Electronic Journal of Qualitative Theory of Differential Equations
2020, No. **56**, 1–18; https://doi.org/10.14232/ejqtde.2020.1.56 www.math.u-szeged.hu/ejqtde/

Ground state solutions for nonlinearly coupled systems of Choquard type with lower critical exponent

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Received 13 February 2020, appeared 20 September 2020
Communicated by Petru Jebelean

Abstract. In this paper, we study the existence of ground state solutions for the following nonlinearly coupled systems of Choquard type with lower critical exponent by variational methods

$$\begin{cases} -\Delta u + V(x)u = (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}-1}u + p|u|^{p-2}u|v|^q, & \text{in } \mathbb{R}^N, \\ -\Delta v + V(x)v = (I_\alpha * |v|^{\frac{\alpha}{N}+1})|v|^{\frac{\alpha}{N}-1}v + q|v|^{q-2}v|u|^p, & \text{in } \mathbb{R}^N. \end{cases}$$

Where $N \geq 3$, $\alpha \in (0, N)$, I_α is the Riesz potential, $p, q \in \left(1, \sqrt{\frac{N}{N-2}}\right)$ and $Np + (N+2)q < 2N+4$, $\frac{N+\alpha}{N}$ is the lower critical exponent in the sense of Hardy–Littlewood–Sobolev inequality and $V \in C(\mathbb{R}^N, (0, \infty))$ is a bounded potential function. As far as we have known, little research has been done on this type of coupled systems up to now. Our research is a promotion and supplement to previous research.

Keywords: nonlinearly coupled systems, lower critical exponent, Choquard type equation, ground state solutions, variational methods.

2020 Mathematics Subject Classification: 35J10, 35J60, 35J65.

1 Introduction and main result

We are interested in the following nonlinearly coupled systems of Choquard type with lower critical exponent

$$\begin{cases} -\Delta u + V(x)u = (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}-1}u + p|u|^{p-2}u|v|^q, & \text{in } \mathbb{R}^N, \\ -\Delta v + V(x)v = (I_\alpha * |v|^{\frac{\alpha}{N}+1})|v|^{\frac{\alpha}{N}-1}v + q|v|^{q-2}v|u|^p, & \text{in } \mathbb{R}^N. \end{cases} \tag{1.1}$$

Where the dimension $N \geq 3$ of $\mathbb{R}^N$ is given and function $I_\alpha : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ is a Riesz potential of order $\alpha \in (0, N)$ defined for each $x \in \mathbb{R}^N \setminus \{0\}$,

$$I_\alpha(x) = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{\frac{N}{2}}2^\alpha |x|^{N-\alpha}},$$

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Γ denotes the classical Gamma function, $*$ represents the convolution product on $\mathbb{R}^N$, $p, q \in \left(1, \sqrt{\frac{N}{N-2}}\right)$ and $Np + (N+2)q < 2N + 4$, $V \in C(\mathbb{R}^N, (0, \infty))$ is a bounded potential function. More precisely, we make the following assumptions on V,

(V_1) $V_0 := \inf_{x \in \mathbb{R}^N} V(x) > 0$;

(V_2) $V(x) < \lim_{|y| \to \infty} V(y) = V_\infty < \infty$.

For the following Choquard equation

$$-\Delta u + V(x)u = (I_\alpha * |u|^p)|u|^{p-2}u, \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

when $N = 3$, $\alpha = 2$, $p = 2$ and V is a positive constant, this equation appears in several physical contexts, such as standing waves for the Hartree equation, the description of the quantum physics of a polaron at rest by S. I. Pekar in [13] and the modeling of an electron trapped in its own hole in 1976 in the work of Choquard, as a certain approximation to Hartree–Fock theory of one–component plasma (see [4]). In some particular cases, this equation is also known as the Schrödinger–Newton equation, which was introduced by R. Penrose [14] in his discussion on the selfgravitational collapse of a quantum mechanical wave function. The existence and uniqueness of positive solutions for equation (1.2) with $N = 3$, $V(x) \equiv 1$, $\alpha = 2$ and $p = 2$ was firstly obtained by E. H. Lieb in [4]. Later, P. L. Lions [6,7] got the existence and multiplicity results of normalized solution on the same topic. Since then, the existence and qualitative properties of solutions for equation (1.2) have been widely studied by variational methods in the recent decades. For related topics, we refer the reader to the recent survey paper [12].

To study equation (1.2) variationally, the well-known Hardy–Littlewood–Sobolev inequality is the starting point. Particularly, V. Moroz and J. Van Schaftingen [9] established the existence, qualitative properties and decay estimates of ground state solutions for the autonomous case of equation (1.2) with $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$ and $V(x) \equiv 1$. In view of the Pohožaev identity [9–11], Choquard equation (1.2) with V is a positive constant has no nontrivial smooth H^1 solution when either $p \leq \frac{N+\alpha}{N}$ or $p \geq \frac{N+\alpha}{N-2}$. Usually, $\frac{N+\alpha}{N}$ is called the lower critical exponent and $\frac{N+\alpha}{N-2}$ is the upper critical exponent for Choquard equation in the sense of Hardy–Littlewood–Sobolev inequality. The upper critical exponent plays a similar role as the Sobolev critical exponent in the local semilinear equations. C. O. Alves, S. Gao, M. Squassina and M. Yang[1] established the existence of ground states for a type of critical Choquard equation with constant coefficients and also studied the existence and multiplicity of semi–classical solutions and characterized the concentration behavior by variational methods. G. Li and C. Tang [8] obtained a positive ground state solution for Choquard equation with upper critical exponent when the nonlinear perturbation satisfies the general subcritical growth conditions. The lower critical exponent seems to be a new feature for Choquard equation, which is related to a new phenomenon of "bubbling at infinity" (for more details see [10]).

J. Van Schaftingen and J. Xia [15] studied the ground state solutions of the following Choquard equation with lower critical exponent and coercive potential V,

$$-\Delta u + V(x)u = (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}-1}u, \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

Later, J. Van Schaftingen and J. Xia [16] also obtained a ground state solution for the following Choquard equation with lower critical exponent and a local nonlinear perturbation

$$-\Delta u + u = (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}-1}u + f(x, u), \quad \text{in } \mathbb{R}^N. \tag{1.4}$$

For the autonomous case $f(x,u) = f(u)$ satisfies some superlinear assumptions, the existence and symmetry of ground state for equation (1.4) were also got. Furthermore, they derived a ground state solution of equation (1.4) for the nonautonomous case $f(x,u) = K(x)|u|^{q-2}u$ with $q \in (2, 2+\frac{4}{N})$ and $K \in L^\infty(\mathbb{R}^N)$ satisfying $\inf_{x\in\mathbb{R}^N} K(x) = K_\infty = \lim_{|x|\to\infty} K(x) > 0$.

As we mentioned above, all the results in the literature are concerned with a single equation. More recently, P. Chen and X. Liu [2] obtained the existence of ground state solutions for the following linearly coupled systems of Choquard type with subcritical exponent $p \in (\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2})$,

$$\begin{cases} -\Delta u + u = (I_\alpha * |u|^p)|u|^{p-2}u + \lambda v, & \text{in } \mathbb{R}^N, \\ -\Delta v + v = (I_\alpha * |v|^p)|v|^{p-2}v + \lambda u, & \text{in } \mathbb{R}^N. \end{cases}$$

Later, M. Yang, J. de Albuquerque, E. Silva and M. Silva [19] obtained the existence of positive ground state solutions for the following linearly coupled systems of Choquard type

$$\begin{cases} -\Delta u + u = (I_\alpha * |u|^p)|u|^{p-2}u + \lambda v, & \text{in } \mathbb{R}^N, \\ -\Delta v + v = (I_\alpha * |v|^q)|v|^{q-2}v + \lambda u, & \text{in } \mathbb{R}^N. \end{cases} \tag{1.5}$$

when the exponents satisfy one of case 1, case 2 and case 3, and also obtained that there is no nontrivial solution for system (1.5) in case 4, where

$$\begin{aligned} &\text{case 1,} \quad \tfrac{N+\alpha}{N} < p < \tfrac{N+\alpha}{N-2} &&\text{and} \quad q = \tfrac{N+\alpha}{N-2}, \\ &\text{case 2,} \quad p = \tfrac{N+\alpha}{N} &&\text{and} \quad \tfrac{N+\alpha}{N} < q < \tfrac{N+\alpha}{N-2}, \\ &\text{case 3,} \quad p = \tfrac{N+\alpha}{N} &&\text{and} \quad q = \tfrac{N+\alpha}{N-2}, \\ &\text{case 4,} \quad p, q \le \tfrac{N+\alpha}{N} &&\text{or} \quad p, q \ge \tfrac{N+\alpha}{N-2}. \end{aligned}$$

Motivated by [2, 15, 16, 19], in this paper, we will study the existence of ground state solutions for system (1.1). Our main result reads as followed.

Theorem 1.1. *Let $N \ge 3$, $\alpha \in (0, N)$, $p, q \in \left(1, \sqrt{\frac{N}{N-2}}\right)$, $Np + (N+2)q < 2N+4$ and V satisfies $(V_1), (V_2)$, then system (1.1) admits at least one ground state solution.*

Remark 1.2. The assumption $Np + (N+2)q < 2N+4$ is mainly used to get the energy estimate of c_0 in Lemma 2.6. In particular, $p, q \in \left(1, \frac{N+2}{N+1}\right)$ satisfy our assumptions on p, q.

