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Bifurcation curves of positive solutions for the Minkowski-curvature problem with cubic nonlinearity

Shao-Yuan Huang ✉ [1] and **Min-Shu Hwang** [2]

[1] Department of Mathematics and Information Education National Taipei University of Education, Taipei 106, Taiwan
[2] Department of Mathematics, National Tsing Hua University, Hsinchu 300, Taiwan

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Abstract. In this paper, we study the shape of bifurcation curve S_L of positive solutions for the Minkowski-curvature problem

$$\begin{cases} -\left(\dfrac{u'(x)}{\sqrt{1-(u'(x))^2}}\right)' = \lambda\left(-\varepsilon u^3+u^2+u+1\right), & -L<x<L, \\ u(-L)=u(L)=0, \end{cases}$$

where $\lambda, \varepsilon > 0$ are bifurcation parameters and $L > 0$ is an evolution parameter. We prove that there exists $\varepsilon_0 > 0$ such that the bifurcation curve S_L is monotone increasing for all $L > 0$ if $\varepsilon \geq \varepsilon_0$, and the bifurcation curve S_L is from monotone increasing to S-shaped for varying $L > 0$ if $0 < \varepsilon < \varepsilon_0$.

Keywords: bifurcation curve, positive solution, Minkowski-curvature problem.

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1 Introduction and main result

In this paper, we study the shapes of bifurcation curves of positive solutions $u \in C^2(-L,L) \cap C[-L,L]$ for the one-dimensional Minkowski-curvature problem

$$\begin{cases} -\left(\dfrac{u'(x)}{\sqrt{1-(u'(x))^2}}\right)' = \lambda f(u), & -L<x<L, \\ u(-L)=u(L)=0, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a bifurcation parameter, $L > 0$ is an evolution parameter and the nonlinearity

$$f(u) \equiv -\varepsilon u^3 + u^2 + u + 1, \qquad \varepsilon > 0. \tag{1.2}$$

✉ Corresponding author. Email: syhuang@mail.ntue.edu.tw

It is well-known that studying the multiplicity of positive solutions of problem (1.1) is equivalent to studying the shape of bifurcation curve S_L of (1.1) where

$$S_L \equiv \{(\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of (1.1)}\} \quad \text{for } L > 0. \tag{1.3}$$

Thus this investigation is essential.

Before going into further discussions on problems (1.1), we give some terminologies in this paper for the shape of bifurcation curve S_L on the $(\lambda, \|u\|_\infty)$-plane.

Definition 1.1. Let S_L be the bifurcation curve of (1.1) on the $(\lambda, \|u\|_\infty)$-plane.

(i) **S-like shaped:** The curve S_L is said to be *S-like shaped* if S_L has at least two turning points at some points $(\lambda_1, \|u_{\lambda_1}\|_\infty)$ and $(\lambda_2, \|u_{\lambda_2}\|_\infty)$ where $\lambda_1 < \lambda_2$ are two positive numbers such that:

(a) at $(\lambda_1, \|u_{\lambda_1}\|_\infty)$ the bifurcation curve S_L turns to the right,

(b) $\|u_{\lambda_2}\|_\infty < \|u_{\lambda_1}\|_\infty$,

(c) at $(\lambda_2, \|u_{\lambda_2}\|_\infty)$ the bifurcation curve S_L turns to the left.

(ii) **S-shaped:** The curve S_L is said to be *S-shaped* if S_L is S-like shaped, has exactly two turning points, and has at most three intersection points with any vertical line on the $(\lambda, \|u\|_\infty)$-plane.

(iii) **Monotone increasing:** The curve S_L is said to be *monotone increasing* if $\lambda_1 < \lambda_2$ for any two points $(\lambda_i, \|u_{\lambda_i}\|_\infty)$, $i = 1, 2$, lying in S_L with $\|u_{\lambda_1}\|_\infty \le \|u_{\lambda_2}\|_\infty$.

Crandall and Rabinowitz [2, p. 177] first considered shape of bifurcation curve of positive solutions for the n-dimensional *semilinear* problem

$$\begin{cases} -\Delta u(x) = \lambda\left(-\varepsilon u^3 + u^2 + u + 1\right) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where Ω is a general bounded domain in $\mathbb{R}^n$ $(n \ge 1)$ with smooth boundary $\partial\Omega$. They applied the implicit function theorem and perturbation arguments to prove that the bifurcation curve of positive solutions of (1.4) is S-like shaped on the $(\lambda, \|u_\lambda\|_\infty)$-plane when $\varepsilon > 0$ is sufficiently small. Shi [17, Theorem 4.1] proved that the bifurcation curve of positive solutions of (1.4) is S-shaped when $\varepsilon > 0$ is small and Ω is a ball in $\mathbb{R}^n$ with $1 \le n \le 6$. Hung and Wang [6] consider the one-dimensional case

$$\begin{cases} -u''(x) = \lambda\left(-\varepsilon u^3 + u^2 + u + 1\right), & -1 < x < 1, \\ u(-1) = u(1) = 0. \end{cases} \tag{1.5}$$

Then they provided the complete variational process of shape of bifurcation curve $\bar{S}$ of (1.5) with varying $\varepsilon > 0$ where

$$\bar{S} \equiv \{(\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of (1.5)}\}, \tag{1.6}$$

see Theorem 1.2.

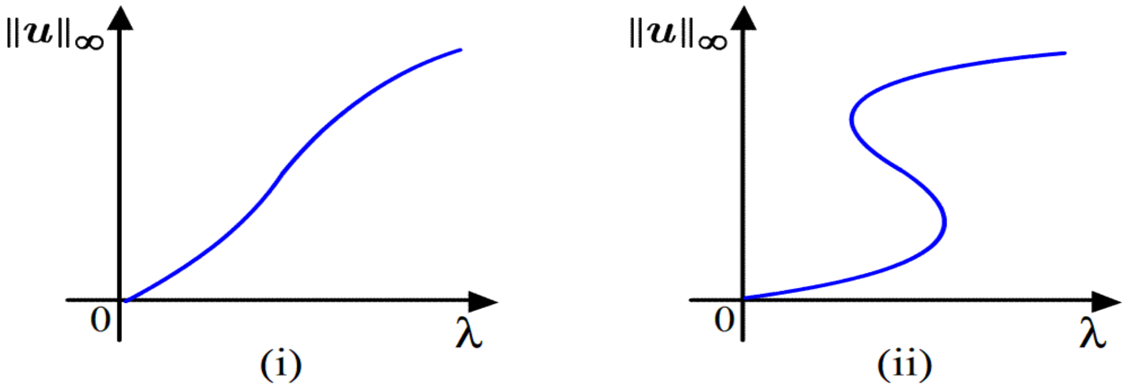

Figure 1.1: Graphs of bifurcation curves $\bar{S}$ of (1.4). (i) $\varepsilon \geq \varepsilon_0$ and (ii) $0 < \varepsilon < \varepsilon_0$.

Theorem 1.2 ([6, Theorem 3.1])**.** *Consider (1.5). Then the bifurcation curve $\bar{S}$ is continuous on the $(\lambda, \|u_\lambda\|_\infty)$-plane, starts from $(0,0)$ and goes to infinity. Furthermore, there exists a critical bifurcation value $\varepsilon_0 \in (0, 1/\sqrt{27})$ such that the bifurcation curve $\bar{S}$ is monotone increasing if $\varepsilon \geq \varepsilon_0$, and $\bar{S}$ is S-shaped if $0 < \varepsilon < \varepsilon_0$, see Figure 1.1.*

To the best of my knowledge, there are no manuscripts to describe the variational process for S_L of (1.5) with varying $\varepsilon, L > 0$. Hence we start to concern this issue. In addition, references [7, 8, 16] provided some sufficient conditions to determine the shape of bifurcation curve or multiplicity of positive solutions of problem (1.1) with general $f(u) \in C[0, \infty)$. However, these results can not be applied in our problem (1.1) because the cubic nonlinearity $f(u)$ defined by (1.2) is not always positive in $[0, \infty)$. So studying the problem (1.1) is worth and interesting.

By elementary analysis, we find that $f(u)$ has unique zero β_ε in $[0, \infty)$. Then the main result is as follows:

Theorem 1.3 (See Figure 1.2)**.** *Consider (1.1). Let ε_0 be defined in Theorem 1.2. Then the following statements (i)–(iii) hold:*

(i) For $L > 0$, the bifurcation curve S_L is continuous on the $(\lambda, \|u_\lambda\|_\infty)$-plane, starts from $(0,0)$ and goes to infinity along the horizontal line $\|u\|_\infty = \rho_{L,\varepsilon}$ where $\rho_{L,\varepsilon} \equiv \min\{L, \beta_\varepsilon\}$.

(ii) If $\varepsilon \geq \varepsilon_0$, then the bifurcation curve S_L is monotone increasing for all $L > 0$.

(iii) If $0 < \varepsilon < \varepsilon_0$, then there exist two positive numbers $L_\varepsilon < \tilde{L}_\varepsilon$ such that

(a) the bifurcation curve S_L is monotone increasing for $0 < L \leq L_\varepsilon$.

(b) the bifurcation curve S_L is S-like shaped for $L_\varepsilon < L \leq \tilde{L}_\varepsilon$.

(c) the bifurcation curve S_L is S-shaped for $L > \tilde{L}_\varepsilon$.

Furthermore, L_ε is a continuous function of $\varepsilon \in (0, \varepsilon_0)$, $\lim_{\varepsilon \to 0^+} L_\varepsilon \in (0, \infty)$ and $\lim_{\varepsilon \to \varepsilon_0^-} L_\varepsilon = \infty$.

Remark 1.4. By numerical simulations to bifurcation curves S_L of (1.1), we conjecture that the bifurcation curve S_L is also S-shaped on the $(\lambda, \|u_\lambda\|_\infty)$-plane for $L_\varepsilon < L \leq \tilde{L}_\varepsilon$ and $0 < \varepsilon < \varepsilon_0$. Further investigations are needed. In addition, by Theorems 1.2 and 1.3, we make a list which shows the different properties for Minkowski-curvature problem (1.1) and semilinear problem (1.4), see Table 1.

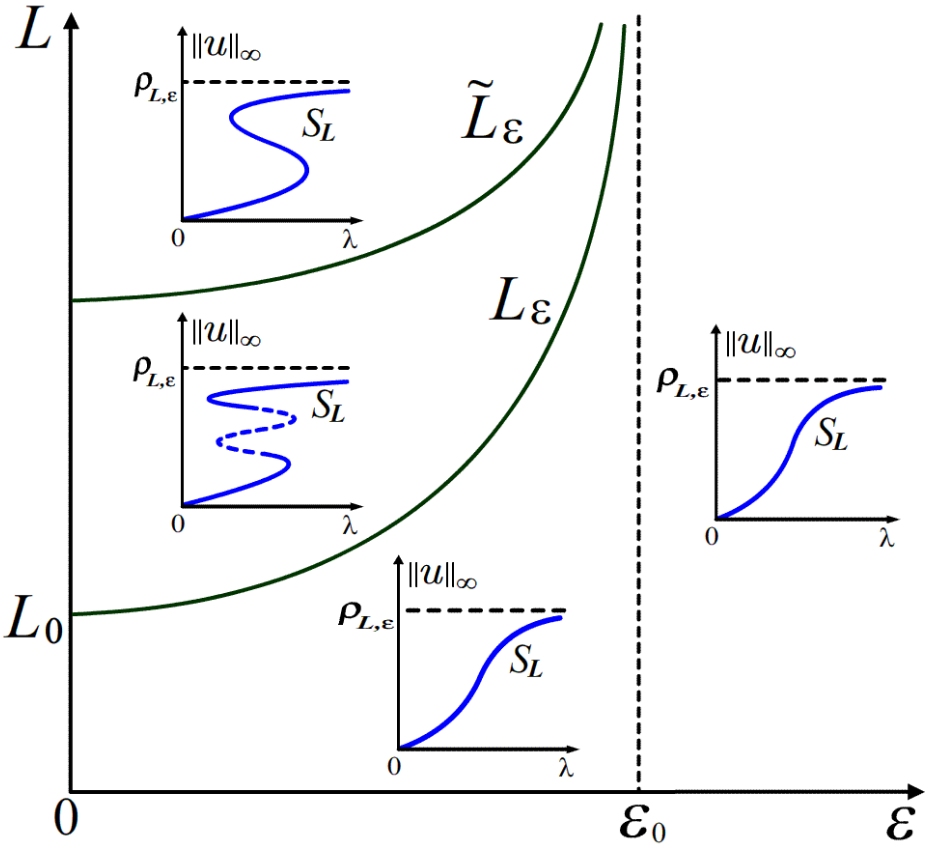

Figure 1.2: Graphs of bifurcation curve S_L of (1.1) for $\varepsilon > 0$.

