Electronic Journal of Qualitative Theory of Differential Equations
2020, No. **34**, 1–7; https://doi.org/10.14232/ejqtde.2020.1.34 www.math.u-szeged.hu/ejqtde/

Polynomial differential systems with hyperbolic algebraic limit cycles

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Received 30 March 2020, appeared 29 May 2020
Communicated by Gabriele Villari

Abstract. For a given algebraic curve of degree n, we exhibit differential systems of degree greater than or equal to n, by introducing functions which are solutions of certain partial differential equations. These systems admit precisely the bounded components of the curve as limit cycles.

Keywords: sixteenth problem of Hilbert, planar differential system, invariant curve, periodic solution, hyperbolic limit cycle.

2020 Mathematics Subject Classification: 34C25, 34C05, 34C07.

1 Introduction

The second part of the sixteenth problem of Hilbert still persists as a research area. It aims to find the maximum number of limit cycles of the differential system:

$$\begin{aligned} \dot{x} &= \frac{dx}{dt} = P(x,y), \\ \dot{y} &= \frac{dy}{dt} = Q(x,y), \end{aligned} \tag{1.1}$$

where P and Q are polynomials.

Several articles and books have been published on the analysis of the existence, number and stability of limit cycles of equation (1.1) (see for instance [5,6,8,9,15,18]).

Generally, the exact analytical expressions of limit cycles for a given differential system are unknown, except in specific cases.

This paper is a contribution in the direction of determining the number of limit cycles and giving their explicit form.

Motivated by some publications [1–4,7,11–14,16], we will exhibit polynomial vector fields, where just by choosing the components of the system satisfying certain conditions, we can conclude directly the number and the explicit form of limit cycles.

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2 Introductory concepts

Let us recall some useful notions.

For $U \in \mathbb{R}[x,y]$, the algebraic curve $U=0$ is called an invariant curve of the polynomial system (1.1), if for some polynomial $K \in \mathbb{R}[x,y]$, called the cofactor of the algebraic curve, we have

$$P(x,y)\frac{\partial U}{\partial x} + Q(x,y)\frac{\partial U}{\partial y} = KU. \tag{2.1}$$

Simple analysis of equation (2.1) shows that when $\max(\deg P, \deg Q) = n$, the degree of the cofactor K is at most $n-1$ and that the curve $U=0$ is formed by trajectories of the system (1.1).

The curve $\Omega = \left\{(x,y) \in \mathbb{R}^2, U(x,y) = 0\right\}$ is a non-singular curve of system (1.1), if the equilibrium points of the system that satisfy

$$\begin{aligned} P(x,y) &= 0, \\ Q(x,y) &= 0 \end{aligned} \tag{2.2}$$

are not contained on the curve Ω.

A limit cycle $\Gamma = \{(x(t), y(t)), t \in [0,T]\}$ is a T-periodic solution isolated with respect to all other possible periodic solutions of the system.

A T-periodic solution Γ is a hyperbolic limit cycle if $\int_0^T \operatorname{div}(\Gamma)dt$ is different from zero.

By using the method of characteristics to solve partial differential equations, we conclude that, the solution of equation

$$\alpha\frac{\partial f}{\partial x} + \beta\frac{\partial f}{\partial y} = 0 \tag{2.3}$$

is

$$f(x,y) = \Phi(\beta x - \alpha y), \tag{2.4}$$

where α, β are nonzero reals and Φ is an arbitrary function.

The solution of the equation

$$\alpha\frac{\partial f}{\partial x} + \beta\frac{\partial f}{\partial y} = \gamma \tag{2.5}$$

is the function f solving the equation

$$\Psi(\beta x - \alpha y, \gamma x - \alpha f) = 0, \tag{2.6}$$

where α, β, γ are nonzero reals and Ψ is an arbitrary function. In the polynomial case

$$f(x,y) = \frac{\gamma}{\alpha}x + \sum_{k=0}^{n} c_k (\beta x - \alpha y)^k \tag{2.7}$$

or

$$f(x,y) = \frac{\gamma}{\beta}y + \sum_{k=0}^{n} c_k (\beta x - \alpha y)^k \tag{2.8}$$

the solution of the equation

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = f \tag{2.9}$$

is the function f solving the equation

$$\Psi\left(\frac{x}{f},\frac{y}{f}\right)=0. \tag{2.10}$$

In the polynomial case it can be taken as

$$f(x,y)=ax+by. \tag{2.11}$$

Colin Christopher in his article [7] gives the following theorem.