The method used to prove Theorem 1.1 is as follows. Firstly, we establish the variational framework for system (1.1). Let $H^1(\mathbb{R}^N)$ denote the normal Sobolev space equipped with the norm

$$\|u\| := \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) dx \right)^{\frac{1}{2}}.$$

Define $X = H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ equipped with norm

$$\|(u,v)\| = (\|u\|^2 + \|v\|^2)^{\frac{1}{2}}.$$

Similar to $H^1(\mathbb{R}^N)$, X is a Hilbert space and satisfies

$$X \hookrightarrow L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N), \qquad p \in [2, 2^*], \quad \text{where } 2^* = \frac{2N}{N-2}.$$

By Hardy–Littlewood–Sobolev inequality and Sobolev embedding theorem, the energy functional associated to system (1.1)

$$
\begin{aligned}
J_V(u,v) = &\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)|u|^2)dx + \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla v|^2 + V(x)|v|^2)dx \\
&- \frac{N}{2(N+\alpha)}\int_{\mathbb{R}^N}(I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}+1}dx - \frac{N}{2(N+\alpha)}\int_{\mathbb{R}^N}(I_\alpha * |v|^{\frac{\alpha}{N}+1})|v|^{\frac{\alpha}{N}+1}dx \\
&- \int_{\mathbb{R}^N}|u|^p|v|^q dx
\end{aligned}
$$

is $C^1(X,\mathbb{R})$ and

$$
\begin{aligned}
\langle J_V'(u,v),(\phi,\varphi)\rangle = &\int_{\mathbb{R}^N}(\nabla u\nabla\phi + V(x)u\phi)dx + \int_{\mathbb{R}^N}(\nabla v\nabla\varphi + V(x)v\varphi)dx \\
&- \int_{\mathbb{R}^N}(I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}-1}u\phi)dx - \int_{\mathbb{R}^N}(I_\alpha * |v|^{\frac{\alpha}{N}+1}|v|^{\frac{\alpha}{N}-1}v\varphi)dx \\
&- p\int_{\mathbb{R}^N}|v|^q|u|^{p-2}u\phi dx - q\int_{\mathbb{R}^N}|u|^p|v|^{q-2}v\varphi dx, \quad \text{for } (\phi,\varphi)\in X.
\end{aligned}
$$

Thus, any critical point of J_V is a weak solution of system (1.1). As usual, a nontrivial solution $(u,v)\in X$ of system (1.1) is called a ground state solution if

$$
J_V(u,v) = c_g^V := \inf\{J_V(u,v) : (u,v)\in X\setminus\{(0,0)\} \quad \text{and} \quad J_V'(u,v)=0\}.
$$

Secondly, in the process of finding ground state solutions for system (1.1), the following limiting problem plays a significant role

$$
\begin{cases}
-\Delta u + V_\infty u = (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}-1}u + p|u|^{p-2}u|v|^q, & \text{in } \mathbb{R}^N, \\
-\Delta v + V_\infty v = (I_\alpha * |v|^{\frac{\alpha}{N}+1})|v|^{\frac{\alpha}{N}-1}v + q|v|^{q-2}v|u|^p, & \text{in } \mathbb{R}^N.
\end{cases}
\tag{1.6}
$$

Compared with the autonomous system (1.6), the potential V in system (1.1) breaks down the invariance under translations in $\mathbb{R}^N$, then we cannot use the translation-invariant concentration–compactness argument. The strategy to prove Theorem 1.1 is a comparison of the energy of the functional J_V with the functional J_{V_∞} associated to system (1.6). On the one hand, we construct a Palais–Smale sequence $\{(u_n,v_n)\}$ of J_{V_∞} at the level c_0 defined in (2.4), that is, a sequence $\{(u_n,v_n)\}$ in X such that $J_{V_\infty}(u_n,v_n)\to c_0$ and $J'_{V_\infty}(u_n,v_n)\to 0$ as $n\to\infty$. On the other hand, we prove that up to translations the sequence $\{(u_n,v_n)\}$ converges to a nontrivial solution (u,v) of system (1.6). Then, in the same way we obtain a $(\mathrm{PS})_{c_V}$ sequence of J_V. Furthermore, by the equivalent characterization of c_0, we can show that $c_V < c_0$ under the assumptions on the potential V. Based on $c_V < c_0$, the compactness maintains and a ground state solution for system (1.1) is obtained.

The rest of the paper is organized as follows. We give some preliminaries in Section 2. We obtain a ground state solution for system (1.6) in Section 3. Theorem 1.1 is proved in Section 4.

2 Preliminary

In this section, we first provide some preliminary results.

The following well-known Hardy–Littlewood–Sobolev inequality will be frequently used in this paper.

Lemma 2.1 (Hardy–Littlewood–Sobolev inequality, [5])**.** *Let $p, q > 1$, $\alpha \in (0, N)$, $1 \le r < s < \infty$ and $s \in (1, \frac{N}{\alpha})$ such that*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{\alpha}{N}, \qquad \frac{1}{r} - \frac{1}{s} = \frac{\alpha}{N}.$$

(i) Let $f \in L^p(\mathbb{R}^N)$ and $g \in L^q(\mathbb{R}^N)$, we have

$$\Big| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^{N-\alpha}} dxdy \Big| \le C(N, \alpha, p) \|f\|_{L^p(\mathbb{R}^N)} \|g\|_{L^q(\mathbb{R}^N)}.$$

(ii) For any $f \in L^r(\mathbb{R}^N)$, $I_\alpha * f \in L^s(\mathbb{R}^N)$ and

$$\|I_\alpha * f\|_{L^s(\mathbb{R}^N)} \le C(N, \alpha, r) \|f\|_{L^r(\mathbb{R}^N)}.$$

By Hardy–Littlewood–Sobolev inequality mentioned above and the classical Sobolev embedding theorem, we obtain

$$\int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{\alpha}{N}+1}) |u|^{\frac{\alpha}{N}+1} dx \le C(N, \alpha) \Big(\int_{\mathbb{R}^N} |u|^2 dx \Big)^{\frac{\alpha}{N}+1}. \tag{2.1}$$

This inequality can be restated as the following minimization problem

$$S = \inf \left\{ \int_{\mathbb{R}^N} |u|^2 dx : u \in H^1(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{\alpha}{N}+1}) |u|^{\frac{\alpha}{N}+1} dx = 1 \right\}.$$

By Theorem 4.3 in [5], the infimum S is achieved by a function $u \in H^1(\mathbb{R}^N)$ if and only if

$$u(x) = A \left(\frac{\varepsilon}{\varepsilon^2 + |x-a|^2} \right)^{\frac{N}{2}}, \qquad x \in \mathbb{R}^N, \tag{2.2}$$

for some given constants $A \in \mathbb{R}$, and $a \in \mathbb{R}^N$, $\varepsilon \in (0, \infty)$. The form of the minimizers in (2.2) suggests that a loss of compactness in equation (1.3) with V is a positive constant may occur by both of translations and dilations.

First, we recall that pointwise convergence of a bounded sequence implies weak convergence.

Lemma 2.2 ([18, Proposition 5.4.7])**.** *Let $N \ge 3$, $q \in (1, \infty)$ and $\{u_n\}$ be a bounded sequence in $L^q(\mathbb{R}^N)$. If $u_n(x) \to u(x)$ almost everywhere in $\mathbb{R}^N$ as $n \to \infty$, then $u_n \rightharpoonup u$ weakly in $L^q(\mathbb{R}^N)$.*

Similarly as in [3], we can get the following lemma.

Lemma 2.3. *Assume that $\{u_n\} \subset H^1(\mathbb{R}^N)$ is a sequence satisfying that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$, then for any $\varphi \in H^1(\mathbb{R}^N)$,*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{\alpha}{N}+1}) |u_n|^{\frac{\alpha}{N}-1} u_n \varphi dx = \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{\alpha}{N}+1}) |u|^{\frac{\alpha}{N}-1} u \varphi dx.$$

Proof. For the reader's convenience, we give a complete proof here. Up to a subsequence, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$, $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$ and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$. By Sobolev's embedding theorem, $\{u_n\}$ is bounded in $L^2(\mathbb{R}^N) \cap L^{2^*}(\mathbb{R}^N)$, the sequence $\{|u_n|^{\frac{N+\alpha}{N}}\}$ is bounded in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. Then by Lemma 2.2

$$|u_n|^{\frac{\alpha}{N}+1} \rightharpoonup |u|^{\frac{\alpha}{N}+1}, \quad \text{in } L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N).$$

$$|u_n|^{\frac{\alpha}{N}-1}u_n\varphi \to |u|^{\frac{\alpha}{N}-1}u\varphi, \quad \text{in } L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N), \text{ for any } \varphi \in H^1(\mathbb{R}^N).$$

By Lemma 2.1, the Riesz potential defines a linear continuous map from $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ to $L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$. We know that,

$$I_\alpha * (|u_n|^{\frac{\alpha}{N}-1}u_n\varphi) \to I_\alpha * (|u|^{\frac{\alpha}{N}-1}u\varphi), \quad \text{in } L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N).$$

Thus,

$$\begin{aligned}
&\int_{\mathbb{R}^N}(I_\alpha * |u_n|^{\frac{\alpha}{N}+1})|u_n|^{\frac{\alpha}{N}-1}u_n\varphi dx - \int_{\mathbb{R}^N}(I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}-1}u\varphi dx \\
&= \int_{\mathbb{R}^N}|u_n|^{\frac{\alpha}{N}+1}(I_\alpha * (|u_n|^{\frac{\alpha}{N}-1}u_n\varphi)dx - \int_{\mathbb{R}^N}|u|^{\frac{\alpha}{N}+1}(I_\alpha * (|u|^{\frac{\alpha}{N}-1}u\varphi)dx \\
&= \int_{\mathbb{R}^N}|u_n|^{\frac{\alpha}{N}+1}(I_\alpha * (|u_n|^{\frac{\alpha}{N}-1}u_n\varphi) - I_\alpha * (|u|^{\frac{\alpha}{N}-1}u\varphi))dx \\
&\quad + \int_{\mathbb{R}^N}(|u_n|^{\frac{\alpha}{N}+1} - |u|^{\frac{\alpha}{N}+1})(I_\alpha * (|u|^{\frac{\alpha}{N}-1}u\varphi))dx \\
&\to 0, \quad \text{as } n \to \infty.
\end{aligned} \tag{2.3}$$

The proof is complete. □

Lemma 2.4 ([17, Lemma 1.21]). *Let $r_0 > 0$ and $s \in [2, 2^*)$. If $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and*

$$\sup_{y\in\mathbb{R}^N}\int_{B(y,r_0)}|u_n|^s \to 0, \quad \text{as } n \to \infty,$$

then $u_n \to 0$ in $L^t(\mathbb{R}^N)$ for $t \in (2, 2^*)$.

