), so they determine a unique solution on $\mathbb{R}_+ = [0,\infty)$. We will try to study the uniqueness of (1.1) with conditions (1.2) and (1.3).

Theorem 3.8. *For any constant $\bar{k} > 0$ there is an $\bar{\epsilon} > 0$ such that for any continuous function $k : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying*

$$\sup_{z\in\mathbb{R}_+} |\bar{k} - k(z)| < \bar{\epsilon},$$

there is a unique solution of (1.1) with conditions (1.2) and (1.3).

Proof. To study conditions (1.2) and (1.3), we introduce

$$x = k\frac{\partial u}{\partial z},$$
$$y = k\frac{\partial v}{\partial z},$$

and (1.1) is replaced by

$$\begin{cases} \frac{\partial u}{\partial z} = \hat{k}x, \\ \frac{\partial v}{\partial z} = \hat{k}y, \\ \frac{\partial x}{\partial z} = -f(v - v_g), \\ \frac{\partial y}{\partial z} = f(u - u_g) \end{cases} \tag{3.16}$$

for $\hat{k} = \frac{1}{k}$. The affine system (3.16) has a unique equilibrium

$$(u_g, v_g, 0, 0)$$

with the linearization

$$\begin{cases} \frac{\partial u}{\partial z} = \hat{k}x, \\ \frac{\partial v}{\partial z} = \hat{k}y, \\ \frac{\partial x}{\partial z} = -fv, \\ \frac{\partial y}{\partial z} = fu. \end{cases} \tag{3.17}$$

If $\sup_{z\in\mathbb{R}_+} \hat{k}(z) < \infty$, then the asymptotic property of (3.17) is determined by its Lyapunov exponents. When k is a constant function, then the matrix

$$\begin{pmatrix} 0 & 0 & \hat{k} & 0 \\ 0 & 0 & 0 & \hat{k} \\ 0 & -f & 0 & 0 \\ f & 0 & 0 & 0 \end{pmatrix}$$

has eigenvalues

$$\lambda_1 = -\sqrt{\frac{f\hat{k}}{2}}(1+i), \qquad \lambda_2 = -\sqrt{\frac{f\hat{k}}{2}}(1-i), \qquad \lambda_3 = \sqrt{\frac{f\hat{k}}{2}}(1-i), \qquad \lambda_4 = \sqrt{\frac{f\hat{k}}{2}}(1+i)$$

with the corresponding eigenvectors

$$\left(-\frac{(1+i)\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}, -\frac{(1-i)\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}, i, 1\right), \qquad \left(-\frac{(1-i)\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}, -\frac{(1+i)\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}, -i, 1\right),$$
$$\left(\frac{(1-i)\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}, \frac{(1+i)\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}, -i, 1\right), \qquad \left(\frac{(1+i)\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}, \frac{(1-i)\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}, i, 1\right).$$

So the linear system (3.17) has a stable space

$$S = \left[\begin{array}{c} \left(-\frac{\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}, -\frac{\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}, 0, 1\right) \\ \left(-\frac{\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}, \frac{\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}, 1, 0\right) \end{array} \right]$$

and (3.16) has a stable manifold

$$W_s = (u_g, v_g, 0, 0) + S.$$

Thus condition (1.2) holds if [17,18]

$$(0,0) \in (u_g, v_g) + \left[\begin{array}{c} \left(-\frac{\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}, -\frac{\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}\right) \\ \left(-\frac{\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}, \frac{\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}\right) \end{array} \right],$$

which is uniquely satisfied

$$(0,0) = (u_g, v_g) + c_1\left(-\frac{\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}, -\frac{\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}\right) + c_2\left(-\frac{\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}, \frac{\sqrt{\hat{k}}}{\sqrt{2}\sqrt{f}}\right)$$
$$c_1 = \frac{\sqrt{\hat{k}}(u_g + v_g)}{\sqrt{2}\sqrt{f}}, \quad c_2 = \frac{\sqrt{\hat{k}}(u_g - v_g)}{\sqrt{2}\sqrt{f}}.$$

Consequently, there is a unique solution of (1.1) with conditions (1.2) and (1.3). This is already shown above in Remark 3.6. By using a roughness result [3, Proposition 1, p. 34], we see that for any constant $\bar{k} > 0$ there is an $\bar{\epsilon} > 0$ such that for any continuous function $k : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\sup_{z\in\mathbb{R}_+} |\bar{k} - k(z)| < \bar{\epsilon},$$

there is a unique solution of (1.1) with conditions (1.2) and (1.3). $\bar{\epsilon}$ can be estimated in the term of $\bar{k}$ and f, but we do not go into details. Since $k(z)$ is just continuous, here we have a solution $u(z)$, $v(z)$ of (1.1) such that $u(z)$, $v(z)$, $\frac{\partial u(z)}{\partial z}$, $\frac{\partial v(z)}{\partial z}$, $\frac{\partial}{\partial z}(k(z)\frac{\partial u(z)}{\partial z})$ and $\frac{\partial}{\partial z}(k(z)\frac{\partial u(z)}{\partial z})$ exist and continuous on $\mathbb{R}_+$. The proof is complete. □

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