Theorem 2.1. *Let $U=0$ be a non-singular algebraic curve of degree m, and D a first degree polynomial, chosen so that the line $D=0$ lies outside all bounded components of $U=0$. Choose the constants α and β so that $\alpha D_x+\beta D_y\neq 0$, then the polynomial vector field of degree m,*

$$\begin{aligned}\dot{x}&=\alpha U+DU_y,\\ \dot{y}&=\beta U-DU_x\end{aligned} \tag{2.12}$$

has all the bounded components of $U=0$ as hyperbolic limit cycles. Furthermore, the vector field has no other limit cycles.

Our contribution is a generalization, which consists in introducing polynomial functions to system (2.12) and in the study of the existence of limit cycles.

3 The main result

We start by adding a polynomial function of any degree to system (2.12), which becomes,

$$\begin{aligned}\dot{x}&=\alpha U+(ax+by+\Phi(\beta x-\alpha y))U_y,\\ \dot{y}&=\beta U-(ax+by+\Phi(\beta x-\alpha y))U_x\end{aligned} \tag{3.1}$$

and we show that system (3.1) has all the bounded components of $U=0$ as hyperbolic limit cycles if the conditions of Theorem 1 of [7] are satisfied.

Theorem 3.1. *Let $U=0$ be a non-singular algebraic curve of degree m, and Φ a polynomial function of degree n, chosen so that the curve $ax+by+\Phi(\beta x-\alpha y)=0$ lies outside all bounded components of $U=0$. Choose the constants a and b so that $a\alpha+b\beta\neq 0$, then the polynomial vector field of degree $m+n-1$,*

$$\begin{cases}\dot{x}=\alpha U+(ax+by+\Phi(\beta x-\alpha y))U_y,\\ \dot{y}=\beta U-(ax+by+\Phi(\beta x-\alpha y))U_x\end{cases}$$

has all the bounded components of $U=0$ as hyperbolic limit cycles.

Proof. Let Γ be the curve of $U=0$.

Note that Γ is a non-singular curve of system (3.1) and the curve $ax+by+\Phi(\beta x-\alpha y)=0$ lies outside all bounded components of Γ.

To show that all the bounded components of Γ are hyperbolic limit cycles of system (3.1), we will prove that Γ is an invariant curve of the system (3.1), and $\int_0^T \operatorname{div}(\Gamma)dt\neq 0$ (see for instance Perko [17]).

i) Γ is an invariant curve of the system (3.1):

$$\begin{aligned}\frac{dU}{dt} &= U_x\left(\alpha U + (ax+by+\Phi(\beta x-\alpha y))U_y\right) + U_y\left(\beta U - (ax+by+\Phi(\beta x-\alpha y))U_x\right)\\ &= \left(\alpha U_x + \beta U_y\right)U\end{aligned}$$

where the cofactor is $K(x,y) = \alpha U_x + \beta U_y$.

ii) $\int_0^T \operatorname{div}(\Gamma)dt$ is nonzero.

To see this, first note that

$$\int_0^T \operatorname{div}(\Gamma)dt = \int_0^T K(x(t),y(t))dt, \tag{3.2}$$

see for instance Giacomini & Grau [10]. Then one has

$$\begin{aligned}\int_0^T K(x(t),y(t))dt &= \oint_\Gamma \frac{\alpha U_x}{-(ax+by+\Phi(\beta x-\alpha y))U_x}dy + \oint_\Gamma \frac{\beta U_y}{(ax+by+\Phi(\beta x-\alpha y))U_y}dx\\ &= \oint_\Gamma \frac{\alpha}{-(ax+by+\Phi(\beta x-\alpha y))}dy + \oint_\Gamma \frac{\beta}{(ax+by+\Phi(\beta x-\alpha y))}dx.\end{aligned}$$