Lemma 2.5. *The functional J_{V_∞} satisfies the following properties:*

(1) there exists $\rho > 0$ such that $\inf_{(u,v)\in X,\ \|(u,v)\|=\rho} J_{V_\infty}(u,v) > 0$;

(2) for any $(u,v) \in X\backslash\{(0,0)\}$, it holds $\lim_{t\to\infty} J_{V_\infty}(tu,tv) = -\infty$.

Proof. (1) By (2.1) and the classical Sobolev inequality, we can deduce that

$$\begin{aligned}
J_{V_\infty}(u,v) &\geq \frac{1}{2}\min\{1, V_\infty\}(\|u\|^2 + \|v\|^2) - \frac{N}{2(N+\alpha)}\int_{\mathbb{R}^N}(I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}+1}dx \\
&\quad - \frac{N}{2(N+\alpha)}\int_{\mathbb{R}^N}(I_\alpha * |v|^{\frac{\alpha}{N}+1})|v|^{\frac{\alpha}{N}+1}dx - \int_{\mathbb{R}^N}|u|^p|v|^q dx \\
&\geq \frac{1}{2}\min\{1, V_\infty\}\|(u,v)\|^2 - C_1(\|u\|^{\frac{2\alpha}{N}+2} + ||v||^{\frac{2\alpha}{N}+2}) - \int_{\mathbb{R}^N}(|u|^{2p} + |v|^{2q})dx \\
&\geq \frac{1}{2}\min\{1, V_\infty\}\|(u,v)\|^2 - C_1\|(u,v)\|^{\frac{2\alpha}{N}+2} - C_2\|(u,v)\|^{2p} - C_3\|(u,v)\|^{2q},
\end{aligned}$$

where C_1, C_2 are positive constants. Since $p, q > 1$ and $\alpha > 0$, we have that

$$\inf_{(u,v)\in X,\ \|(u,v)\|=\rho} J_{V_\infty}(u,v) > 0,$$

provided that $\rho > 0$ is sufficiently small.

(2) For any $(u,v) \in X\backslash\{(0,0)\}$, we have

$$\begin{aligned}
J_{V_\infty}(tu,tv) \leq & \frac{t^2}{2}\max\{1,V_\infty\}(\|u\|^2+\|v\|^2) - \frac{Nt^{\frac{2\alpha}{N}+2}}{2(N+\alpha)}\int_{\mathbb{R}^N}(I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}+1}dx \\
& - \frac{Nt^{\frac{2\alpha}{N}+2}}{2(N+\alpha)}\int_{\mathbb{R}^N}(I_\alpha * |v|^{\frac{\alpha}{N}+1})|v|^{\frac{\alpha}{N}+1}dx - t^{p+q}\int_{\mathbb{R}^N}|u|^p|v|^q dx \\
\leq & \frac{t^2}{2}\max\{1,V_\infty\}\|(u,v)\|^2 - \frac{Nt^{\frac{2\alpha}{N}+2}}{2(N+\alpha)}\Big(\int_{\mathbb{R}^N}(I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}+1}dx \\
& + \int_{\mathbb{R}^N}(I_\alpha * |v|^{\frac{\alpha}{N}+1})|v|^{\frac{\alpha}{N}+1}dx\Big).
\end{aligned}$$

Then the conclusion (2) follows. $\square$

By the classical Mountain Pass theorem [17], we have a minimax description at the energy level c_0 defined by

$$c_0 = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_{V_\infty}(\gamma(t)), \tag{2.4}$$

where

$$\Gamma = \{\gamma \in C([0,1],X) : \gamma(0) = (0,0), J_{V_\infty}(\gamma(1)) < 0\}.$$

Lemma 2.6. *Let $N \geq 3$, $\alpha \in (0,N)$, $p,q \in \big(1,\sqrt{\frac{N}{N-2}}\big)$ and $Np+(N+2)q < 2N+4$, then $c_0 < c_* := \frac{\alpha}{2(N+\alpha)}(V_\infty S)^{\frac{N}{\alpha}+1}$.*

Proof. We first show that $c_0 \leq c_1$, where

$$c_1 = \inf_{(u,v)\in X\backslash\{(0,0)\}}\max_{t\geq 0} J_{V_\infty}(tu,tv).$$

Indeed, for any $(u,v) \in X\backslash\{(0,0)\}$, by Lemma 2.5 (2), there exists $t_{u,v} > 0$ such that

$$J_{V_\infty}(t_{u,v}u, t_{u,v}v) < 0.$$

Hence, by (2.4), we have

$$c_0 \leq \max_{\tau\in[0,1]} J_{V_\infty}(\tau t_{u,v}u, \tau t_{u,v}v) \leq \max_{t\geq 0} J_{V_\infty}(tu,tv). \tag{2.5}$$

It leads to $c_0 \leq c_1$.

By the representation formula (2.2) for the optimal functions of Hardy–Littlewood–Sobolev inequality, for each $\varepsilon > 0$, we set

$$U(x) = A(1+|x|^2)^{-\frac{N}{2}}, \qquad x \in \mathbb{R}^N,$$

$U_\varepsilon(x) = \varepsilon^{\frac{N}{2}}U(\varepsilon x)$ and $V_\varepsilon(x) = \varepsilon^{\frac{N+\beta}{2}}U(\varepsilon x)$, where $\beta \in \big(\frac{N(p+q-2)}{2-q}, \frac{4-N(p+q-2)}{q}\big)$. For each $\varepsilon > 0$ the function U_ε satisfies

$$\int_{\mathbb{R}^N}|U_\varepsilon|^2dx = S \quad \text{and} \quad \int_{\mathbb{R}^N}(I_\alpha * |U_\varepsilon|^{\frac{\alpha}{N}+1})|U_\varepsilon|^{\frac{\alpha}{N}+1}dx = 1.$$

Through direct computations, we have that

$$\int_{\mathbb{R}^N}|V_\varepsilon|^2dx = \varepsilon^\beta\int_{\mathbb{R}^N}|U|^2dx, \qquad \int_{\mathbb{R}^N}(I_\alpha * |V_\varepsilon|^{\frac{\alpha}{N}+1})|V_\varepsilon|^{\frac{\alpha}{N}+1}dx = \varepsilon^{\frac{\beta(N+\alpha)}{N}},$$

$$\int_{\mathbb{R}^N} |\nabla U_\varepsilon|^2 dx = \varepsilon^2 \int_{\mathbb{R}^N} |\nabla U|^2 dx, \qquad \int_{\mathbb{R}^N} |\nabla V_\varepsilon|^2 dx = \varepsilon^{\beta+2} \int_{\mathbb{R}^N} |\nabla U|^2 dx.$$

For every $\varepsilon > 0$, we now consider the function $\xi_\varepsilon : [0,\infty) \to \mathbb{R}$ defined by

$$\xi_\varepsilon(t) := J_{V_\infty}(tU_\varepsilon, tV_\varepsilon) = g(t) + h_\varepsilon(t) + f_\varepsilon(t), \qquad t \in [0,\infty),$$

where the functions $g, h_\varepsilon, f_\varepsilon : [0,\infty) \to \mathbb{R}$ are defined by

$$\begin{aligned}
g(t) &= \frac{1}{2} V_\infty S t^2 - \frac{N}{2(N+\alpha)} t^{\frac{2\alpha}{N}+2},\\
h_\varepsilon(t) &= \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla V_\varepsilon|^2 dx + \frac{t^2}{2} V_\infty \int_{\mathbb{R}^N} |V_\varepsilon|^2 dx - \frac{N t^{\frac{2(N+\alpha)}{N}}}{2(N+\alpha)} \int_{\mathbb{R}^N} (I_\alpha * |V_\varepsilon|^{\frac{\alpha}{N}+1}) |V_\varepsilon|^{\frac{\alpha}{N}+1} dx,\\
f_\varepsilon(t) &= \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla U_\varepsilon|^2 dx - t^{p+q} \int_{\mathbb{R}^N} |U_\varepsilon|^p |V_\varepsilon|^q dx.
\end{aligned}$$