Bifurcation curve	S_L of (1.1)	$\bar{S}$ of (1.4)
1. Shapes ($0 < \varepsilon < \varepsilon_0$)	from monotone increasing to S-shaped with varying ε	S-shaped
2. Shapes ($\varepsilon \geq \varepsilon_0$)	monotone increasing	monotone increasing
3. Numbers of turning points	(1). from 0 to 2 varying $L > 0$ if $0 < \varepsilon < \varepsilon_0$ (2). 0 if $\varepsilon \geq \varepsilon_0$	(1). 2 if $0 < \varepsilon < \varepsilon_0$ (2). 0 if $\varepsilon \geq \varepsilon_0$
4. Continuity	continuous	continuous
5. Evolution parameter(s)	ε and L	ε
6. Starting point	$(0, 0)$	$(0, 0)$
7. "End point"	$(\infty, \rho_{L,\varepsilon})$	(∞, ∞)

Table 1.1: Comparison of properties of S_L and $\bar{S}$.

The paper is organized as follows: Section 2 contains the lemmas used for proving the main result. Section 3 contains the proof of main result (Theorem 1.3). Section 4 contains the proof of assertion (2.31).

2 Lemmas

To prove Theorem 1.3, we first introduce the time-map method used in Corsato [4, p. 127]. We define the time-map formula for (1.1) by

$$T_\lambda(\alpha) \equiv \int_0^\alpha \frac{\lambda\,[F(\alpha)-F(u)]+1}{\sqrt{\{\lambda\,[F(\alpha)-F(u)]+1\}^2-1}}du \quad \text{for } 0<\alpha<\beta_\varepsilon \text{ and } \lambda>0, \tag{2.1}$$

where $F(u) \equiv \int_0^u f(t)dt$. Observe that positive solutions $u_\lambda \in C^2(-L,L) \cap C[-L,L]$ for (1.1) correspond to

$$\|u_\lambda\|_\infty = \alpha \quad \text{and} \quad T_\lambda(\alpha) = L.$$

So by definition of S_L in (1.3), we have that

$$S_L = \{(\lambda,\alpha) : T_\lambda(\alpha) = L \text{ for some } 0<\alpha<\beta_\varepsilon \text{ and } \lambda>0\}. \tag{2.2}$$

Thus, it is important to understand fundamental properties of the time-map $T_\lambda(\alpha)$ on $(0,\beta_\varepsilon)$ in order to study the shape of the bifurcation curve S_L of (1.1) for any fixed $L>0$. Note that it can be proved that $T_\lambda(\alpha)$ is a triple differentiable function of $\varepsilon \in (0,\beta_\varepsilon)$ for $\varepsilon,\lambda>0$, and $T_\lambda(\alpha)$, $T'_\lambda(\alpha)$ are differentiable function of $\lambda>0$ for $0<\alpha<\beta_\varepsilon$ and $a>0$. The proofs are easy but tedious and hence we omit them. Similarly, we define the time-map formula for (1.5) by

$$\bar{T}(\alpha) \equiv \frac{1}{\sqrt{2}}\int_0^\alpha \frac{1}{\sqrt{F(\alpha)-F(u)}}du \quad \text{for } \alpha>0, \tag{2.3}$$

see [12, p. 779]. Then we have that $\|u_\lambda\|_\infty = \alpha$ and $\bar{T}(\alpha) = \sqrt{\lambda}$. So by the definition of $\bar{S}$ in (1.6), we see that

$$\bar{S} = \left\{(\lambda,\alpha) : \sqrt{\lambda} = \bar{T}(\alpha) \text{ for some } \alpha>0\right\}. \tag{2.4}$$

For the sake of convenience, we let

$$A = A(\alpha,u) \equiv \alpha f(\alpha) - uf(u), \qquad B = B(\alpha,u) \equiv F(\alpha)-F(u),$$

$$C = C(\alpha,u) \equiv \alpha^2 f'(\alpha) - u^2 f'(u) \quad \text{and} \quad D = D(\alpha,u) \equiv \alpha^3 f''(\alpha) - u^3 f''(u).$$

Obviously, we have

$$B(\alpha,u) = \int_u^\alpha f(t)dt > 0 \quad \text{for } 0<u<\alpha<\beta_\varepsilon \tag{2.5}$$

because $f(u)>0$ for $0<u<\beta_\varepsilon$.

Lemma 2.1. *Consider (1.1) with $\varepsilon>0$. Then the following statements (i)–(iii) hold:*

(i) $\lim_{\alpha\to 0^+} T_\lambda(\alpha) = 0$ *and* $\lim_{\alpha\to\beta_\varepsilon^-} T_\lambda(\alpha) = \infty$ *for* $\lambda>0$.

(ii) $\lim_{\lambda\to 0^+} \sqrt{\lambda}T_\lambda^{(i)}(\alpha) = \bar{T}^{(i)}(\alpha)$ *and* $\lim_{\lambda\to\infty} T'_\lambda(\alpha) = 1$ *for* $0<\alpha<\beta_\varepsilon$ *and* $i=1,2,3$.

(iii) $\partial T_\lambda(\alpha)/\partial\lambda < 0$ *for* $0<\alpha<\beta_\varepsilon$ *and* $\lambda>0$.

Proof. Since

$$\lim_{u\to 0^+} \frac{F(u)}{u^2} = \infty,$$

and by [7, Lemma 3.1], we obtain that $\lim_{\alpha\to 0^+} T_\lambda(\alpha) = 0$. Since $f(\beta_\varepsilon) = 0$, there exist $b, c \in \mathbb{R}$ such that $f(u) = (\beta_\varepsilon - u)(\varepsilon u^2 + bu + c)$. Since $f(u) > 0$ on $(0, \beta_\varepsilon)$, there exists $M > 0$ such that $0 < \varepsilon u^2 + bu + c < M$ for $0 < u < \beta_\varepsilon$. For $0 < t < 1$, by the mean-value theorem, there exists $\eta_t \in (\beta_\varepsilon t, \beta_\varepsilon)$ such that

$$\begin{aligned} B(\beta_\varepsilon, \beta_\varepsilon t) &= \int_{\beta_\varepsilon t}^{\beta_\varepsilon} f(t)dt = f(\eta_t)\beta_\varepsilon (1-t) = (\beta_\varepsilon - \eta_t)\left(\varepsilon\eta_t^2 + b\eta_t + c\right)\beta_\varepsilon (1-t) \\ &< (\beta_\varepsilon - \beta_\varepsilon t) M\beta_\varepsilon (1-t) = M\beta_\varepsilon^2 (1-t)^2 . \end{aligned} \tag{2.6}$$

Then there exists $t^* \in (0,1)$ such that $B(\beta_\varepsilon, \beta_\varepsilon t) < 1$ for $t^* < t < 1$. So by (2.5) and (2.6), we see that

$$\begin{aligned} \lim_{\alpha\to\beta_\varepsilon^-} T_\lambda(\alpha) &= \lim_{\alpha\to\beta_\varepsilon^-} \alpha \int_0^1 \frac{\lambda B(\alpha,\alpha t) + 1}{\sqrt{\lambda^2 B^2(\alpha,\alpha t) + 2\lambda B(\alpha,\alpha t)}} dt \\ &\geq \lim_{\alpha\to\beta_\varepsilon^-} \alpha \int_{t^*}^1 \frac{1}{\sqrt{\lambda^2 B^2(\alpha,\alpha t) + 2\lambda B(\alpha,\alpha t)}} dt \\ &\geq \beta_\varepsilon \int_{t^*}^1 \frac{1}{\sqrt{(\lambda^2 + 2\lambda)\, B(\beta_\varepsilon, \beta_\varepsilon t)}} dt \geq \frac{1}{\sqrt{(\lambda^2+2\lambda)\, M}} \int_{t^*}^1 \frac{1}{1-t} dt = \infty, \end{aligned}$$

which implies that statement (i) holds. In addition, we compute that, for $0 < \alpha < \beta_\varepsilon$ and $\lambda > 0$,

$$T_\lambda'(\alpha) = \frac{1}{\alpha}\int_0^\alpha \frac{\lambda^3 B^3 + 3\lambda^2 B^2 + \lambda\,(2B - A)}{\left(\lambda^2 B^2 + 2\lambda B\right)^{3/2}} du, \tag{2.7}$$

$$T_\lambda''(\alpha) = \frac{1}{\alpha^2}\int_0^\alpha \frac{\left(3A^2 B - B^2 C - 2AB^2\right)\lambda^3 + \left(3A^2 - 4AB - 2BC\right)\lambda^2}{\left(\lambda^2 B^2 + 2\lambda B\right)^{5/2}} du, \tag{2.8}$$

$$\begin{aligned} T_\lambda'''(\alpha) = \frac{1}{\alpha^3}\int_0^\alpha \frac{\lambda^3}{\left[\lambda^2 B^2 + 2\lambda B\right]^{7/2}} \Big[& B^2\left(9A^2 B - 3B^2 C - B^2 D - 12A^3 + 9ABC\right)\lambda^2 \\ & + B(27A^2 B - 12B^2 C - 4B^2 D - 24A^3 + 27ABC)\lambda + 18A^2 B - 12B^2 C \\ & - 4B^2 D - 15A^3 + 18ABC \Big] du. \end{aligned} \tag{2.9}$$

So we observe that, for $0 < \alpha < \beta_\varepsilon$,

$$\lim_{\lambda\to 0^+} \sqrt{\lambda} T_\lambda'(\alpha) = \frac{1}{\alpha}\int_0^\alpha \frac{2B - A}{(2B)^{3/2}} du = \bar{T}'(\alpha),$$

$$\lim_{\lambda\to 0^+} \sqrt{\lambda} T_\lambda''(\alpha) = \frac{1}{\alpha^2}\int_0^\alpha \frac{3A^2 - 4AB - 2BC}{(2B)^{5/2}} du = \bar{T}''(\alpha),$$

$$\lim_{\lambda\to 0^+} \sqrt{\lambda} T_\lambda'''(\alpha) = \frac{1}{\alpha^3}\int_0^\alpha \frac{18A^2 B - 12B^2 C - 4B^2 D - 15A^3 + 18ABC}{(2B)^{5/2}} du = \bar{T}'''(\alpha).$$

Furthermore, $\lim_{\lambda\to\infty} T_\lambda'(\alpha) = 1$. So statement (ii) holds. The statement (iii) follows immediately by [7, Lemma 4.2(ii)]. The proof is complete. □

Lemma 2.2. *Consider (1.1) with $\varepsilon > 0$. Then the following statements (i) and (ii) hold:*

(i) $T_\lambda'(\alpha) > 0$ for $0 < \alpha \leq 1$ and $\lambda > 0$.

(ii) $T_\lambda(\alpha)$ has at most one critical point, a local minimum, on $[\frac{5}{12\varepsilon}, \beta_\varepsilon)$.