Let $\omega = \beta x - \alpha y$. By applying Green's formula we obtain

$$\begin{aligned}&\oint_\Gamma \frac{\beta}{(ax+by+\Phi(\omega))}dx - \oint_\Gamma \frac{\alpha}{(ax+by+\Phi(\omega))}dy\\ &= \int\int_{\operatorname{int}(\Gamma)} \left(\frac{\partial\left(\frac{\beta}{(ax+by+\Phi(\omega))}\right)}{\partial y} + \frac{\partial\left(\frac{\alpha}{(ax+by+\Phi(\omega))}\right)}{\partial x}\right)dxdy\\ &= \int\int_{\operatorname{int}(\Gamma)} \left(\frac{-\beta\left(b+\frac{\partial\Phi}{\partial\omega}(-\alpha)\right)}{(ax+by+\Phi(\omega))^2} + \frac{-\alpha\left(a+\frac{\partial\Phi}{\partial\omega}(\beta)\right)}{(ax+by+\Phi(\omega))^2}\right)dxdy\\ &= -\int\int_{\operatorname{int}(\Gamma)} \left(\frac{\beta\left(b+\frac{\partial\Phi}{\partial\omega}(-\alpha)\right)}{(ax+by+\Phi(\omega))^2} + \frac{\alpha\left(a+\frac{\partial\Phi}{\partial\omega}(\beta)\right)}{(ax+by+\Phi(\omega))^2}\right)dxdy\\ &= -\int\int_{\operatorname{int}(\Gamma)} \left(\frac{\beta b+\alpha a}{(ax+by+\Phi(\omega))^2}\right)dxdy,\end{aligned}$$

where $\operatorname{int}(\Gamma)$ denotes the interior of Γ.

As $\alpha a + \beta b \neq 0$, $\int_0^T K(x(t),y(t))dt$ is nonzero. □

Remark 3.2. When $\Phi(\beta x - \alpha y)$ is constant, we find ourselves in the case of Cristopher's theorem (i.e. Theorem 2.1).

When $\Phi(\beta x - \alpha y)$ is of first degree, the line $ax + by + c = 0$ in Christopher's theorem will be replaced by the line $(a+\beta)x + (b-\alpha)y + d = 0$.

Example 3.3 (Quintic system with exactly one limit cycle)**.** Let $\alpha = 1, \beta = 2, a = 1, b = 2$, $\Phi(\beta x - \alpha y) = \Phi(2x - y) = (2x-y)^2 + 1$.

The system

$$\begin{aligned}\dot{x} &= x^4 + y^2 - 4y - 3x + 5 + (x + 2y + (2x-y)^2 + 1)(2y-4),\\ \dot{y} &= 2\left(x^4 + y^2 - 4y - 3x + 5\right) - (x + 2y + (2x-y)^2 + 1)\left(4x^3 - 3\right)\end{aligned} \tag{3.3}$$

admits one hyperbolic limit cycle represented by the curve $x^4 + y^2 - 4y - 3x + 5 = 0$. See Figure 3.1.