Since $\xi_\varepsilon(t) > 0$ whenever $t > 0$ is small enough, $\lim_{t\to 0} \xi_\varepsilon(t) = 0$ and $\lim_{t\to\infty} \xi_\varepsilon(t) = -\infty$, for each $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that

$$\xi_\varepsilon(t_\varepsilon) = \max_{t\geq 0} \xi_\varepsilon(t).$$

By the definition of the function g, we have

$$c_1 \leq \max_{t\geq 0} \xi_\varepsilon(t) = \xi_\varepsilon(t_\varepsilon) = g(t_\varepsilon) + h_\varepsilon(t_\varepsilon) + f_\varepsilon(t_\varepsilon) \leq g(t_*) + h_\varepsilon(t_\varepsilon) + f_\varepsilon(t_\varepsilon), \tag{2.6}$$

where $t_* = (V_\infty S)^{\frac{N}{2\alpha}}$ satisfies that

$$g(t_*) = \max_{t\geq 0} g(t) = \frac{\alpha}{2(N+\alpha)} V_\infty^{\frac{N}{\alpha}+1} S^{\frac{N}{\alpha}+1} = c_*.$$

Since $\xi_\varepsilon'(t_\varepsilon) = 0$, we have

$$\begin{aligned}
&\varepsilon^2 \int_{\mathbb{R}^N} |\nabla U|^2 dx + \varepsilon^{\beta+2} \int_{\mathbb{R}^N} |\nabla U|^2 dx + \varepsilon^{\beta} \int_{\mathbb{R}^N} |U|^2 dx + V_\infty S\\
&\quad = t_\varepsilon^{\frac{2\alpha}{N}} + t_\varepsilon^{\frac{2\alpha}{N}} \int_{\mathbb{R}^N} (I_\alpha * |V_\varepsilon|^{\frac{\alpha}{N}+1}) |V_\varepsilon|^{\frac{\alpha}{N}+1} dx + (p+q) t_\varepsilon^{p+q-2} \int_{\mathbb{R}^N} |U_\varepsilon|^p |V_\varepsilon|^q dx\\
&\quad \geq t_\varepsilon^{\frac{2\alpha}{N}}.
\end{aligned} \tag{2.7}$$

Hence, we have $\limsup_{\varepsilon\to 0} t_\varepsilon^{\frac{2\alpha}{N}} \leq V_\infty S$, which is equivalent to $\limsup_{\varepsilon\to 0} t_\varepsilon \leq V_\infty^{\frac{N}{2\alpha}} S^{\frac{N}{2\alpha}}$. Notice that

$$\begin{aligned}
&t_\varepsilon^{\frac{2\alpha}{N}} \int_{\mathbb{R}^N} (I_\alpha * |V_\varepsilon|^{\frac{\alpha}{N}+1}) |V_\varepsilon|^{\frac{\alpha}{N}+1} dx + (p+q) t_\varepsilon^{p+q-2} \int_{\mathbb{R}^N} |U_\varepsilon|^p |V_\varepsilon|^q dx\\
&\qquad = \varepsilon^{\frac{\beta(N+\alpha)}{N}} t_\varepsilon^{\frac{2\alpha}{N}} + (p+q) \varepsilon^{\frac{N(p+q-2)+\beta q}{2}} t_\varepsilon^{p+q-2} \int_{\mathbb{R}^N} |U|^{p+q} dx,
\end{aligned}$$

we can obtain that

$$\lim_{\varepsilon\to 0} \left(t_\varepsilon^{\frac{2\alpha}{N}} \int_{\mathbb{R}^N} (I_\alpha * |V_\varepsilon|^{\frac{\alpha}{N}+1}) |V_\varepsilon|^{\frac{\alpha}{N}+1} dx + (p+q) t_\varepsilon^{p+q-2} \int_{\mathbb{R}^N} |U_\varepsilon|^p |V_\varepsilon|^q dx \right) = 0. \tag{2.8}$$

Then (2.7) and (2.8) imply $\liminf_{\varepsilon\to 0} t_\varepsilon^{\frac{2\alpha}{N}} \geq V_\infty S$. Therefore, $\lim_{\varepsilon\to 0} t_\varepsilon^{\frac{2\alpha}{N}} = V_\infty S$. It leads to $\lim_{\varepsilon\to 0} t_\varepsilon = t_*$.

We now observe that

$$f_\varepsilon(t_\varepsilon)+h_\varepsilon(t_\varepsilon)\le \frac{1}{2}\varepsilon^{\beta+2}t_\varepsilon^2\int_{\mathbb{R}^N}|\nabla U|^2dx+\frac{1}{2}\varepsilon^2 t_\varepsilon^2\int_{\mathbb{R}^N}|\nabla U|^2dx$$
$$+\frac{1}{2}\varepsilon^\beta t_\varepsilon^2 V_\infty\int_{\mathbb{R}^N}|U|^2dx-\varepsilon^{\frac{N(p+q-2)+\beta q}{2}}t_\varepsilon^{p+q-2}\int_{\mathbb{R}^N}|U|^{p+q}dx.$$

Since $p,q\in\left(1,\sqrt{\frac{N}{N-2}}\right)$, $Np+(N+2)q<2N+4$ and $\beta\in\left(\frac{N(p+q-2)}{2-q},\frac{4-N(p+q-2)}{q}\right)$, through direct computations, we can get that $\frac{N(p+q-2)+\beta q}{2}<\min\{\beta,2\}$. Thus

$$f_\varepsilon(t_\varepsilon)+h_\varepsilon(t_\varepsilon)<0,\quad \text{when } \varepsilon>0 \text{ is small enough.}$$

Then it follows from (2.6) that $c_1<c_*$ and thus $c_0<c_*$ in view of (2.5). □

3 Existence of ground state solutions for the limiting problem (1.6)

In this section, we will prove that the limiting problem (1.6) admits at least one ground state solution.

Before giving a complete proof, we state the following lemmas, which will be frequently used in the sequel proofs. Set

$$\|(u,v)\|_{V_\infty}=\left(\int_{\mathbb{R}^N}(|\nabla u|^2+V_\infty u^2)dx+\int_{\mathbb{R}^N}(|\nabla v|^2+V_\infty v^2)dx\right)^{\frac{1}{2}}.$$

Define

$$c_g^{V_\infty}:=\inf\{J_{V_\infty}(u,v):(u,v)\in X\setminus\{(0,0)\}\quad\text{and}\quad J'_{V_\infty}(u,v)=0\}.$$

Lemma 3.1. *If $\{(u_n,v_n)\}$ is a sequence in X such that*

$$\liminf_{n\to\infty}\|(u_n,v_n)\|_{V_\infty}>0,\quad \textit{and}\quad \lim_{n\to\infty}\langle\Phi'(u_n,v_n),(u_n,v_n)\rangle=0,$$

where the functional $\Phi:X\to\mathbb{R}$ is defined by

$$\Phi(u,v)=\frac{1}{2}\|(u,v)\|_{V_\infty}^2-\frac{N}{2(N+\alpha)}\left(\int_{\mathbb{R}^N}(I_\alpha*|u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}+1}dx+\int_{\mathbb{R}^N}(I_\alpha*|v|^{\frac{\alpha}{N}+1})|v|^{\frac{\alpha}{N}+1}dx\right),$$

then $\liminf_{n\to\infty}\Phi(u_n,v_n)\ge c_*$.

Proof. From $\lim\limits_{n\to\infty}\langle\Phi'(u_n,v_n),(u_n,v_n)\rangle=0$, we observe that

$$\|(u_n,v_n)\|_{V_\infty}^2=\int_{\mathbb{R}^N}(I_\alpha*|u_n|^{\frac{\alpha}{N}+1})|u_n|^{\frac{\alpha}{N}+1}dx+\int_{\mathbb{R}^N}(I_\alpha*|v_n|^{\frac{\alpha}{N}+1})|v_n|^{\frac{\alpha}{N}+1}dx+o_n(1).$$

By the assumption $\liminf_{n\to\infty}\|(u_n,v_n)\|_{V_\infty}>0$ and (2.1), we can deduce that

$$\liminf_{n\to\infty}\int_{\mathbb{R}^N}(|u_n|^2+|v_n|^2)dx>0.$$

It follows from the definition of S that

$$\begin{aligned}&\int_{\mathbb{R}^N}(I_\alpha*|u_n|^{\frac{\alpha}{N}+1})|u_n|^{\frac{\alpha}{N}+1}dx+\int_{\mathbb{R}^N}(I_\alpha*|v_n|^{\frac{\alpha}{N}+1})|v_n|^{\frac{\alpha}{N}+1}dx+o_n(1)\\&\quad\ge\int_{\mathbb{R}^N}(V_\infty|u_n|^2+V_\infty|v_n|^2)dx\\&\quad\ge V_\infty S\left[\left(\int_{\mathbb{R}^N}(I_\alpha*|u_n|^{\frac{\alpha}{N}+1})|u_n|^{\frac{\alpha}{N}+1}dx\right)^{\frac{N}{N+\alpha}}+\left(\int_{\mathbb{R}^N}(I_\alpha*|v_n|^{\frac{\alpha}{N}+1})|v_n|^{\frac{\alpha}{N}+1}dx\right)^{\frac{N}{N+\alpha}}\right]\\&\quad\ge V_\infty S\left(\int_{\mathbb{R}^N}(I_\alpha*|u_n|^{\frac{\alpha}{N}+1})|u_n|^{\frac{\alpha}{N}+1}dx+\int_{\mathbb{R}^N}(I_\alpha*|v_n|^{\frac{\alpha}{N}+1})|v_n|^{\frac{\alpha}{N}+1}dx\right)^{\frac{N}{N+\alpha}},\end{aligned}$$

which leads to

$$\begin{aligned}
&\liminf_{n\to\infty}\|(u_n,v_n)\|_{V_\infty}^2\\
&\quad=\liminf_{n\to\infty}\left(\int_{\mathbb{R}^N}(I_\alpha*|u_n|^{\frac{\alpha}{N}+1})|u_n|^{\frac{\alpha}{N}+1}dx+\int_{\mathbb{R}^N}(I_\alpha*|v_n|^{\frac{\alpha}{N}+1})|v_n|^{\frac{\alpha}{N}+1}dx\right)\\
&\quad\geq (V_\infty S)^{\frac{N}{\alpha}+1}.
\end{aligned}\tag{3.1}$$