Proof. We can see that $2B(\alpha,u)-A(\alpha,u)>0$ for $0<u<\alpha\leq 1$ because $2B(\alpha,\alpha)-A(\alpha,\alpha)=0$ and

$$\frac{\partial}{\partial u}\left[2B(\alpha,u)-A(\alpha,u)\right]=-2\varepsilon u^3+\left(u^2-1\right)<0 \text{ for } 0<u<\alpha<1.$$

So by (2.5) and (2.7), we obtain that $T_\lambda'(\alpha)>0$ for $0<\alpha\leq 1$ and $\lambda>0$. Then statement (i) holds. By (2.5), (2.7) and (2.8), we observe that, for $0<\alpha<\beta_\varepsilon$ and $\lambda>0$,

$$\begin{aligned}
&\alpha T_\lambda''(\alpha)+2T_\lambda'(\alpha)\\
&=\frac{1}{\alpha}\int_0^\alpha \frac{B^5\lambda^3+5B^4\lambda^2+\lambda B\left(3A^2+16B^2-4AB-BC\right)+3A^2+8B^2-8AB-2BC}{\sqrt{\lambda}\left(\lambda B^2+2B\right)^{5/2}}du\\
&>\frac{1}{\alpha}\int_0^\alpha \frac{\lambda B\left(3A^2+16B^2-4AB-BC\right)+3A^2+8B^2-8AB-2BC}{\sqrt{\lambda}\left(\lambda B^2+2B\right)^{5/2}}du\\
&=\frac{1}{\alpha}\int_0^\alpha \frac{\lambda B\left[3\left(A-B\right)^2+5B^2+B\left(2A-2B-C\right)\right]+3\left(A-2B\right)^2+2B\left(2A-2B-C\right)}{\sqrt{\lambda}\left(\lambda B^2+2B\right)^{5/2}}du\\
&>\frac{1}{\alpha}\int_0^\alpha \frac{\lambda B^2\left(2A-2B-C\right)+2B\left(2A-2B-C\right)}{\sqrt{\lambda}\left(\lambda B^2+2B\right)^{5/2}}du\\
&=\frac{1}{\alpha}\int_0^\alpha \frac{\left(\lambda B^2+2B\right)\left(2A-2B-C\right)}{\sqrt{\lambda}\left(\lambda B^2+2B\right)^{5/2}}du=\frac{1}{\alpha}\int_0^\alpha \frac{2A-2B-C}{\sqrt{\lambda}\left(\lambda B^2+2B\right)^{3/2}}du\\
&=\frac{1}{6\alpha}\int_0^\alpha \frac{\phi(\alpha)-\phi(u)}{\sqrt{\lambda}\left(\lambda B^2+2B\right)^{3/2}}du,
\end{aligned}\tag{2.10}$$

where $\phi(u)\equiv u^3\left(9\varepsilon u-4\right)$. Clearly, $\phi'(u)=12u^2\left(3\varepsilon u-1\right)$. Since

$$f\left(\frac{4}{9\varepsilon}\right)=1+\frac{324\varepsilon+80}{729\varepsilon^2}>0,$$

we see that

$$\frac{1}{3\varepsilon}<\frac{4}{9\varepsilon}<\beta_\varepsilon.\tag{2.11}$$

So we observe that

$$\phi(u)\begin{cases}<0 & \text{for } 0<u<\frac{4}{9\varepsilon},\\ =0 & \text{for } u=\frac{4}{9\varepsilon},\\ >0 & \text{for } \frac{4}{9\varepsilon}<u<\beta_\varepsilon,\end{cases} \quad\text{and}\quad \phi'(u)\begin{cases}<0 & \text{for } 0<u<\frac{1}{3\varepsilon},\\ =0 & \text{for } u=\frac{1}{3\varepsilon},\\ >0 & \text{for } \frac{1}{3\varepsilon}<u<\beta_\varepsilon.\end{cases}\tag{2.12}$$

Let $\alpha\in\left[\frac{5}{12\varepsilon},\beta_\varepsilon\right)$ be given. Then we consider two cases.

Case 1. Assume that $\frac{4}{9\varepsilon}\leq\alpha<\beta_\varepsilon$. Since $\phi(0)=0$, and by (2.12), we see that $\phi(\alpha)-\phi(u)>0$ for $0<u<\alpha$. So by (2.10), we obtain $\alpha T_\lambda''(\alpha)+2T_\lambda'(\alpha)>0$ for $\lambda>0$.

Case 2. Assume that $\frac{5}{12\varepsilon}\leq\alpha<\frac{4}{9\varepsilon}$. Since $\phi(0)=0$, and by (2.12), there exists $\tilde{\alpha}\in\left(0,\frac{1}{3\varepsilon}\right)$ such that

$$\phi(\alpha)-\phi(u)\begin{cases}<0 & \text{for } 0<u<\tilde{\alpha},\\ =0 & \text{for } u=\tilde{\alpha},\\ >0 & \text{for } \tilde{\alpha}<u<\alpha.\end{cases}$$

So by (2.10), we observe that, for $\lambda > 0$,

$$\begin{aligned}
&\alpha T_\lambda''(\alpha) + 2T_\lambda'(\alpha) \\
&> \frac{1}{6\alpha\sqrt{\lambda}}\left[\int_0^{\tilde{\alpha}} \frac{\phi(\alpha)-\phi(u)}{[\lambda B^2+2B]^{3/2}}du + \int_{\tilde{\alpha}}^{\alpha} \frac{\phi(\alpha)-\phi(u)}{[\lambda B^2+2B]^{3/2}}du\right] \\
&> \frac{1}{6\alpha\sqrt{\lambda}\,[\lambda B^2(\alpha,\tilde{\alpha})+2B(\alpha,\tilde{\alpha})]^{3/2}}\left\{\int_0^{\tilde{\alpha}} [\phi(\alpha)-\phi(u)]\,du + \int_{\tilde{\alpha}}^{\alpha} [\phi(\alpha)-\phi(u)]\,du\right\} \\
&= \frac{1}{6\alpha\sqrt{\lambda}\,[\lambda B^2(\alpha,\tilde{\alpha})+2B(\alpha,\tilde{\alpha})]^{3/2}}\int_0^{\alpha} [\phi(\alpha)-\phi(u)]\,du \\
&= \frac{6\varepsilon\alpha^3}{5\sqrt{\lambda}\,[\lambda B^2(\alpha,\tilde{\alpha})+2B(\alpha,\tilde{\alpha})]^{3/2}}\left(\alpha-\frac{5}{12\varepsilon}\right) \geq 0.
\end{aligned}$$

Thus by Cases 1–2, we have

$$\alpha T_\lambda''(\alpha) + 2T_\lambda'(\alpha) > 0 \ \text{ for } \frac{5}{12\varepsilon} \leq \alpha < \beta_\varepsilon \text{ and } \lambda > 0. \tag{2.13}$$

Fixed $\lambda > 0$. If $T_\lambda(\alpha)$ has a critical point $\breve{\alpha}$ in $[\frac{5}{12\varepsilon}, \beta_\varepsilon)$, by (2.13), then $\breve{\alpha}T_\lambda''(\breve{\alpha}) = \breve{\alpha}T_\lambda''(\breve{\alpha}) + 2T_\lambda'(\breve{\alpha}) > 0$. It implies that $T_\lambda(\alpha)$ has at most one critical point, a local minimum, on $\left[\frac{5}{12\varepsilon}, \beta_\varepsilon\right)$ for $\lambda > 0$. Then the statement (ii) holds. The proof is complete. □

Lemma 2.3. *Consider* (1.1) *with* $\varepsilon > 0$. *Then*

$$\frac{\partial}{\partial\lambda}\left[\sqrt{\lambda}T_\lambda'(\alpha)\right] > 0 \quad \text{for } 0 < \alpha \leq \frac{5}{12\varepsilon} \text{ and } \lambda > 0. \tag{2.14}$$

Proof. By (2.5) and (2.7), we compute and find that

$$\frac{\partial}{\partial\lambda}\left[\sqrt{\lambda}T_\lambda'(\alpha)\right] = \frac{1}{2\alpha}\int_0^{\alpha} \frac{B^2\left(B^3\lambda^2+5B^2\lambda+3A+6B\right)}{\left(\lambda B^2+2B\right)^{5/2}}du > \frac{1}{2\alpha}\int_0^{\alpha} \frac{3B^2\left(A+2B\right)}{\left(\lambda B^2+2B\right)^{5/2}}du. \tag{2.15}$$

In addition, we compute that

$$\frac{\partial}{\partial u}\left[A(\alpha,u)+2B(\alpha,u)\right] = R(u),$$

where $R(u) \equiv 3\varepsilon u^3 - 3(1-\varepsilon)u^2 - 6u - 4$. Clearly, $R'(u) = 9\varepsilon u^2 - 6(1-\varepsilon)u - 6$ is a quadratic polynomial of u with positive leading coefficient. Furthermore,

$$R'(0) = -6 < 0 \quad \text{and} \quad R'\left(\frac{5}{12\varepsilon}\right) \equiv -\frac{56\varepsilon+15}{16\varepsilon} < 0.$$

Thus we observe that $R'(u) < 0$ for $0 \leq u \leq \frac{5}{12\varepsilon}$. It follows that

$$\frac{\partial}{\partial u}\left[A(\alpha,u)+2B(\alpha,u)\right] = R(u) \leq R(0) = -4 < 0 \quad \text{for } 0 \leq u \leq \frac{5}{12\varepsilon}.$$

Then we have

$$A(\alpha,u)+2B(\alpha,u) > A(\alpha,\alpha)+2B(\alpha,\alpha) = 0 \quad \text{for } 0 < u < \alpha \leq \frac{5}{12\varepsilon}.$$

So by (2.15), we obtain (2.14). The proof is complete. □

Lemma 2.4. *Consider* (1.1) *with* $\varepsilon > 0$. *Let* I *be a closed interval in* $(0, \beta_\varepsilon)$. *Then the following statements (i)–(iii) hold:*

(i) If $\bar{T}'(\alpha) < 0$ *for* $\alpha \in I$, *then there exists* $\check{\lambda} > 0$ *such that* $T'_\lambda(\alpha) < 0$ *for* $\alpha \in I$ *and* $0 < \lambda < \check{\lambda}$.

(ii) If $\alpha \bar{T}''(\alpha) + k\bar{T}'(\alpha) < 0$ *for* $\alpha \in I$ *and some* $k > 0$, *then there exists* $\hat{\lambda} > 0$ *such that* $\alpha T''_\lambda(\alpha) + kT'_\lambda(\alpha) < 0$ *for* $\alpha \in I$ *and* $0 < \lambda < \hat{\lambda}$.

(iii) If $[2\alpha \bar{T}''(\alpha) + 3\bar{T}'(\alpha)]' > 0$ *for* $\alpha \in I$, *then there exists* $\bar{\lambda} > 0$ *such that* $[2\alpha T''_\lambda(\alpha) + 3T'_\lambda(\alpha)]' > 0$ *for* $\alpha \in I$ *and* $0 < \lambda < \bar{\lambda}$.