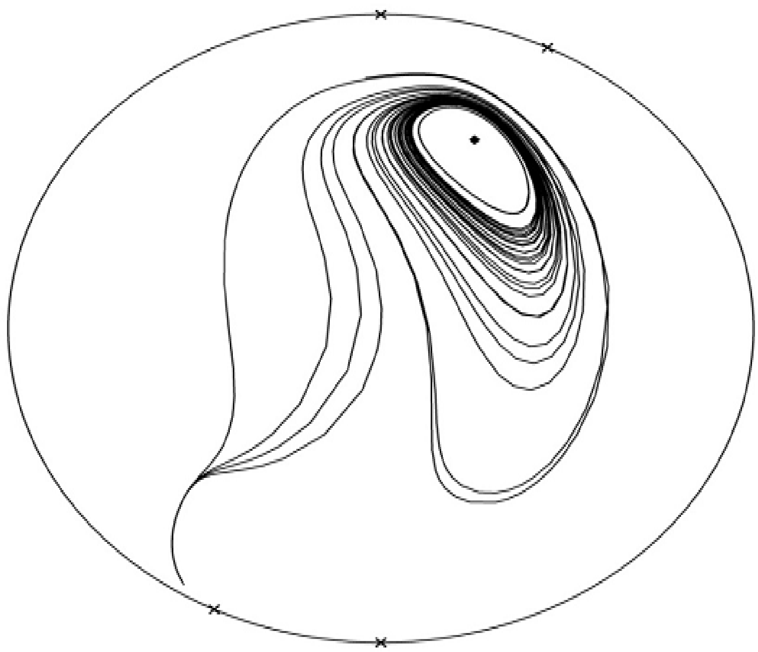

Figure 3.1: Limit cycle of system (3.3).

Remark 3.4. Let us consider the system

$$\begin{aligned} \dot{x} &= \alpha U + f(x,y)U_y, \\ \dot{y} &= \beta U - f(x,y)U_x, \end{aligned} \tag{3.4}$$

where U and f are C^1 functions on an open subset V of $\mathbb{R}^2$. To have all the bounded components of $U = 0$ as limit cycles it is necessary that f satisfies the partial differential equation

$$\alpha \frac{\partial f}{\partial x} + \beta \frac{\partial f}{\partial y} = \gamma, \quad \text{where } \gamma \neq 0. \tag{3.5}$$

In the polynomial case $f(x,y) = \frac{\gamma}{\alpha} x + \Phi(\beta x - \alpha y)$ or $f(x,y) = \frac{\gamma}{\beta} y + \Phi(\beta x - \alpha y)$, which are just particular cases of Theorem 3.1.

Example 3.5 (Quintic system with exactly two limit cycles)**.** Let $\alpha = 1$, $\beta = -1$, $\gamma = 3$, $f(x,y) = 3x + (x+y)^2$.

The system

$$\begin{aligned} \dot{x} &= x^3 - 2xy^2 + 10xy - 15x + y^4 - 10y^3 + 35y^2 - 50y + 30 \\ &\quad + \left((x+y)^2 + 3x\right)\left(4y^3 - 30y^2 - 4xy + 10x + 70y - 50\right), \\ \dot{y} &= 2\left(x^3 - 2xy^2 + 10xy - 15x + y^4 - 10y^3 + 35y^2 - 50y + 30\right) \\ &\quad - \left((x+y)^2 + 3x\right)\left(3x^2 - 2y^2 + 10y - 15\right) \end{aligned} \tag{3.6}$$

admits two hyperbolic limit cycles represented by the curve $x^3 - 2xy^2 + 10xy - 15x + y^4 - 10y^3 + 35y^2 - 50y + 30 = 0$. See Figure 3.2.

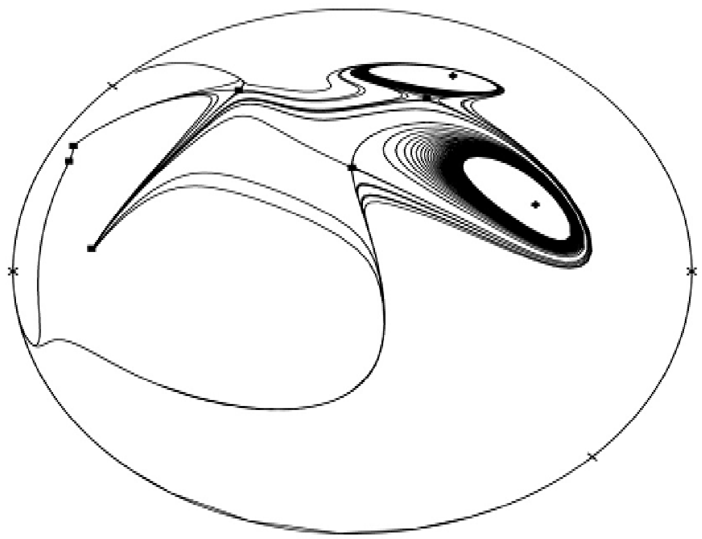

Figure 3.2: Limit cycles of system (3.6).

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