Therefore,

$$\begin{aligned}
\Phi(u_n,v_n)&=\Phi(u_n,v_n)-\frac{N}{2(N+\alpha)}\langle\Phi'(u_n,v_n),(u_n,v_n)\rangle+o_n(1)\\
&=\frac{\alpha}{2(N+\alpha)}\|u_n,v_n\|_{V_\infty}^2+o_n(1).
\end{aligned}\tag{3.2}$$

Then combine (3.1) with (3.2),

$$\liminf_{n\to\infty}\Phi(u_n,v_n)=\liminf_{n\to\infty}\frac{\alpha}{2(N+\alpha)}\|u_n,v_n\|_{V_\infty}^2\geq\frac{\alpha}{2(N+\alpha)}(V_\infty S)^{1+\frac{N}{\alpha}}=c_*.$$

The proof is complete. □

Lemma 3.2. *Let $\{(u_n,v_n)\}$ be a bounded $(PS)_c$ sequence with $c\in(0,c_*)$ for functional J_{V_∞}, then up to a subsequence and translations, the sequence $\{(u_n,v_n)\}$ converges weakly to some $(u,v)\in X\backslash\{(0,0)\}$ such that*

$$J'_{V_\infty}(u,v)=0\quad\textit{and}\quad J_{V_\infty}(u,v)\in(0,c].$$

Proof. First we show that

$$\limsup_{n\to\infty}\frac{1}{2}\int_{\mathbb{R}^N}(|u_n|^{2p}+|v_n|^{2q})dx>0.\tag{3.3}$$

Otherwise, up to a subsequence, we have

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^p|v_n|^qdx\leq\limsup_{n\to\infty}\int_{\mathbb{R}^N}(|u_n|^{2p}+|v_n|^{2q})dx=0.\tag{3.4}$$

Since $\lim\limits_{n\to\infty}\langle J'_{V_\infty}(u_n,v_n),(u_n,v_n)\rangle=0$, we have

$$\|(u_n,v_n)\|_{V_\infty}^2=\int_{\mathbb{R}^N}(I_\alpha*|u_n|^{\frac{\alpha}{N}+1})|u_n|^{\frac{\alpha}{N}+1}dx+\int_{\mathbb{R}^N}(I_\alpha*|v_n|^{\frac{\alpha}{N}+1})|v_n|^{\frac{\alpha}{N}+1}dx+o_n(1).$$

While, $J_{V_\infty}(u_n,v_n)\to c>0,\ n\to\infty$, together with (3.4) and (2.1), imply that

$$\liminf_{n\to\infty}\|(u_n,v_n)\|_{V_\infty}>0.$$

Then we deduce from Lemma 3.1 that

$$\begin{aligned}
c&=\liminf_{n\to\infty}J_{V_\infty}(u_n,v_n)\\
&=\liminf_{n\to\infty}\Phi(u_n,v_n)-\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^p|v_n|^qdx\\
&=\liminf_{n\to\infty}\Phi(u_n,v_n)\\
&\geq c_*,
\end{aligned}$$

which contradicts with the fact $c \in (0, c_*)$. Thus (3.3) holds. It implies that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{2p} dx > 0, \quad \text{or} \quad \limsup_{n\to\infty} \int_{\mathbb{R}^N} |v_n|^{2q} dx > 0.$$

By the Lions inequality (Lemma 1.21 in [17]),

$$\int_{\mathbb{R}^N} |u_n|^s dx \le C \left(\int_{\mathbb{R}^N} |\nabla u_n|^2 + |u_n|^2 \right) dx \left(\sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |u_n|^s dx \right)^{1-\frac{2}{s}},$$

$$\int_{\mathbb{R}^N} |v_n|^s dx \le C \left(\int_{\mathbb{R}^N} |\nabla v_n|^2 + |v_n|^2 \right) dx \left(\sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |v_n|^s dx \right)^{1-\frac{2}{s}},$$

where $s \in (2, 2^*)$. Then there exists sequences of points $\{y_n\} \in \mathbb{R}^N$ such that

$$\limsup_{n\to\infty} \int_{B_1(y_n)} |u_n|^{2p} dx > 0, \quad \text{or} \quad \limsup_{n\to\infty} \int_{B_1(y_n)} |v_n|^{2q} dx > 0.$$

Thus we have

$$\limsup_{n\to\infty} \int_{B_1(y_n)} (|u_n|^{2p} + |v_n|^{2q}) dx > 0. \tag{3.5}$$

Define $\widetilde{u}_n := u_n(\cdot + y_n)$, $\widetilde{v}_n := v_n(\cdot + y_n)$. Since the functional J_{V_∞} is invariant under translations, the sequence $\{(\widetilde{u}_n, \widetilde{v}_n)\} \subset X$ is also a bounded $(\mathrm{PS})_c$ sequence of J_{V_∞}. Then by (3.5) there exists some $(u, v) \in X \backslash \{(0,0)\}$ such that

$$\begin{aligned}
&(\widetilde{u}_n, \widetilde{v}_n) \rightharpoonup (u, v) \ \text{ in } X, \\
&\widetilde{u}_n \rightharpoonup u, \ \widetilde{v}_n \rightharpoonup v \ \text{ in } H^1(\mathbb{R}^N), \\
&\widetilde{u}_n \to u, \ \widetilde{v}_n \to v \ \text{ in } L^r_{loc}(\mathbb{R}^N), \ r \in [1, 2^*), \\
&\widetilde{u}_n(x) \to u(x), \ \widetilde{v}_n(x) \to v(x), \ \text{a.e. } x \in \mathbb{R}^N.
\end{aligned}$$

Since $1 < p, q < \sqrt{\frac{N}{N-2}}$ implies that $2 < 2p, 2q, 2pq < 2^*$, we have

$$\int_{\mathbb{R}^N} |\widetilde{v}_n|^{2q} |\widetilde{u}_n|^{2(p-1)} dx \le \left(\int_{\mathbb{R}^N} |\widetilde{v}_n|^{2pq} dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^N} |\widetilde{u}_n|^{2p} dx \right)^{\frac{p-1}{p}} < \infty.$$

That is to say $\{|\widetilde{v}_n|^q |\widetilde{u}_n|^{p-2} \widetilde{u}_n\}$ is bounded in $L^2(\mathbb{R}^N)$. Then by Proposition 5.4.7 in [18],

$$|\widetilde{v}_n|^q |\widetilde{u}_n|^{p-2} \widetilde{u}_n \rightharpoonup |\widetilde{v}|^q |\widetilde{u}|^{p-2} \widetilde{u}, \quad \text{in } L^2(\mathbb{R}^N).$$

Since $\phi \in H^1(\mathbb{R}^N) \subset L^2(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} |\widetilde{v}_n|^q |\widetilde{u}_n|^{p-2} \widetilde{u}_n \phi dx \to \int_{\mathbb{R}^N} |\widetilde{v}|^q |\widetilde{u}|^{p-2} \widetilde{u} \phi dx, \qquad n \to \infty. \tag{3.6}$$

Similarly, we can also get

$$\int_{\mathbb{R}^N} |\widetilde{u}_n|^p |\widetilde{v}_n|^{q-2} \widetilde{v}_n \varphi dx \to \int_{\mathbb{R}^N} |\widetilde{u}|^p |\widetilde{v}|^{q-2} \widetilde{v} \varphi dx, \qquad n \to \infty. \tag{3.7}$$

We now claim that $J'_{V_\infty}(u,v)=0$. For any $(\phi,\varphi)\in X$, by Lemma 2.3, (3.6) and (3.7), we have

$$\begin{aligned}
&\langle J'_{V_\infty}(u_n,v_n),(\phi(x-y_n),\varphi(x-y_n))\rangle\\
&\quad=\int_{\mathbb{R}^N}(\nabla u_n\cdot\nabla\phi(x-y_n)+V_\infty u_n\phi(x-y_n)+\nabla v_n\cdot\nabla\varphi(x-y_n)+V_\infty v_n\varphi(x-y_n))dx\\
&\qquad-\int_{\mathbb{R}^N}(I_\alpha*|u_n|^{\frac{\alpha}{N}+1})|u_n|^{\frac{\alpha}{N}-1}u_n\phi(x-y_n)dx-\int_{\mathbb{R}^N}(I_\alpha*|v_n|^{\frac{\alpha}{N}+1})|v_n|^{\frac{\alpha}{N}-1}v_n\varphi(x-y_n)dx\\
&\qquad-\int_{\mathbb{R}^N}|v_n|^q|u_n|^{p-2}u_n\phi(x-y_n)dx-\int_{\mathbb{R}^N}|u_n|^p|v_n|^{q-2}v_n\varphi(x-y_n)dx\\
&\quad=\int_{\mathbb{R}^N}(\nabla\widetilde{u}_n\cdot\nabla\phi+V_\infty\widetilde{u}_n\phi+\nabla\widetilde{v}_n\cdot\nabla\varphi+V_\infty\widetilde{v}_n\varphi)dx\\
&\qquad-\int_{\mathbb{R}^N}(I_\alpha*|\widetilde{u}_n|^{\frac{\alpha}{N}+1})|\widetilde{u}_n|^{\frac{\alpha}{N}-1}\widetilde{u}_n\phi dx-\int_{\mathbb{R}^N}(I_\alpha*|\widetilde{v}_n|^{\frac{\alpha}{N}+1})|\widetilde{v}_n|^{\frac{\alpha}{N}-1}\widetilde{v}_n\varphi dx\\
&\qquad-\int_{\mathbb{R}^N}|\widetilde{v}_n|^q|\widetilde{u}_n|^{p-2}\widetilde{u}_n\phi dx-\int_{\mathbb{R}^N}|\widetilde{u}_n|^p|\widetilde{v}_n|^{q-2}\widetilde{v}_n\varphi dx\\
&\quad=\langle J'_{V_\infty}(u,v),(\phi,\varphi)\rangle+o_n(1).
\end{aligned}$$

Thus $J'_{V_\infty}(u,v)=0$.