Proof. (I) Assume that $\bar{T}'(\alpha) < 0$ for $\alpha \in I$. By Lemma 2.1(ii), we have

$$\lim_{\lambda \to 0^+} \sqrt{\lambda} T'_\lambda(\alpha) = \bar{T}'(\alpha) < 0 \quad \text{for } \alpha \in I. \tag{2.16}$$

For $\alpha \in I$, by (2.16), we define λ_α by

$$\lambda_\alpha \equiv \begin{cases} 1 & \text{if } T'_\lambda(\alpha) < 0 \text{ for all } \lambda > 0, \\ \sup\{\lambda_1 : T'_\lambda(\alpha) < 0 \text{ for } 0 < \lambda < \lambda_1\} & \text{if } T'_\lambda(\alpha) \geq 0 \text{ for some } \lambda > 0. \end{cases} \tag{2.17}$$

Clearly, $T'_\lambda(\alpha) < 0$ for $\alpha \in I$ and $0 < \lambda < \lambda_\alpha$. Let $\check{\lambda} \equiv \inf\{\lambda_\alpha : \alpha \in I\}$. Assume that $\check{\lambda} = 0$. By (2.17), there exists a sequence $\{\alpha_k\}_{k \in \mathbb{N}} \subset I$ such that

$$\lim_{k \to \infty} \lambda_{\alpha_k} = 0 \quad \text{and} \quad T'_{\lambda_{\alpha_k}}(\alpha_k) \geq 0 \quad \text{for } k \in \mathbb{N}. \tag{2.18}$$

Without loss of generality, we assume that $\lim_{k \to \infty} \alpha_k = \check{\alpha} \in I$. So by (2.16) and (2.18), we observe that

$$0 \leq \lim_{k \to \infty} \sqrt{\lambda_{\alpha_k}} T'_{\lambda_{\alpha_k}}(\alpha_k) = \lim_{k \to \infty} \sqrt{\lambda_{\alpha_k}} T'_{\lambda_{\alpha_k}}(\check{\alpha}) = \bar{T}'(\check{\alpha}) < 0,$$

which is a contradiction. It implies that $\check{\lambda} > 0$. So statement (i) holds.

(II) Assume that $\alpha \bar{T}''(\alpha) + k\bar{T}'(\alpha) < 0$ for $\alpha \in I$ and some $k > 0$. Let $G_1(\alpha, \lambda) \equiv \alpha T''_\lambda(\alpha) + kT'_\lambda(\alpha)$. By Lemma 2.1(ii), we see that

$$\lim_{\lambda \to 0^+} \sqrt{\lambda} G_1(\alpha, \lambda) = \alpha \bar{T}''(\alpha) + k\bar{T}'(\alpha) < 0 \quad \text{for } \alpha \in I. \tag{2.19}$$

For $\alpha \in I$, by (2.19), we define λ_α by

$$\lambda_\alpha \equiv \begin{cases} 1 & \text{if } G_1(\alpha, \lambda) < 0 \text{ for all } \lambda > 0, \\ \sup\{\lambda_2 : G_1(\alpha, \lambda) < 0 \text{ for } 0 < \lambda < \lambda_2\} & \text{if } G_1(\alpha, \lambda) \geq 0 \text{ for some } \lambda > 0. \end{cases}$$

Clearly, $G_1(\alpha, \lambda) < 0$ for $\alpha \in I$ and $0 < \lambda < \lambda_\alpha$. Let $\hat{\lambda} \equiv \inf\{\lambda_\alpha : \alpha \in I\}$. We use the similar argument in (I) to obtain that $\hat{\lambda} > 0$. So statement (ii) holds.

(III) Assume that $[2\alpha \bar{T}''(\alpha) + 3\bar{T}'(\alpha)]' > 0$ for $\alpha \in I$. Let $G_2(\alpha, \lambda) \equiv [2\alpha T''(\alpha) + 3T'(\alpha)]'$. By Lemma 2.1(ii), we see that

$$\begin{aligned} \lim_{\lambda \to 0^+} \sqrt{\lambda} G_2(\alpha, \lambda) &= \lim_{\lambda \to 0^+} \left[2\alpha \sqrt{\lambda} T'''_\lambda(\alpha) + 5\sqrt{\lambda} T''_\lambda(\alpha) \right] = 2\alpha \bar{T}'''(\alpha) + 5\bar{T}''(\alpha) \\ &= [2\alpha \bar{T}''(\alpha) + 3\bar{T}'(\alpha)]' > 0 \quad \text{for } \alpha \in I. \end{aligned} \tag{2.20}$$

For $\alpha \in I$, by (2.20), we define λ_α by

$$\lambda_\alpha \equiv \begin{cases} 1 & \text{if } G_2(\alpha,\lambda) < 0 \text{ for all } \lambda > 0, \\ \sup\{\lambda_3 : G_2(\alpha,\lambda) < 0 \text{ for } 0 < \lambda < \lambda_3\} & \text{if } G_2(\alpha,\lambda) \geq 0 \text{ for some } \lambda > 0. \end{cases}$$

Clearly, $G_2(\alpha,\lambda) < 0$ for $\alpha \in I$ and $0 < \lambda < \lambda_\alpha$. Let $\bar{\lambda} \equiv \inf\{\lambda_\alpha : \alpha \in I\}$. We use the similar argument in (I) to obtain that $\bar{\lambda} > 0$. So statement (iii) holds. The proof is complete. □

Lemma 2.5. *Consider (1.5) with $\varepsilon > 0$. Let ε_0 be defined in Theorem 1.2. Then the following statements (i)–(iii) hold:*

(i) $\bar{T}'(\alpha) \geq 0$ for $0 < \alpha < \beta_\varepsilon$ and $\varepsilon \geq \varepsilon_0$.

(ii) $[2\alpha\bar{T}''(\alpha) + 3\bar{T}'(\alpha)]' > 0$ for $\frac{1}{3\varepsilon} \leq \alpha \leq \frac{5}{12\varepsilon}$ and $\varepsilon \leq \varepsilon_0$.

(iii) There exists $\hat{\varepsilon} \in (0,\varepsilon_0)$ such that $\bar{T}'(\alpha) \geq 0$ for $0 < \alpha \leq \frac{1}{3\varepsilon}$ and $\hat{\varepsilon} \leq \varepsilon < \varepsilon_0$. Furthermore, $\hat{\varepsilon} < \sqrt{31/1000}$.

Proof. The statement (i) follows immediately by Theorem 1.2 and (2.4). The statement (ii) follows immediately by [6, Lemma 3.5]. By [11, Theorem 2.1], there exists $\hat{\varepsilon} > 0$ satisfying

$$\hat{\varepsilon} < \sqrt{\frac{31}{1000}} < \varepsilon_0$$

such that

$$\bar{T}'\left(\frac{1}{3\varepsilon}\right) \begin{cases} < 0 & \text{for } 0 < \varepsilon < \hat{\varepsilon}, \\ = 0 & \text{for } \varepsilon = \hat{\varepsilon}, \\ > 0 & \text{for } \hat{\varepsilon} < \varepsilon < \varepsilon_0. \end{cases} \tag{2.21}$$

By Theorem 1.2, (2.4) and [6, Lemma 3.3], we see that, for $0 < \varepsilon < \varepsilon_0$, there exist two positive numbers $\alpha_* < \alpha^* < \beta_\varepsilon$ such that

$$\bar{T}'(\alpha) \begin{cases} > 0 & \text{on } (0,\alpha_*) \cup (\alpha^*,\beta_\varepsilon), \\ = 0 & \text{when } \alpha = \alpha_* \text{ or } \alpha = \alpha^*, \\ < 0 & \text{for } (\alpha_*,\alpha^*). \end{cases} \tag{2.22}$$

Since f is a convex function on $[0,\frac{1}{3\varepsilon}]$, and by [15, Lemma 3.2], we see that $\bar{T}(\alpha)$ is either strictly increasing on $(0,\frac{1}{3\varepsilon})$, or strictly increasing and then strictly decreasing on $(0,\frac{1}{3\varepsilon})$. So by (2.21) and (2.22), we observe that $\frac{1}{3\varepsilon} \leq \alpha_*$ for $\hat{\varepsilon} \leq \varepsilon < \varepsilon_0$. It follows that $\bar{T}'(\alpha) \geq 0$ for $0 < \alpha \leq \frac{1}{3\varepsilon}$ and $\hat{\varepsilon} \leq \varepsilon < \varepsilon_0$. So the statement (iii) holds. The proof is complete. □

Lemma 2.6. *Consider (1.5) with $0 < \varepsilon \leq \hat{\varepsilon}$ where $\hat{\varepsilon}$ is defined in Lemma 2.5. Then $\alpha\bar{T}''(\alpha) + \bar{T}'(\alpha) < 0$ for $1 \leq \alpha \leq 1.7$.*

Proof. Let $\bar{A} \equiv \varepsilon\left(\alpha^4 - u^4\right)$, $\bar{B} \equiv \alpha^3 - u^3$, $\bar{C} \equiv \alpha^2 - u^2$ and $\bar{D} \equiv \alpha - u$. We compute that

$$\alpha\bar{T}''(\alpha) + \bar{T}'(\alpha) = \frac{1}{4\sqrt{2}\alpha}\int_0^\alpha \frac{N_1(\alpha,u)}{[F(\alpha)-F(u)]^{5/2}}du, \tag{2.23}$$

where

$$N_1(\alpha,u) \equiv \frac{1}{72}\left(9\bar{A}^2 + 4\bar{B}^2 + 36\bar{D}^2 - 6\bar{A}\bar{B} + 198\bar{A}\bar{D} - 120\bar{B}\bar{D} + 36\bar{A}\bar{C} - 12\bar{B}\bar{C} - 36\bar{C}\bar{D}\right).$$

Let $\alpha \in [1, 1.7]$, $u \in (0, \alpha)$ and $\varepsilon \in (0, \tilde{\varepsilon}]$ be given. By Lemma [11, Lemma 3.6], we have

$$\bar{A} < \frac{4\varepsilon\alpha}{3}\bar{B} \quad \text{and} \quad \bar{D} > \frac{1}{3\alpha^2}\bar{B} > \frac{1}{3\alpha^2}\left(\frac{3}{4\varepsilon\alpha}\bar{A}\right) = \frac{\bar{A}}{4\alpha^3\varepsilon}.$$

Then

$$1 < \alpha^2 < \frac{\left(\alpha^2 + \alpha u + u^2\right)\bar{D}}{\bar{D}} = \frac{\bar{B}}{\bar{D}} < 3\alpha^2 \leq 3\,(1.7)^2 = 8.67, \tag{2.24}$$

$$\bar{A} < \frac{4\varepsilon\alpha}{3}\bar{B} < \frac{4\hat{\varepsilon}}{3}\,(1.7)\,\bar{B} = \frac{34\hat{\varepsilon}}{15}\bar{B} \quad \text{and} \quad \bar{D} > \frac{\bar{A}}{4\alpha^3\varepsilon} > \frac{\bar{A}}{4\,(1.7)^3\,\hat{\varepsilon}} = \frac{250}{4913\hat{\varepsilon}}\bar{A}. \tag{2.25}$$

In addition, by Lemma 2.5(iii), we compute and find that

$$\frac{34}{15}\hat{\varepsilon} - \frac{2}{3} < \frac{34}{15}\sqrt{\frac{31}{1000}} - \frac{2}{3}\,(\approx -0.26) < 0, \tag{2.26}$$

$$198\left(\frac{34}{15}\hat{\varepsilon} - \frac{20}{33}\right) < 198\left(\frac{34}{15}\sqrt{\frac{31}{1000}} - \frac{20}{33}\right)(\approx -40.98) < -0.40, \tag{2.27}$$

$$1 - \frac{5}{34\hat{\varepsilon}} - \frac{250}{4913\hat{\varepsilon}} < 1 - \frac{5}{34\sqrt{\frac{31}{1000}}} - \frac{250}{4913\sqrt{\frac{31}{1000}}}\,(\approx -0.88) < 0. \tag{2.28}$$

By (2.24)–(2.28), we observe that

$$\begin{aligned}
N_1(\alpha, u) &= \frac{1}{72}\left(9\bar{A}^2 + 4\bar{B}^2 + 36\bar{D}^2 - 6\bar{A}\bar{B} + 198\bar{A}\bar{D} - 120\bar{B}\bar{D} + 36\bar{A}\bar{C} - 12\bar{B}\bar{C} - 36\bar{C}\bar{D}\right) \\
&= \frac{1}{72}\left[9\bar{A}\left(\bar{A} - \frac{2}{3}\bar{B}\right) + 198\bar{D}\left(\bar{A} - \frac{20}{33}\bar{B}\right) + 36\bar{C}\left(\bar{A} - \frac{1}{3}\bar{B} - \bar{D}\right) + 4\bar{B}^2 + 36\bar{D}^2\right] \\
&< \frac{1}{72}\left[9\bar{A}\bar{B}\left(\frac{34}{15}\hat{\varepsilon} - \frac{2}{3}\right) + 198\bar{B}\bar{D}\left(\frac{34}{15}\hat{\varepsilon} - \frac{20}{33}\right)\right. \\
&\quad \left. + 36\bar{A}\bar{C}\left(1 - \frac{5}{34\hat{\varepsilon}} - \frac{250}{4913\hat{\varepsilon}}\right) + 4\bar{B}^2 + 36\bar{D}^2\right] \\
&< \frac{1}{72}\left(-40\bar{B}\bar{D} + 4\bar{B}^2 + 36\bar{D}^2\right) = \frac{\bar{D}^2}{18}\left[\left(\frac{\bar{B}}{\bar{D}} - 5\right)^2 - 16\right] \\
&< \frac{\bar{D}^2}{18}\left[(1-5)^2 - 16\right] = 0.
\end{aligned}$$

So by (2.23), we obtain that $\alpha\bar{T}''(\alpha) + \bar{T}'(\alpha) < 0$ for $1 \leq \alpha \leq 1.7$ and $0 < \varepsilon \leq \hat{\varepsilon}$. The proof is complete. □

Lemma 2.7. *Consider* (1.5) *with* $0.07 \leq \varepsilon \leq \hat{\varepsilon}$*. Then* $\alpha\bar{T}''(\alpha) + \frac{5}{2}\bar{T}'(\alpha) < 0$ *for* $1.7 \leq \alpha \leq \frac{1}{3\varepsilon}$*.*

Proof. We compute that

$$\alpha\bar{T}''(\alpha) + \frac{5}{2}\bar{T}'(\alpha) = \frac{1}{4\sqrt{2}\alpha}\int_0^\alpha \frac{N_2(\alpha,u)}{[F(\alpha) - F(u)]^{5/2}}du, \tag{2.29}$$

where

$$\begin{aligned}
N_2(\alpha, u) \equiv \frac{1}{144}\Big(&-9\bar{A}^2 + 42\bar{A}\bar{B} + 450\bar{A}\bar{D} + 126\bar{A}\bar{C} - 16\bar{B}^2 - 240\bar{B}\bar{D} \\
&- 60\bar{B}\bar{C} + 288\bar{D}^2 + 36\bar{C}\bar{D}\Big).
\end{aligned} \tag{2.30}$$

Then we assert that

$$N_2(\alpha,u) < 0 \quad \text{for } 0<u<\alpha,\ 1.7 \le \alpha \le \frac{1}{3\varepsilon} \text{ and } 0.07 \le \varepsilon \le \hat{\varepsilon}. \tag{2.31}$$

The proof of assertion (2.31) is easy but tedious. Thus, we put it in Appendix. So by (2.29)–(2.31), we see that $\alpha\bar{T}''(\alpha) + \frac{5}{2}\bar{T}'(\alpha) < 0$ for $1.7 \le \alpha \le \frac{1}{3\varepsilon}$ and $0.07 \le \varepsilon \le \hat{\varepsilon}$. □

Lemma 2.8. *Consider (1.5) with $0<\varepsilon<0.07$. Then $\bar{T}'(\alpha) < 0$ for $1.7 \le \alpha \le \frac{1}{3\varepsilon}$.*

Proof. We compute that

$$\bar{T}'(\alpha) = \frac{1}{2\sqrt{2}\alpha}\int_0^\alpha \frac{2B(\alpha,u)-A(\alpha,u)}{B^{3/2}(\alpha,u)}du = \frac{1}{2\sqrt{2}\alpha}\int_0^\alpha \frac{\theta(\alpha)-\theta(u)}{B^{3/2}(\alpha,u)}du, \tag{2.32}$$

where $\theta(u) \equiv 2F(u) - uf(u)$ for $0 \le u < \beta_\varepsilon$. Since $0<\varepsilon<0.07$, and by [11, Lemma 3.1], there exists $p \in \left(0,\frac{1}{3\varepsilon}\right)$ such that $\theta'(u) > 0$ for $(0,p)$ and $\theta'(u) < 0$ for $\left(p,\frac{1}{3\varepsilon}\right)$. Let $\alpha \in \left[1.7,\frac{1}{3\varepsilon}\right]$ be given. Assume that $\theta(\alpha) \le 0$, see Figure 2.1(i). Since $\theta(0)=0$, we see that $\theta(\alpha)-\theta(u) < 0$ for $0<u<\alpha$. So by (2.32), we obtain that $\bar{T}'(\alpha) < 0$. Assume that $\theta(\alpha) > 0$, see Figure 2.1(ii). We compute and find that

$$\theta'(1.7) = 2\varepsilon u^3 - u^2 + 1\big|_{u=1.7} = \frac{4913}{500}\varepsilon - \frac{189}{100} < 0 \quad \text{for } 0<\varepsilon<0.07.$$

Since $1.7 \le \alpha \le \frac{1}{3\varepsilon}$, there exists $\bar{\alpha} \in (0,p)$ such that

$$\theta(\alpha)-\theta(u) \begin{cases} >0 & \text{for } 0<u<\bar{\alpha}, \\ =0 & \text{for } u=\bar{\alpha}, \\ <0 & \text{for } \bar{\alpha}<u<\alpha. \end{cases}$$

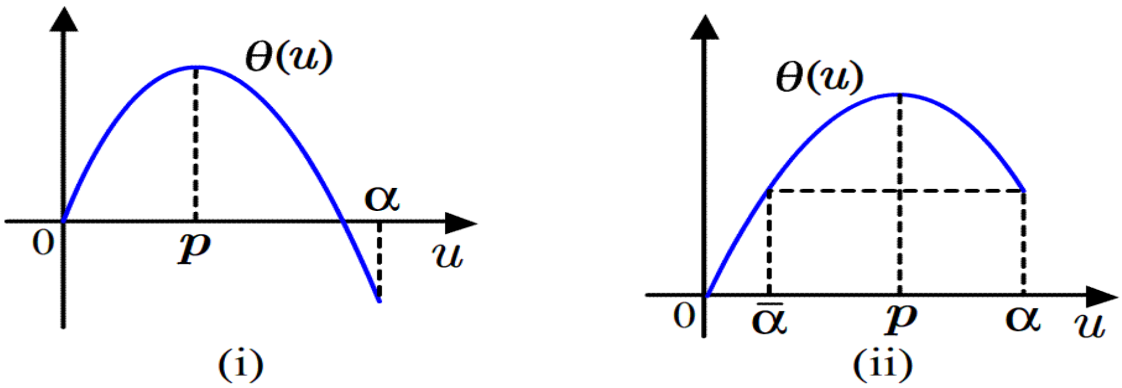

Figure 2.1: Graphs of $\theta(u)$ on $[0,\alpha]$ where $1.7 \le \alpha \le \frac{1}{3\varepsilon}$ and $0<\varepsilon<0.07$.

So by (2.32) and similar argument of [14, (3.11)], we observe that

$$\bar{T}'(\alpha) < \frac{1}{2\sqrt{2}\alpha B^{3/2}(\alpha,\bar{\alpha})}\int_0^\alpha u\theta'(u)du = \frac{\alpha\left(8\varepsilon\alpha^3 - 5\alpha^2 + 10\right)}{40\sqrt{2}B^{3/2}(\alpha,\bar{\alpha})}. \tag{2.33}$$

Since

$$\frac{\partial}{\partial u}\left(8\varepsilon u^3 - 5u^2 + 10\right) = 2u\left(12\varepsilon u - 5\right) < 0 \quad \text{for } 1.7 \le u \le \frac{1}{3\varepsilon},$$

we see that, for $1.7 \le u \le \frac{1}{3\varepsilon}$ and $0 < \varepsilon < 0.07$,

$$8\varepsilon u^3 - 5u^2 + 10 < 8\varepsilon u^3 - 5u^2 + 10\big|_{u=1.7} = \frac{4913}{125}\varepsilon - \frac{89}{20} < 0.$$

So by (2.33), we obtain that $\bar{T}'(\alpha) < 0$. The proof is complete. □

Lemma 2.9. *Consider* (1.1) *with* $0 < \varepsilon < \varepsilon_0$. *Then there exists* $\xi_\varepsilon > 0$ *such that*

$$\Gamma_\varepsilon \equiv \{\lambda > 0 : T'_\lambda(\alpha) < 0 \text{ for some } \alpha \in (0, \beta_\varepsilon)\} = (0, \xi_\varepsilon).$$

Proof. Let $\varepsilon \in (0, \varepsilon_0)$ be given. By (2.22), there exist two positive numbers $\alpha_* < \alpha^* < \beta_\varepsilon$ such that

$$\lim_{\lambda \to 0^+} \sqrt{\lambda} T'_\lambda(\alpha) = \bar{T}'(\alpha) \begin{cases} > 0 & \text{on } (0, \alpha_*) \cup (\alpha^*, \beta_\varepsilon), \\ = 0 & \text{when } \alpha = \alpha_* \text{ or } \alpha^*, \\ < 0 & \text{on } (\alpha_*, \alpha^*). \end{cases} \tag{2.34}$$

Then we divide this proof into the next four steps.

Step 1. We prove that $\alpha_* < \frac{5}{12\varepsilon}$. Assume that $\alpha_* \ge \frac{5}{12\varepsilon}$. By (2.34) and Lemma 2.3, we see that

$$0 \le \bar{T}'(\alpha) = \lim_{\lambda \to 0^+} \sqrt{\lambda} T'_\lambda(\alpha) < \sqrt{\lambda} T'_\lambda(\alpha) \quad \text{for } 0 < \alpha \le \frac{5}{12\varepsilon} \text{ and } \lambda > 0. \tag{2.35}$$

By Lemma 2.2(ii) and (2.35), we further see that $T'_\lambda(\alpha) > 0$ for $0 < \alpha < \beta_\varepsilon$ for $\lambda > 0$. So by (2.34), we obtain that

$$0 \le \lim_{\lambda \to 0^+} \sqrt{\lambda} T'_\lambda\left(\frac{\alpha_* + \alpha^*}{2}\right) = \bar{T}'\left(\frac{\alpha_* + \alpha^*}{2}\right) < 0,$$

which is a contradiction. It implies that $\alpha_* < \frac{5}{12\varepsilon}$.

Step 2. We prove that, for $\alpha \in (\alpha_*, \alpha^*) \cap \left(0, \frac{5}{12\varepsilon}\right]$, there exists a continuously differential function $\tilde{\lambda}_\alpha > 0$ of α such that

$$\sqrt{\lambda} T'_\lambda(\alpha) \begin{cases} < 0 & \text{if } 0 < \lambda < \tilde{\lambda}_\alpha, \\ = 0 & \text{if } \lambda = \tilde{\lambda}_\alpha, \\ > 0 & \text{if } \lambda > \tilde{\lambda}_\alpha. \end{cases} \tag{2.36}$$

By Lemma 2.1(ii), we see that

$$\lim_{\lambda \to \infty} \sqrt{\lambda} T'_\lambda(\alpha) = \infty \cdot 1 = \infty \quad \text{for } \alpha \in (0, \beta_\varepsilon). \tag{2.37}$$

By (2.34), (2.37), Lemma 2.3 and implicit function theorem, we observe that, for $\alpha \in (\alpha_*, \alpha^*) \cap \left(0, \frac{5}{12\varepsilon}\right]$, there exists a continuously differential function $\tilde{\lambda}_\alpha > 0$ of α such that (2.36) holds.