By the Fatou lemma,

$$\begin{aligned}
J_{V_\infty}(u,v)&=J_{V_\infty}(u,v)-\frac{1}{2}\langle J'_{V_\infty}(u,v),(u,v)\rangle\\
&=\frac{\alpha}{2(N+\alpha)}\int_{\mathbb{R}^N}((I_\alpha*|u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}+1}+(I_\alpha*|v|^{\frac{\alpha}{N}+1})|v|^{\frac{\alpha}{N}+1})dx\\
&\quad+\Big(\frac{p+q}{2}-1\Big)\int_{\mathbb{R}^N}|u|^p|v|^q dx\\
&\le\liminf_{n\to\infty}\Big(\frac{\alpha}{2(N+\alpha)}\int_{\mathbb{R}^N}((I_\alpha*|\widetilde{u}_n|^{\frac{\alpha}{N}+1})|\widetilde{u}_n|^{\frac{\alpha}{N}+1}+(I_\alpha*|\widetilde{v}_n|^{\frac{\alpha}{N}+1})|\widetilde{v}_n|^{\frac{\alpha}{N}+1})dx\\
&\quad+\Big(\frac{p+q}{2}-1\Big)\int_{\mathbb{R}^N}|\widetilde{u}_n|^p|\widetilde{v}_n|^q dx\Big)\\
&=\liminf_{n\to\infty}\Big(J_{V_\infty}(\widetilde{u}_n,\widetilde{v}_n)-\frac{1}{2}\langle J'_{V_\infty}(\widetilde{u}_n,v_n),(\widetilde{u}_n,\widetilde{v}_n)\rangle\Big)\\
&=c.
\end{aligned}$$

Thus $J_{V_\infty}(u,v)\le c$.

We finally conclude that

$$\begin{aligned}
J_{V_\infty}(u,v)&=J_{V_\infty}(u,v)-\frac{1}{2}\langle J'_{V_\infty}(u,v),(u,v)\rangle\\
&=\frac{\alpha}{2(N+\alpha)}\int_{\mathbb{R}^N}((I_\alpha*|u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}+1}+(I_\alpha*|v|^{\frac{\alpha}{N}+1})|v|^{\frac{\alpha}{N}+1})dx\\
&\quad+\Big(\frac{p+q}{2}-1\Big)\int_{\mathbb{R}^N}|u|^p|v|^q dx\\
&>0.
\end{aligned}\tag{3.8}$$

Therefore, the lemma follows. □

By Lemma 2.5 and Mountain Pass theorem, there exists a Palais–Smale sequence $\{(u_n,v_n)\}$ of J_{V_∞} at the energy level c_0. It then follows lemma 2.6 that $c_0\in(0,c_*)$. The sequence $\{(u_n,v_n)\}$

is bounded in X. In fact, by taking $\mu \in \left(2, \min\left\{\frac{2(N+\alpha)}{N}, p+q\right\}\right]$, we can get

$$
\begin{aligned}
c_0 + o_n(1) &= J_{V_\infty}(u_n, v_n) - \frac{1}{\mu}\langle J'_{V_\infty}(u_n, v_n), (u_n, v_n)\rangle \\
&= \left(\frac{1}{2} - \frac{1}{\mu}\right) \|(u_n, v_n)\|^2_{V_\infty} \\
&\quad + \left(\frac{1}{\mu} - \frac{N}{2(N+\alpha)}\right)\left(\int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{\alpha}{N}+1})|u_n|^{\frac{\alpha}{N}+1}dx + \int_{\mathbb{R}^N} (I_\alpha * |v_n|^{\frac{\alpha}{N}+1})|v_n|^{\frac{\alpha}{N}+1}dx\right) \\
&\quad + \left(\frac{p+q}{\mu} - 1\right) \int_{\mathbb{R}^N} |u_n|^p |v_n|^q dx \\
&\geq \left(\frac{1}{2} - \frac{1}{\mu}\right) \|(u_n, v_n)\|^2_{V_\infty}.
\end{aligned}
$$

Thus $\{(u_n, v_n)\}$ is bounded in X. Up to a subsequence if necessary, there exists $(u, v) \in X$ such that

$$(u_n, v_n) \rightharpoonup (u, v) \quad \text{in } X, \qquad (u_n(x), v_n(x)) \to (u(x), v(x)) \quad \text{a.e. in } \mathbb{R}^N.$$

Then Lemma 3.2 infers that (u, v) is a nontrivial critical point of functional J_{V_∞} and $J_{V_\infty}(u, v) \in (0, c_0]$.

Let$\{(z_n, w_n)\}$ be a sequence of nontrivial critical points of J_{V_∞} such that

$$\lim_{n\to\infty} J_{V_\infty}(z_n, w_n) = c_g^{V_\infty}.$$

It is easy to see that $c_g^{V_\infty} \leq c_0 < c_*$. By using the same arguments as above, we can get that $\{(z_n, w_n)\}$ is bounded in X. In view of $\langle J'_{V_\infty}(z_n, w_n), (z_n, w_n)\rangle = 0$, it follows that $\{\|(z_n, w_n)\|\}$ has a positive lower bound, which together with (3.8) implies that $c_g^{V_\infty} > 0$. Therefore, $\{(z_n, w_n)\}$ is a $(\mathrm{PS})_{c_g^{V_\infty}}$ sequence of J_{V_∞} with $c_g^{V_\infty} \in (0, c_0]$. It follows from Lemma 3.2 that up to a sequence of $\{(z_n, w_n)\}$ and translations,

$$(z_n, w_n) \rightharpoonup (z, w) \neq 0 \quad \text{in } X, \quad \text{as } n \to \infty, \qquad J'_{V_\infty}(z, w) = 0 \quad \text{and} \quad J_{V_\infty}(z, w) \in (0, c_g^{V_\infty}].$$

Furthermore, by the definition of $c_g^{V_\infty}$, we conclude that $J_{V_\infty}(z, w) = c_g^{V_\infty}$. Hence, (z, w) is a ground state solution of system (1.6). □

4 Proof of Theorem 1.1

Lemma 4.1. *For any solution $(u, v) \in X\backslash\{(0, 0)\}$ of system (1.6), the function $J_{V_\infty}(tu, tv)$, $t \geq 0$ achieves its unique strict global maximum at $t = 1$, that is to say*

$$J_{V_\infty}(u, v) = \max_{t\geq 0} J_{V_\infty}(tu, tv) > J_{V_\infty}(tu, tv), \quad \textit{for } t \geq 0 \textit{ and } t \neq 1.$$

Proof. Let $(u, v) \in X\backslash\{(0, 0)\}$ be a solution of system (1.6), for every $t \geq 0$, we have

$$
\begin{aligned}
J_{V_\infty}(tu, tv) &= \frac{t^2}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V_\infty |u|^2 + |\nabla v|^2 + V_\infty |v|^2) dx \\
&\quad - \frac{N}{2(N+\alpha)} t^{\frac{2\alpha}{N}+2} \int_{\mathbb{R}^N} ((I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}+1} + (I_\alpha * |v|^{\frac{\alpha}{N}+1})|v|^{\frac{\alpha}{N}+1}) dx \\
&\quad - t^{p+q} \int_{\mathbb{R}^N} |u|^p |v|^q dx \\
&= \frac{A}{2} t^2 - \frac{BN}{2(N+\alpha)} t^{\frac{2\alpha}{N}+2} - C t^{p+q},
\end{aligned}
\tag{4.1}
$$

where

$$A := \int_{\mathbb{R}^N} (|\nabla u|^2 + V_\infty |u|^2 + |\nabla v|^2 + V_\infty |v|^2)dx;$$

$$B := \int_{\mathbb{R}^N} ((I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}+1} + (I_\alpha * |v|^{\frac{\alpha}{N}+1})|v|^{\frac{\alpha}{N}+1})dx;$$

$$C := \int_{\mathbb{R}^N} |u|^p |v|^q dx.$$

By (4.1), it is easy to get that $J_{V_\infty}(tu,tv) \in C^1([0,\infty),\mathbb{R})$ and $\lim_{t\to\infty} J_{V_\infty}(tu,tv) = -\infty$. Thus $J_{V_\infty}(tu,tv)$ can achieve its global maximum. Since $0 = \langle J'_{V_\infty}(u,v),(u,v)\rangle = A - B - (p+q)C$, by a direct calculation, we can get that $t = 1$ is the only point such that $\frac{dJ_{V_\infty}(tu,tv)}{dt} = 0$. Then $J_{V_\infty}(tu,tv)$ achieves the unique strict global maximum at $t = 1$. □

Lemma 4.2. *Assume $(V_1), (V_2)$ hold, then there exists a $(PS)_{c_V}$ sequence for J_V with $0 < c_V < c_g^{V_\infty}$.*

Proof. Firstly, we claim that there exists $(u^0, v^0) \in X$ such that $J_V(u^0, v^0) < 0$. Indeed, for any $(u,v) \in X\backslash\{(0,0)\}$, we have $J_V(u,v) < J_{V_\infty}(u,v)$. In view of (4.1), by taking $u^0 = tu, v^0 = tv$ with t large enough, where (u,v) is a ground state solution of system (1.6). Then we get that $J_V(u^0,v^0) < J_{V_\infty}(u^0,v^0) < 0$.