Step 3. We prove that

$$\xi_\varepsilon \equiv \sup\left\{\tilde{\lambda}_\alpha : \alpha \in (\alpha_*, \alpha^*) \cap \left(0, \frac{5}{12\varepsilon}\right]\right\} \in (0, \infty).$$

Clearly, $\xi_\varepsilon > 0$. By (2.34) and Lemma 2.3, we see that

$$0 = \lim_{\lambda \to 0^+} \sqrt{\lambda} T'_\lambda(\alpha_*) < T'_{\lambda=1}(\alpha_*).$$

So by Lemma 2.3 and continuity of $T'_{\lambda=1}(\alpha)$ with respect to α, there exists $\delta > 0$ such that

$$0 < T'_{\lambda=1}(\alpha) \leq \sqrt{\lambda} T'_{\lambda}(\alpha) \quad \text{for } \alpha_* < \alpha < \alpha_* + \delta < \frac{5}{12\varepsilon} \text{ and } \lambda \geq 1,$$

from which it follows that $\tilde{\lambda}_\alpha < 1$ for $\alpha_* < \alpha < \alpha_* + \delta$. Thus $\lim_{\alpha \to \alpha_*^+} \tilde{\lambda}_\alpha \leq 1 < \infty$. By similar argument, we obtain that

$$\lim_{\alpha \to (\alpha^*)^-} \tilde{\lambda}_\alpha < \infty \quad \text{if } \alpha^* < \frac{5}{12\varepsilon}.$$

So by Step 2, we observe that $\xi_\varepsilon \in (0, \infty)$.

Step 4. We prove that $\Gamma_\varepsilon = (0, \xi_\varepsilon)$. Let $\lambda_1 \in (0, \xi_\varepsilon)$. There exists $\alpha_1 \in (\alpha_*, \alpha^*) \cap \left(0, \frac{5}{12\varepsilon}\right]$ such that $\lambda_1 < \tilde{\lambda}_{\alpha_1}$. Then by (2.36), we see that $T'_{\lambda_1}(\alpha_1) < 0$, which implies that $\lambda_1 \in \Gamma_\varepsilon$. Thus $(0, \xi_\varepsilon) \subseteq \Gamma_\varepsilon$. Let $\lambda_2 \in \Gamma_\varepsilon$. There exists $\alpha_2 \in (0, \beta_\varepsilon)$ such that $T'_{\lambda_2}(\alpha_2) < 0$. Next, we consider two cases.

Case 1. Assume that $\frac{5}{12\varepsilon} < \alpha^*$. By (2.34) and Lemma 2.3, we see that

$$0 \leq \lim_{\lambda \to 0^+} \sqrt{\lambda} T'_{\lambda}(\alpha) < \sqrt{\lambda} T'_{\lambda}(\alpha) \quad \text{for } \alpha \in (0, \alpha_*] \text{ and } \lambda > 0. \tag{2.38}$$

By Steps 2 and 3, we see that

$$\sqrt{\lambda} T'_{\lambda}(\alpha) \geq 0 \quad \text{for } \alpha \in \left(\alpha_*, \frac{5}{12\varepsilon}\right] \quad \text{if } \lambda \geq \xi_\varepsilon. \tag{2.39}$$

By (2.39) and Lemma 2.2, we see that

$$T'_{\lambda}(\alpha) > 0 \quad \text{for } \frac{5}{12\varepsilon} \leq \alpha < \beta_\varepsilon \text{ and } \lambda \geq \xi_\varepsilon. \tag{2.40}$$

So by (2.38)–(2.40), we obtain that $T'_{\lambda}(\alpha) \geq 0$ for $\alpha \in (0, \beta_\varepsilon)$ if $\lambda \geq \xi_\varepsilon$. It implies that $\lambda_2 < \xi_\varepsilon$. Thus $\Gamma_\varepsilon \subseteq (0, \xi_\varepsilon)$.

Case 2. Assume that $\alpha^* < \frac{5}{12\varepsilon}$. By (2.34) and Lemma 2.3, we see that

$$0 \leq \lim_{\lambda \to 0^+} \sqrt{\lambda} T'_{\lambda}(\alpha) < \sqrt{\lambda} T'_{\lambda}(\alpha) \quad \text{for } \alpha \in (0, \alpha_*] \cup \left[\alpha^*, \frac{5}{12\varepsilon}\right] \text{ and } \lambda > 0. \tag{2.41}$$

By Steps 2 and 3, we see that

$$\sqrt{\lambda} T'_{\lambda}(\alpha) \geq 0 \quad \text{for } \alpha \in (\alpha_*, \alpha^*) \quad \text{if } \lambda \geq \xi_\varepsilon. \tag{2.42}$$

By (2.41) and Lemma 2.2(ii), we see that

$$T'_{\lambda}(\alpha) > 0 \quad \text{for } \frac{5}{12\varepsilon} \leq \alpha < \beta_\varepsilon \text{ and } \lambda > 0. \tag{2.43}$$

So by (2.41)–(2.43), we obtain that $T'_{\lambda}(\alpha) \geq 0$ for $\alpha \in (0, \beta_\varepsilon)$ if $\lambda \geq \xi_\varepsilon$. It implies that $\lambda_2 < \xi_\varepsilon$. Thus $\Gamma_\varepsilon \subseteq (0, \xi_\varepsilon)$.

By the above discussions, we obtain that $\Gamma_\varepsilon = (0, \xi_\varepsilon)$. The proof is complete. □

Lemma 2.10. *Consider (1.1) with $0 < \varepsilon < \varepsilon_0$. Then there exists $\kappa_\varepsilon \in (0, \xi_\varepsilon)$ such that $T_\lambda(\alpha)$ has exactly two critical points, a local maximum at $\alpha_M(\lambda)$ and a local minimum at $\alpha_m(\lambda)$ $(> \alpha_M(\lambda))$, on $(0, \beta_\varepsilon)$ if $0 < \lambda < \kappa_\varepsilon$.*

Proof. Let $\varepsilon \in (0, \varepsilon_0)$ be given. By (2.34) and Lemma 2.1(ii), there exists $\lambda_1 > 0$ such that

$$T'_\lambda\left(\frac{\alpha_* + \alpha^*}{2}\right) < 0 \quad \text{for } 0 < \lambda < \lambda_1. \tag{2.44}$$

We divide this proof into the next four steps.

Step 1. We prove that there exists $\lambda_2 \in (0, \lambda_1)$ such that, for $0 < \lambda < \lambda_2$, either $T'_\lambda(\alpha) > 0$ on $\left(0, \frac{1}{3\varepsilon}\right]$, or $T_\lambda(\alpha)$ has exactly one critical point, a local maximum, on $\left(0, \frac{1}{3\varepsilon}\right]$, see Figure 2.2. By Lemma 2.2(i), we have

$$T'_\lambda(\alpha) > 0 \quad \text{for } 0 < \alpha \leq 1 \text{ and } \lambda > 0. \tag{2.45}$$

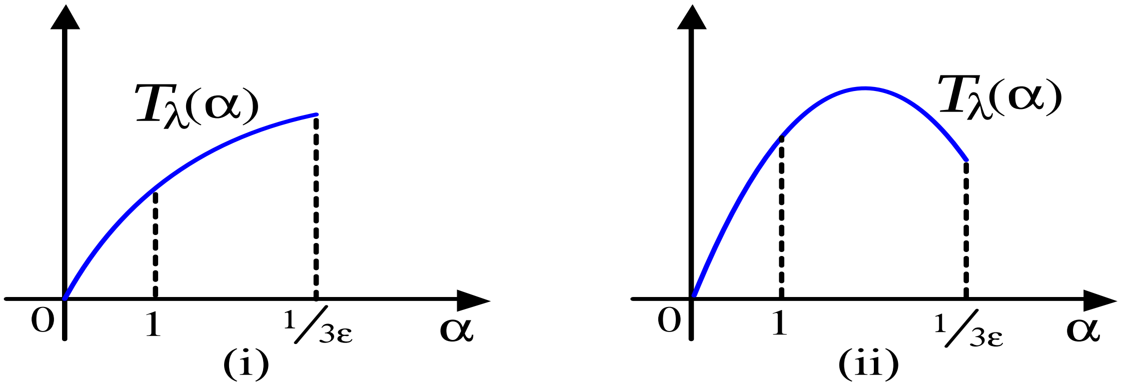

Figure 2.2: Graphs of $T_\lambda(\alpha)$ on $(0, \frac{1}{3\varepsilon}]$ for $0 < \lambda < \lambda_2$.

Then we consider the following three cases.

Case 1. Assume that $\hat{\varepsilon} \leq \varepsilon < \varepsilon_0$. By Lemmas 2.1(ii), 2.3 and 2.5(iii), we see that

$$0 \leq \bar{T}'(\alpha) = \lim_{\lambda \to 0^+} \sqrt{\lambda} T'_\lambda(\alpha) < \sqrt{\lambda} T'_\lambda(\alpha) \quad \text{for } 1 < \alpha \leq \frac{1}{3\varepsilon} \text{ and } \lambda > 0.$$

So by (2.45), $T'_\lambda(\alpha) > 0$ on $\left(0, \frac{1}{3\varepsilon}\right]$ for $\lambda > 0$, see Figure 2.2(i).

Case 2. Assume that $0.07 \leq \varepsilon < \hat{\varepsilon}$. By (2.21), Lemmas 2.1(ii), 2.4(ii), 2.6 and 2.7, there exists $\lambda_2 \in (0, \lambda_1)$ such that

$$T'_\lambda\left(\frac{1}{3\varepsilon}\right) < 0 \quad \text{and} \quad \alpha T''_\lambda(\alpha) + K(\alpha) T'_\lambda(\alpha) < 0 \quad \text{for } 1 \leq \alpha \leq \frac{1}{3\varepsilon} \text{ and } 0 < \lambda < \lambda_2, \tag{2.46}$$

where $K(\alpha) \equiv 1$ if $1 \leq \alpha \leq 1.7$, and $K(\alpha) \equiv 5/2$ if $1.7 < \alpha \leq \frac{1}{3\varepsilon}$. By (2.45) and (2.46), there exists $\alpha_\lambda \in \left(1, \frac{1}{3\varepsilon}\right)$ such that $T'_\lambda(\alpha_\lambda) = 0$ for $0 < \lambda < \lambda_2$. Furthermore,

$$\alpha_\lambda T''_\lambda(\alpha_\lambda) = \alpha_\lambda T''_\lambda(\alpha_\lambda) + K(\alpha_\lambda) T'_\lambda(\alpha_\lambda) < 0 \quad \text{for } 0 < \lambda < \lambda_2.$$

Thus $T_\lambda(\alpha)$ has exactly one local maximum at α_λ on $\left(0, \frac{1}{3\varepsilon}\right]$ for $0 < \lambda < \lambda_2$, see Figure 2.2(ii).

Case 3. Assume that $0 < \varepsilon < 0.07$. By Lemmas 2.4, 2.6 and 2.8, there exists $\lambda_2 \in (0, \lambda_1)$ such that

$$\alpha T''_\lambda(\alpha) + T'_\lambda(\alpha) < 0 \quad \text{for } 1 \leq \alpha \leq 1.7 \text{ and } 0 < \lambda < \lambda_2, \tag{2.47}$$

$$T'_\lambda(\alpha) < 0 \quad \text{for } 1.7 \leq \alpha \leq \frac{1}{3\varepsilon} \text{ and } 0 < \lambda < \lambda_2. \tag{2.48}$$

So by (2.45), (2.47) and (2.48), there exists $\alpha_\lambda \in (1, 1.7)$ such that $T'_\lambda(\alpha_\lambda) = 0$ for $0 < \lambda < \lambda_2$. Furthermore,

$$\alpha_\lambda T''_\lambda(\alpha_\lambda) = \alpha_\lambda T''_\lambda(\alpha_\lambda) + T'_\lambda(\alpha_\lambda) < 0 \quad \text{for } 0 < \lambda < \lambda_2.$$

Thus $T_\lambda(\alpha)$ has exactly one local maximum at α_λ on $\left(0, \frac{1}{3\varepsilon}\right]$ for $0 < \lambda < \lambda_2$, see Figure 2.2(ii).

Step 2. We prove that there exists $\lambda_3 \in (0, \lambda_2)$ such that, for $\lambda \in (0, \lambda_3)$, one of the following cases holds:

(ci) $T'_\lambda(\alpha) > 0$ on $\left(\frac{1}{3\varepsilon}, \frac{5}{12\varepsilon}\right)$.

(cii) $T'_\lambda(\alpha) < 0$ on $\left(\frac{1}{3\varepsilon}, \frac{5}{12\varepsilon}\right)$.

(ciii) $T'_\lambda(\alpha) < 0$ on $\left(\frac{1}{3\varepsilon}, \check{\alpha}\right)$ and $T'_\lambda(\alpha) > 0$ on $\left(\check{\alpha}, \frac{5}{12\varepsilon}\right)$ for some $\check{\alpha} \in \left(\frac{1}{3\varepsilon}, \frac{5}{12\varepsilon}\right)$.

(civ) $T'_\lambda(\alpha) > 0$ on $\left(\frac{1}{3\varepsilon}, \check{\alpha}\right)$ and $T'_\lambda(\alpha) < 0$ on $\left(\check{\alpha}, \frac{5}{12\varepsilon}\right)$ for some $\check{\alpha} \in \left(\frac{1}{3\varepsilon}, \frac{5}{12\varepsilon}\right)$.

(cv) $T'_\lambda(\alpha) > 0$ on $\left(\frac{1}{3\varepsilon}, \check{\alpha}\right) \cup \left(\hat{\alpha}, \frac{5}{12\varepsilon}\right)$ and $T'_\lambda(\alpha) < 0$ on $(\check{\alpha}, \hat{\alpha})$ for some $\check{\alpha}, \hat{\alpha} \in \left(\frac{1}{3\varepsilon}, \frac{5}{12\varepsilon}\right)$.

See Figure 2.3.

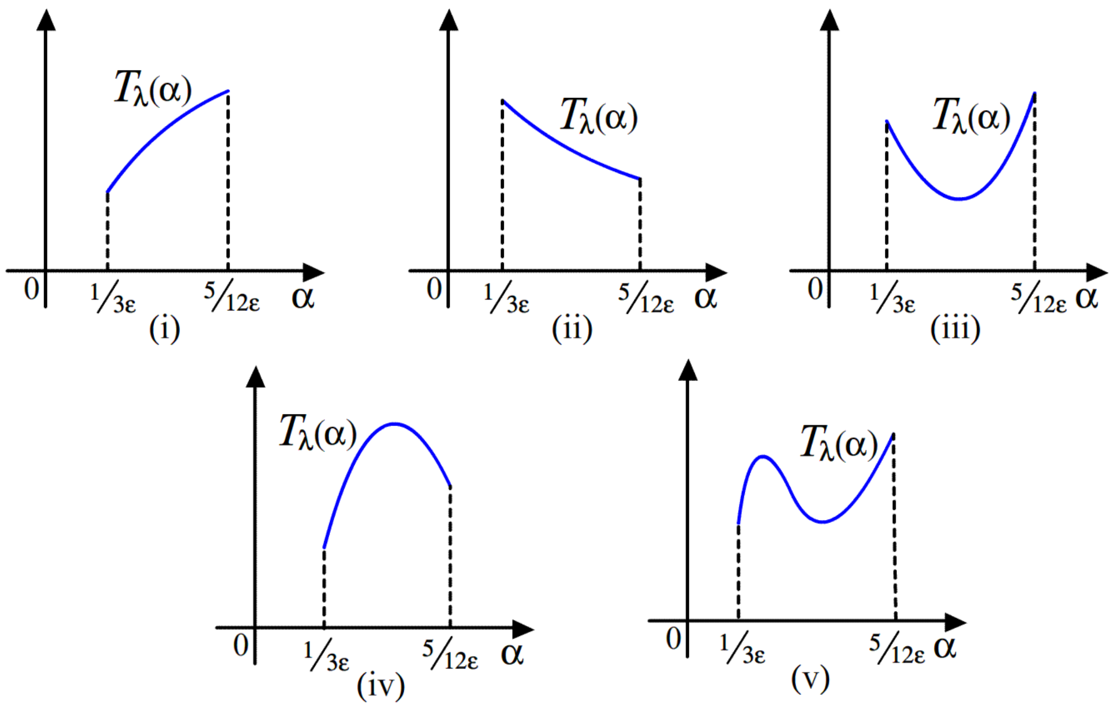

Figure 2.3: Graphs of $T_\lambda(\alpha)$ on $\left(\frac{1}{3\varepsilon}, \frac{5}{12\varepsilon}\right)$ for $0 < \lambda < \lambda_3$.

Let $H(\alpha, \lambda) \equiv 2\alpha T''_\lambda(\alpha) + 3T'_\lambda(\alpha)$. By Lemmas 2.4(iii) and 2.5(ii), there exists $\lambda_3 \in (0, \lambda_2)$ such that

$$\frac{\partial}{\partial \alpha} H(\alpha, \lambda) > 0 \quad \text{for } \frac{1}{3\varepsilon} \le \alpha \le \frac{5}{12\varepsilon} \text{ and } 0 < \lambda \le \lambda_3. \tag{2.49}$$

Fixed $\lambda \in (0, \lambda_3)$. Then we consider three cases.

Case 1. Assume that $H(\alpha, \lambda) < 0$ for $\frac{1}{3\varepsilon} \le \alpha < \frac{5}{12\varepsilon}$. If $T_\lambda(\alpha)$ has a critical point α_1 in $\left(\frac{1}{3\varepsilon}, \frac{5}{12\varepsilon}\right)$, then

$$2\alpha_1 T''_\lambda(\alpha_1) = H(\alpha_1, \lambda) < 0.$$

It implies that $T_\lambda(\alpha)$ has at most one critical point, a local maximum, on $\left(\frac{1}{3\varepsilon}, \frac{5}{12\varepsilon}\right)$. Thus one of (ci), (cii) and (civ) holds.

Case 2. Assume that $H(\alpha,\lambda) > 0$ for $\frac{1}{3\varepsilon} < \alpha \leq \frac{5}{12\varepsilon}$. If $T_\lambda(\alpha)$ has a critical point α_2 in $\left(\frac{1}{3\varepsilon}, \frac{5}{12\varepsilon}\right)$, then

$$2\alpha_2 T''_\lambda(\alpha_2) = H(\alpha_2, \lambda) > 0.$$

It implies that $T_\lambda(\alpha)$ has at most one critical point, a local minimum, on $\left(\frac{1}{3\varepsilon}, \frac{5}{12\varepsilon}\right)$. Thus one of (ci), (cii) and (ciii) holds.

Case 3. Assume that there exists $\alpha_* \in \left(\frac{1}{3\varepsilon}, \frac{5}{12\varepsilon}\right)$ such that $H(\alpha,\lambda) < 0$ for $\frac{1}{3\varepsilon} < \alpha < \alpha_*$ and $H(\alpha,\lambda) > 0$ for $\alpha_* < \alpha < \frac{5}{12\varepsilon}$. If $T_\lambda(\alpha)$ has a critical point in $\left(\frac{1}{3\varepsilon}, \alpha_*\right)$, by above similar argument, $T_\lambda(\alpha)$ has at most one critical point, a local maximum, on $\left(\frac{1}{3\varepsilon}, \alpha_*\right)$. If $T_\lambda(\alpha)$ has a critical point in $\left(\alpha_*, \frac{5}{12\varepsilon}\right)$, by above similar argument, $T_\lambda(\alpha)$ has at most one critical point, a local minimum, on $\left(\alpha_*, \frac{5}{12\varepsilon}\right)$. Thus one of (ci)–(cv) holds.

Step 3. We prove Lemma 2.10. By Lemmas 2.1(i) and 2.2(ii), we see that, for $\lambda > 0$, either $T'_\lambda(\alpha) > 0$ on $\left[\frac{5}{12\varepsilon}, \beta_\varepsilon\right)$, or there exists $\mathring{\alpha} \in \left(\frac{5}{12\varepsilon}, \beta_\varepsilon\right)$ such that $T'_\lambda(\alpha) < 0$ on $\left[\frac{5}{12\varepsilon}, \mathring{\alpha}\right)$ and $T'_\lambda(\alpha) > 0$ on $(\mathring{\alpha}, \beta_\varepsilon)$, see Figure 2.4.

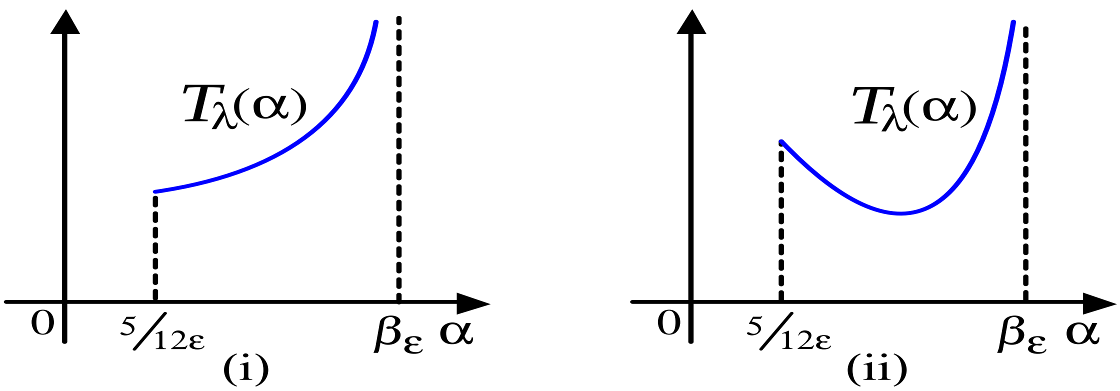

Figure 2.4: Graphs of $T_\lambda(\alpha)$ on $[5/(12\varepsilon), \beta_\varepsilon)$ for $\lambda > 0$.

Then by (2.44) and Steps 1–2, we observe that $T_\lambda(\alpha)$ has exactly two critical points, a local maximum at $\alpha_M(\lambda)$ and a local minimum at $\alpha_m(\lambda)$ $(> \alpha_M(\lambda))$, on $(0, \beta_\varepsilon)$ if $0 < \lambda < \kappa_\varepsilon = \lambda_3$.