Similar to Lemma 2.5, we see that the functional J_V also enjoys the Mountain Pass geometry. Then we have a minimax description at c_V. We show that

$$c_V := \inf_{\gamma\in \Upsilon} \max_{t\in[0,1]} J_V(\gamma(t)) > \max\{J_V(0,0), J_V(u^0,v^0)\},$$

where

$$\Upsilon = \{\gamma \in C([0,1],X) : \gamma(0) = (0,0), \gamma(1) = (u^0,v^0)\}.$$

In fact, (V_1), (V_2) and (2.1) imply that

$$\begin{aligned} J_V(u,v) &\geq \frac{1}{2}\min\{1,V(x)\}(\|u\|^2 + \|v\|^2) - C_1\left(\|u\|^{\frac{2(N+\alpha)}{N}} + \|v\|^{\frac{2(N+\alpha)}{N}}\right) - \frac{1}{2}\int_{\mathbb{R}^N}(|u|^{2p} + |v|^{2q})dx \\ &\geq \frac{1}{2}\min\{1,V(x)\}\|(u,v)\|^2 - C_1\|(u,v)\|^{\frac{2(N+\alpha)}{N}} - C_2\|(u,v)\|^{2p} - C_3\|(u,v)\|^{2q}, \end{aligned}$$

where C_1, C_2 are positive constants. Since $p,q \in \left(1, \sqrt{\frac{N}{N-2}}\right)$, J_V has a strict local minimum at 0 and then $c_V > 0$.

Next, we show that $c_V < c_g^{V_\infty}$. Let (u,v) be the ground state solution of system (1.6) mentioned above. From the proof of Lemma 4.1 and by using (V_2), we see that

$$c_g^{V_\infty} = J_{V_\infty}(u,v) = \max_{t\geq 0} J_{V_\infty}(tu,tv) > \max_{t\geq 0} J_V(tu,tv) \geq c_V.$$

The proof is complete. □

Proof of Theorem 1.1. The proof is divided into four steps.

Step 1. Let $\{(u_n, v_n)\}$ be a $(\mathrm{PS})_{c_V}$ sequence of functional J_V with $0 < c_V < c_g^{V_\infty}$. Then take $\mu \in \big(2, \min\{\frac{2(N+\alpha)}{N}, p+q\}\big]$, we have

$$\begin{aligned}
c_V + o_n(1) &= J_V(u_n, v_n) - \frac{1}{\mu}\langle J_V'(u_n, v_n), (u_n, v_n)\rangle \\
&= \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\int_{\mathbb{R}^N} (|\nabla u_n|^2 + V(x)|u_n|^2 + |\nabla v_n|^2 + V(x)|v_n|^2)dx \\
&\quad + \Big(\frac{1}{\mu} - \frac{N}{2(N+\alpha)}\Big)\int_{\mathbb{R}^N} ((I_\alpha * |u_n|^{\frac{\alpha}{N}+1})|u_n|^{\frac{\alpha}{N}+1} + (I_\alpha * |v_n|^{\frac{\alpha}{N}+1})|v_n|^{\frac{\alpha}{N}+1})dx \\
&\quad + \Big(\frac{p+q}{\mu} - 1\Big)\int_{\mathbb{R}^N} |u_n|^p|v_n|^q dx \\
&\geq \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\int_{\mathbb{R}^N} (|\nabla u_n|^2 + V(x)|u_n|^2 + |\nabla v_n|^2 + V(x)|v_n|^2)dx.
\end{aligned}$$

Thus (V_1) and (V_2) imply that $\{(u_n, v_n)\}$ is bounded in X. Therefore, there exists $(u, v) \in X$ such that up to a subsequence if necessary,

$$(u_n, v_n) \rightharpoonup (u, v) \quad \text{weakly in } X, \quad (u_n(x), v_n(x)) \to (u(x), v(x)), \text{ for almost every } x \in \mathbb{R}^N.$$

By a similar argument as in the proof of Lemma 3.2, we see that there exists $\{y_n\} \subset \mathbb{R}^N$ such that

$$\limsup_{n\to\infty} \int_{B_1(y_n)} (|u_n|^{2p} + |v_n|^{2q})dx > 0. \tag{4.2}$$

Step 2. We can claim that $\{y_n\}$ is bounded in $\mathbb{R}^N$. In fact, suppose that for a subsequence still denoted by $\{y_n\}$ such that

$$\lim_{n\to\infty} |y_n| \to \infty, \tag{4.3}$$

we define $z_n(\cdot) = u_n(\cdot + y_n)$, $w_n(\cdot) = v_n(\cdot + y_n)$, then $\{(z_n, w_n)\}$ is bounded in X, and by (4.2) $(z_n, w_n) \rightharpoonup (z, w) \neq (0, 0)$. In the following, we will show that $J'_{V_\infty}(z, w) = 0$ and $J_{V_\infty}(z, w) \leq c_V$, which contradict that $c_V < c_g^{V_\infty}$. Hence $\{y_n\}$ is bounded.

In order to prove $J'_{V_\infty}(z, w) = 0$, by (4.3), (V_1), (V_2) and Hölder inequality, for any $(\phi, \varphi) \in X$, we have

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^N} (V(x+y_n) - V_\infty)(z_n(x)\phi(x) + w_n(x)\varphi(x))dx\Big| \\
&\leq \Big|\int_{B_{|y_n|/2}} (V(x+y_n) - V_\infty)(z_n(x)\phi(x) + w_n(x)\varphi(x))dx\Big| \\
&\quad + \Big|\int_{\mathbb{R}^N\setminus B_{|y_n|/2}} (V(x+y_n) - V_\infty)(z_n(x)\phi(x) + w_n(x)\varphi(x))dx\Big| \\
&\leq \sup_{B_{|y_n|/2}} |V(x+y_n) - V_\infty|(|z_n|_{L^2(\mathbb{R}^N)}|\phi|_{L^2(\mathbb{R}^N)} + |w_n|_{L^2(\mathbb{R}^N)}|\varphi|_{L^2(\mathbb{R}^N)}) \\
&\quad + C\Big(|z_n|_{L^2(\mathbb{R}^N)}\Big(\int_{\mathbb{R}^N\setminus B_{|y_n|/2}} |\phi|^2dx\Big)^{\frac{1}{2}} + |w_n|_{L^2(\mathbb{R}^N)}\Big(\int_{\mathbb{R}^N\setminus B_{|y_n|/2}} |\varphi|^2dx\Big)^{\frac{1}{2}}\Big) \\
&= o_n(1).
\end{aligned} \tag{4.4}$$

Thus Lemma 2.3 and (4.4) imply that

$$
\begin{aligned}
&\langle J_V'(u_n,v_n),(\phi(x-y_n),\varphi(x-y_n))\rangle\\
&=\int_{\mathbb{R}^N}(\nabla u_n(x)\nabla\phi(x-y_n)+V(x)u_n(x)\phi(x-y_n))dx\\
&\quad-\int_{\mathbb{R}^N}(I_\alpha*|u_n|^{\frac{\alpha}{N}+1})|u_n|^{\frac{\alpha}{N}-1}u_n\phi(x-y_n)dx\\
&\quad+\int_{\mathbb{R}^N}(\nabla v_n(x)\nabla\varphi(x-y_n)+V(x)v_n(x)\varphi(x-y_n))dx\\
&\quad-\int_{\mathbb{R}^N}(I_\alpha*|v_n|^{\frac{\alpha}{N}+1}|v_n|^{\frac{\alpha}{N}-1}v_n\varphi(x-y_n))dx\\
&\quad-p\int_{\mathbb{R}^N}|v_n|^q|u_n|^{p-2}u_n\phi(x-y_n)dx-q\int_{\mathbb{R}^N}|u_n|^p|v_n|^{q-2}v_n\varphi(x-y_n)dx\\
&=\int_{\mathbb{R}^N}(\nabla z_n(x)\nabla\phi(x)+V(x+y_n)z_n(x)\phi(x))dx\\
&\quad+\int_{\mathbb{R}^N}(\nabla w_n(x)\nabla\varphi(x)+V(x+y_n)w_n(x)\varphi(x))dx\\
&\quad-\int_{\mathbb{R}^N}(I_\alpha*|z_n|^{\frac{\alpha}{N}+1})|z_n|^{\frac{\alpha}{N}-1}z_n\phi dx-\int_{\mathbb{R}^N}(I_\alpha*|w_n|^{\frac{\alpha}{N}+1})|w_n|^{\frac{\alpha}{N}-1}w_n\varphi dx\\
&\quad-p\int_{\mathbb{R}^N}|w_n|^q|z_n|^{p-2}z_n\phi dx-q\int_{\mathbb{R}^N}|z_n|^p|w_n|^{q-2}w_n\varphi dx\\
&=\langle J_{V_\infty}'(z_n,w_n),(\phi,\varphi)\rangle+\int_{\mathbb{R}^N}(V(x+y_n)-V_\infty)(z_n(x)\phi(x)+w_n(x)\varphi(x))dx\\
&=\langle J_{V_\infty}'(z,w),(\phi,\varphi)\rangle+o_n(1).
\end{aligned}
\tag{4.5}
$$

Then from (4.5) we deduce that $J_{V_\infty}'(z,w)=0$.

To prove $J_{V_\infty}(z,w)\le c_V$, by the Fatou lemma, we have

$$
\begin{aligned}
J_{V_\infty}(z,w)&=J_{V_\infty}(z,w)-\frac{1}{2}\langle J_{V_\infty}'(z,w),(z,w)\rangle\\
&=\frac{\alpha}{2(N+\alpha)}\int_{\mathbb{R}^N}((I_\alpha*|z|^{\frac{\alpha}{N}+1})|z|^{\frac{\alpha}{N}+1}+(I_\alpha*|w|^{\frac{\alpha}{N}+1})|w|^{\frac{\alpha}{N}+1})dx\\
&\quad+\Big(\frac{p+q}{2}-1\Big)\int_{\mathbb{R}^N}|z|^p|w|^qdx\\
&\le\liminf_{n\to\infty}\Big[\frac{\alpha}{2(N+\alpha)}\Big(\int_{\mathbb{R}^N}(I_\alpha*|z_n|^{\frac{\alpha}{N}+1})|z_n|^{\frac{\alpha}{N}+1}dx+\\
&\quad\int_{\mathbb{R}^N}(I_\alpha*|w_n|^{\frac{\alpha}{N}+1})|w_n|^{\frac{\alpha}{N}+1}dx\Big)+\Big(\frac{p+q}{2}-1\Big)\int_{\mathbb{R}^N}|z_n|^p|w_n|^qdx\Big]\\
&=\liminf_{n\to\infty}\Big[\frac{\alpha}{2(N+\alpha)}\Big(\int_{\mathbb{R}^N}(I_\alpha*|u_n|^{\frac{\alpha}{N}+1})|u_n|^{\frac{\alpha}{N}+1}dx\\
&\quad+\int_{\mathbb{R}^N}(I_\alpha*|v_n|^{\frac{\alpha}{N}+1})|v_n|^{\frac{\alpha}{N}+1}dx\Big)+\Big(\frac{p+q}{2}-1\Big)\int_{\mathbb{R}^N}|u_n|^p|v_n|^qdx\Big]\\
&=\liminf_{n\to\infty}(J_V(u_n,v_n)-\frac{1}{2}\langle J_V'(u_n,v_n),(u_n,v_n)\rangle)=c_V.
\end{aligned}
\tag{4.6}
$$

Therefore, $J_{V_\infty}(z,w)\le c_V$.

Step 3. We show that (u,v) obtained in step 1 is a nontrivial solution of (1.1) and $J_V(u,v)\in(0,c_V]$. By the classical Sobolev embedding theorem, (4.2) and step 2, we have $(u,v)\ne(0,0)$. In view of Lemma 2.3, Lemma 3.2, (V_1) and (V_2), we can show that (u,v) is a critical point of

J_V. Similarly to the proof of (4.6), we have $J_V(u,v) \le c_V$. Direct calculation gives that

$$\begin{aligned} J_V(u,v) &= J_V(u,v) - \frac{1}{2}\langle J_V'(u,v),(u,v)\rangle \\ &= \frac{\alpha}{2(N+\alpha)}\Big(\int_{\mathbb{R}^N}(I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}+1}dx + \int_{\mathbb{R}^N}(I_\alpha * |v|^{\frac{\alpha}{N}+1})|v|^{\frac{\alpha}{N}+1}dx\Big) \\ &\quad + \Big(\frac{p+q}{2}-1\Big)\int_{\mathbb{R}^N}|u|^p|v|^q dx \\ &> 0. \end{aligned}$$

Thus $0 < J_V(u,v) \le c_V < c_g^{V_\infty}$.

Step 4. We show that there exists a ground state solution of system (1.1). By Step 3 and the definition of c_g^V, we see that $c_g^V < c_g^{V_\infty}$. Let $\{(z_n,w_n)\}$ be a sequence of nontrivial critical points of J_V satisfying $J_V(z_n,w_n) \to c_g^V$ as $n \to \infty$. By using the same arguments as in Step 1, we can show that $\{(z_n,w_n)\}$ is bounded in X. In view of $\langle J_V'(z_n,w_n),(z_n,w_n)\rangle = 0$, it follows that $\{\|(z_n,w_n)\|_X\}$ has a positive lower bound. By similar arguments as step 1, we can show that $c_g^V > 0$. Therefore, $\{(z_n,w_n)\}$ is a $(\mathrm{PS})_{c_g^V}$ sequence of functional J_V with $0 < c_g^V < c_g^{V_\infty}$. Repeating Step 1–Step 3, we obtain some $(z,w) \in X \setminus \{(0,0)\}$ such that $J_V'(z,w) = 0$ and $J_V(z,w) \le c_g^V$. Thus (z,w) is a ground state solution of system (1.1). The proof of Theorem 1.1 is complete. □

Acknowledgements

This work is supported by National Natural Science Foundation of China under Grant Numbers: 11701346, 11801338, 11671239; and Research Project Supported by Shanxi Scholarship Council of China under Grant Number: 2020-005.

References

[1] C. O. Alves, F. Gao, M. Squassina, M. Yang, Singularly perturbed critical Choquard equations, *J. Differential Equations* **263**(2017), No. 7, 3943–3988. https://doi.org/10.1016/j.jde.2017.05.009; MR3670042

[2] P. Chen, X. Liu, Ground states of linearly coupled systems of Choquard type, *Appl. Math. Lett.* **84**(2018), 70–75. https://doi.org/10.1016/j.aml.2018.04.016; MR3808499

[3] X. Li, S. Ma, G. Zhang, Existence and qualitative properties of solutions for Choquard equations with a local term, *Nonlinear Anal. Real World Appl.* **45**(2019), 1–25. https://doi.org/10.1016/j.nonrwa.2018.06.007; MR3854292

[4] E. H. Lieb, Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation, *Stud. Appl. Math.* **57**(1976/77), No. 2, 93–105. https://doi.org/10.1002/sapm197757293; MR0471785

[5] E. H. Lieb, M. Loss, *Analysis*, Graduate Studies in Mathematics, Vol. 14, American Mathematical Society, Providence, RI, 1997. MR1415616

[6] P. L. Lions, The Choquard equation and related questions, *Nonlinear Anal.* **4**(1980), No. 6, 1063–1072. https://doi.org/10.1016/0362-546X(80)90016-4; MR0591299

[7] P. L. Lions, The concentration–compactness principle in the calculus of variations. The locally compact case. I, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **1**(1984), No. 2, 109–145. https://doi.org/10.1016/S0294-1449(16)30428-0; MR778970

[8] G. Li, C. Tang, Existence of a ground state solution for Choquard equation with the upper critical exponent, *Comput. Math. Appl.* **76**(2018), No. 11–12, 2635–2647. https://doi.org/10.1016/j.camwa.2018.08.052; MR3907386

[9] V. Moroz, J. Van Schaftingen, Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics, *J. Funct. Anal.* **265**(2013), No. 2, 153–184. https://doi.org/10.1016/j.jfa.2013.04.007; MR3056699

[10] V. Moroz, J. Van Schaftingen, Groundstates of nonlinear Choquard equations: Hardy–Littlewood–Sobolev critical exponent, *Commun. Contemp. Math.* **17**(2015), No. 5, 155005, 12 pp. https://doi.org/10.1142/S0219199715500054; MR3404747

[11] V. Moroz, J. Van Schaftingen, Existence of groundstates for a class of nonlinear Choquard equations, *Trans. Amer. Math. Soc.* **367**(2015), 6557–6579. https://doi.org/10.1090/S0002-9947-2014-06289-2; MR3356947

[12] V. Moroz, J. Van Schaftingen, A guide to the Choquard equation, *J. Fixed Point Theory Appl.* **19**(2017), 773–813. https://doi.org/10.1007/s11784-016-0373-1; MR3625092

[13] S. I. Pekar, *Untersuchungen über die Elektronentheorie der Kristalle* (in German), Akademie-Verlag, Berlin, 1954. Zbl 0058.45503

[14] R. Penrose, On gravity's role in quantum state reduction, *Gen. Relativity Gravitation* **28**(1996), No. 5, 58–600. https://doi.org/10.1007/BF02105068; MR1386305

[15] J. Van Schaftingen, J. Xia, Choquard equations under confining external potentials, *NoDEA Nonlinear Differential Equations Appl.* **24**(2017), No. 1, Paper No. 1, 24 pp. https://doi.org/10.1007/s00030-016-0424-8; MR3582827

[16] J. Van Schaftingen, J. Xia, Groundstates for a local nonlinear perturbation of the Choquard equations with lower critical exponent, *J. Math. Anal. Appl.* **464**(2018), No. 2, 1184–1202. https://doi.org/10.1016/j.jmaa.2018.04.047; MR3798653

[17] M. Willem, *Minimax theorems*, Birkhäuser, Boston, 1996. https://doi.org/10.1007/978-1-4612-4146-1; MR1400007

[18] M. Willem, *Functional analysis. Fundamentals and applications*, Cornerstones, Vol. XIV, Birkhäuser, Basel, 2013. https://doi.org/10.1007/978-1-4614-7004-5; MR3112778

[19] M. Yang, J. de Albuquerque, E. Silva, M. Silva, On the critical cases of linearly coupled Choquard systems, *Appl. Math. Lett.* **91**(2019), 1–8. https://doi.org/10.1016/j.aml.2018.11.005; MR3886985