The proof is complete. □

Lemma 2.11. *Consider* (1.1) *with* $0 < \varepsilon < \varepsilon_0$. *Let* $\alpha_M(\lambda)$ *and* $\alpha_m(\lambda)$ *be defined in Lemma* 2.10. *Then* $\alpha_M(\lambda)$ *is a strictly increasing function of* $\lambda \in (0, \kappa_\varepsilon)$ *and*

$$\lim_{\lambda \to 0^+} \alpha_M(\lambda) < \alpha_M(\lambda) < \lim_{\lambda \to \kappa_\varepsilon^-} \alpha_M(\lambda) \leq \alpha_m(\lambda) \ \text{ for } \lambda \in (0, \kappa_\varepsilon). \tag{2.50}$$

Proof. By Lemma 2.10, we have that

$$T'_\lambda(\alpha) \begin{cases} > 0 & \text{for } \alpha \in (0, \alpha_M(\lambda)) \cup (\alpha_m(\lambda), \infty), \\ = 0 & \text{for } \alpha = \alpha_M(\lambda) \text{ or } \alpha = \alpha_m(\lambda), \\ < 0 & \text{for } \alpha \in (\alpha_M(\lambda), \alpha_m(\lambda)), \end{cases} \quad \text{if } 0 < \lambda < \kappa_\varepsilon. \tag{2.51}$$

By Lemma 2.2, we see that $0 < \alpha_M(\lambda) < \frac{5}{12\varepsilon}$ for $0 < \lambda < \kappa_\varepsilon$. Let $0 < \lambda_1 < \lambda_2 < \kappa_\varepsilon$. By Lemma 2.3, we obtain that

$$\sqrt{\lambda_1} T'_{\lambda_1}(\alpha_M(\lambda_2)) < \sqrt{\lambda_2} T'_{\lambda_2}(\alpha_M(\lambda_2)) = 0,$$

which implies that $\alpha_M(\lambda_1) < \alpha_M(\lambda_2)$ by (2.51). So $\alpha_M(\lambda)$ is a strictly increasing function of $\lambda \in (0, \kappa_\varepsilon)$. It follows that

$$\lim_{\lambda \to 0^+} \alpha_M(\lambda) < \alpha_M(\lambda) < \lim_{\lambda \to \kappa_\varepsilon^-} \alpha_M(\lambda) \quad \text{for } \lambda \in (0, \kappa_\varepsilon).$$

Assume that there exists $\lambda_3 \in (0, \kappa_\varepsilon)$ such that $\lim_{\lambda \to 0^+} \alpha_M(\lambda) < \alpha_m(\lambda_3) < \lim_{\lambda \to \kappa_\varepsilon^-} \alpha_M(\lambda)$. Then there exists $\lambda_4 \in (\lambda_3, \kappa_\varepsilon)$ such that

$$\alpha_M(\lambda_3) < \alpha_m(\lambda_3) < \alpha_M(\lambda_4) < \frac{5}{12\varepsilon}. \tag{2.52}$$

By (2.51), there exists $\alpha_1 \in \left(\alpha_M(\lambda_4), \frac{5}{12\varepsilon}\right)$ such that $T'_{\lambda_4}(\alpha_1) < 0$. Then by (2.51), (2.52) and Lemma 2.3, we observe that

$$0 < \sqrt{\lambda_3} T'_{\lambda_3}(\alpha_1) < \sqrt{\lambda_4} T'_{\lambda_4}(\alpha_1) < 0,$$

which is a contradiction. So (2.50) holds. The proof is complete. □

Lemma 2.12 ([9, Lemma 4.6]). *Consider (1.1) with fixed $L > 0$. Let $\rho_{L,\varepsilon} \equiv \min\{L, \beta_\varepsilon\}$ and $sgn(u)$ be the signum function. Then the following statements (i)–(iii) hold:*

(i) There exists a positive function $\lambda_L(\alpha) \in C^1(0, \rho_{L,\varepsilon})$ such that $T_{\lambda_L(\alpha)}(\alpha) = L$. Moreover, the bifurcation curve $S_L = \{(\lambda_L(\alpha), \alpha) : \alpha \in (0, \rho_{L,\varepsilon})\}$ is continuous on the $(\lambda, \|u\|_\infty)$-plane.

(ii) $\lim_{\alpha \to 0^+} \lambda_L(\alpha) = 0$ and $\lim_{\alpha \to \rho_{L,\varepsilon}^-} \lambda_L(\alpha) = \infty$.

(iii) $\operatorname{sgn}(\lambda'_L(\alpha)) = \operatorname{sgn}(T'_{\lambda_L(\alpha)}(\alpha))$ for $\alpha \in (0, \rho_{L,\varepsilon})$.

Lemma 2.13 ([10, Lemma 3.5]). *Consider (1.1). Let $L > 0$. Then the following statements (i) and (ii) hold:*

(i) If $\lambda_L(\alpha)$ has a local maximum at α_M, then $T_{\lambda_L(\alpha_M)}(\alpha)$ has a local maximum at α_M. Conversely, if $T_\lambda(\alpha)$ has a local maximum at α_M and $T_\lambda(\alpha_M) = L$, then $\lambda_L(\alpha)$ has a local maximum at α_M.

(ii) If $\lambda_L(\alpha)$ has a local minimum at α_m, then $T_{\lambda_L(\alpha_m)}(\alpha)$ has a local minimum at α_m. Conversely, if $T_\lambda(\alpha)$ has a local minimum at α_m and $T_\lambda(\alpha_m) = L$, then $\lambda_L(\alpha)$ has a local minimum at α_m.

Lemma 2.14. *Consider (1.1) with $0 < \varepsilon < \varepsilon_0$. Then there exists a continuous function $L_\varepsilon \in (0, \infty)$ of ε such that*

$$\Lambda_\varepsilon \equiv \left\{L > 0 : \lambda'_L(\alpha) < 0 \text{ for some } \alpha \in (0, \rho_{L,\varepsilon})\right\} = (L_\varepsilon, \infty).$$

Furthermore, $\lambda'_L(\alpha) > 0$ for $\alpha \in (0, \rho_{L,\varepsilon})$ where $0 < L < L_\varepsilon$.

Proof. Let $\varepsilon \in (0, \varepsilon_0)$ be given. By Lemma 2.9 and similar argument in the proof of [7, Lemma 4.7], there exists $L_\varepsilon \in [0, \infty)$ such that $\Lambda_\varepsilon = (L_\varepsilon, \infty)$. We divide the rest of the proof into the next three steps.

Step 1. We prove that $L_\varepsilon > 0$. Assume that $L_\varepsilon = 0$. By Lemma 2.9, we have

$$T'_\lambda(\alpha) \geq 0 \quad \text{for } 0 < \alpha < \beta_\varepsilon \text{ and } \lambda \geq \xi_\varepsilon. \tag{2.53}$$

Let $L = T_{\xi_\varepsilon}(1)$. It implies that $L \in \Lambda_\varepsilon = (0, \infty)$. Then there exists $\alpha_1 \in (0, \rho_{L,\varepsilon})$ such that $\lambda'_L(\alpha_1) < 0$. It follows that $T'_{\lambda_L(\alpha_1)}(\alpha_1) < 0$ by Lemma 2.12(iii). By (2.45) and (2.53), we observe that $\alpha_1 > 1$ and $0 < \lambda_L(\alpha_1) < \xi_\varepsilon$. By Lemmas 2.1(iii), 2.12(i) and (2.53), we further observe that

$$L = T_{\lambda_L(\alpha_1)}(\alpha_1) > T_{\xi_\varepsilon}(\alpha_1) \geq T_{\xi_\varepsilon}(1) = L,$$

which is a contradiction. Thus $L_\varepsilon > 0$.

Step 2. We prove that $\lambda'_L(\alpha) > 0$ for $\alpha \in (0, \rho_{L,\varepsilon})$ where $0 < L < L_\varepsilon$. Let $L \in (0, L_\varepsilon)$ be given. Assume that there exists $\alpha_2 \in (0, \rho_{L,\varepsilon})$ such that $\lambda'_L(\alpha_2) = 0$. So by Lemma 2.12(iii), we obtain that $T'_{\lambda_L(\alpha_2)}(\alpha_2) = 0$. Since

$$0 < \alpha_2 < \rho_{L,\varepsilon} = \min\{L, \beta_\varepsilon\} < \min\{L_\varepsilon, \beta_\varepsilon\} = \rho_{L_\varepsilon,\varepsilon},$$

we see that $T_{\lambda_L(\alpha_2)}(\alpha_2) = L < L_\varepsilon = T_{\lambda_{L_\varepsilon}(\alpha_2)}(\alpha_2)$. So by Lemma 2.1(iii), we obtain that $\lambda_L(\alpha_2) > \lambda_{L_\varepsilon}(\alpha_2)$. Assume that $\alpha_2 \geq \frac{5\varepsilon}{12}$. Since $T'_{\lambda_L(\alpha_2)}(\alpha_2) = 0$, and by Lemma 2.2(ii), $T_{\lambda_L(\alpha_2)}(\alpha)$ has a local minimum at α_2. By Lemma 2.13, we find that $\lambda_L(\alpha)$ has a local minimum at α_2, which is a contradiction since $L < L_\varepsilon$. So $0 < \alpha_2 < \frac{5\varepsilon}{12}$. By Lemma 2.3, we see that

$$\sqrt{\lambda_{L_\varepsilon}(\alpha_2)}T'_{\lambda_{L_\varepsilon}(\alpha_2)}(\alpha_2) < \sqrt{\lambda_L(\alpha_2)}T'_{\lambda_L(\alpha_2)}(\alpha_2) = 0,$$

from which it follows that by Lemma 2.12(iii), $\lambda'_{L_\varepsilon}(\alpha_2) < 0$. It is a contradiction since $\lambda'_{L_\varepsilon}(\alpha) \geq 0$ for $\alpha \in (0, \rho_{L,\varepsilon})$. Thus $\lambda'_L(\alpha) > 0$ for $\alpha \in (0, \rho_{L,\varepsilon})$ where $0 < L < L_\varepsilon$.

Step 3. We prove the continuity of L_ε. Let $\bar{\varepsilon} \in (0, \varepsilon_0)$ be given. For the sake of convenience, we let $T_\lambda(\alpha, \varepsilon) = T_\lambda(\alpha)$ and $\lambda_L(\alpha, \varepsilon) = \lambda_L(\alpha)$. We consider the following two cases and prove they would not occur.

Case 1. Assume that $\liminf_{\varepsilon \to \bar{\varepsilon}} L_\varepsilon < L_{\bar{\varepsilon}}$. Let $L \in (\liminf_{\varepsilon \to \bar{\varepsilon}} L_\varepsilon, L_{\bar{\varepsilon}})$ be given. Then there exists $\{\varepsilon_n\}_{n \in \mathbb{N}} \subset (0, \varepsilon_0)$ such that

$$\lim_{n \to \infty} \varepsilon_n = \bar{\varepsilon} \quad \text{and} \quad L_{\varepsilon_n} < L < L_{\bar{\varepsilon}} \quad \text{for } n \in \mathbb{N}.$$

So there exists $\{\alpha_n\}_{n \in \mathbb{N}} \subset (0, \rho_{L,\varepsilon_n})$ such that

$$\frac{\partial}{\partial \alpha}\lambda_L(\alpha, \bar{\varepsilon}) > 0 \quad \text{for } 0 < \alpha < \rho_{L,\varepsilon} \quad \text{and} \quad \frac{\partial}{\partial \alpha}\lambda_L(\alpha_n, \varepsilon_n) < 0 \quad \text{for } n \in \mathbb{N}. \tag{2.54}$$

By Lemmas 2.2(i) and 2.12(iii), we have

$$\frac{\partial}{\partial \alpha}\lambda_L(\alpha, \varepsilon) > 0 \quad \text{for } 0 < \alpha \leq 1 \text{ and } 0 < \varepsilon < \varepsilon_0. \tag{2.55}$$

By (2.54) and (2.55), we see that $\alpha_n \in (1, \rho_{L,\varepsilon_n})$. We assume without loss of generality that $\lim_{n\to\infty} \alpha_n = \bar{\alpha} \in [1, \rho_{L,\varepsilon_n}]$. If $\bar{\alpha} < \rho_{L,\varepsilon_n}$, by (2.54), we observe that

$$0 < \frac{\partial}{\partial \alpha}\lambda_L(\bar{\alpha}, \bar{\varepsilon}) = \lim_{n \to \infty} \frac{\partial}{\partial \alpha}\lambda_L(\alpha_n, \varepsilon_n) \leq 0,$$

which is a contradiction. If $\bar{\alpha} = \rho_{L,\varepsilon_n}$, by (2.54) and Lemma 2.12(ii), we observe that

$$\lim_{\alpha \to \rho^-_{L,\varepsilon}} \lambda_L(\alpha, \bar{\varepsilon}) = \infty \quad \text{and} \quad \lim_{\alpha \to \rho^-_{L,\varepsilon}} \frac{\partial}{\partial \alpha}\lambda_L(\alpha, \bar{\varepsilon}) \leq 0,$$

which is a contradiction.

Case 2. Assume that $\limsup_{\varepsilon\to\bar{\varepsilon}} L_\varepsilon > L_{\bar{\varepsilon}}$. Let $L \in (L_{\bar{\varepsilon}}, \limsup_{\varepsilon\to\bar{\varepsilon}} L_\varepsilon)$ be given. Then there exists $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0,\varepsilon_0)$ such that

$$\lim_{n\to\infty} \varepsilon_n = \bar{\varepsilon} \quad \text{and} \quad L_{\bar{\varepsilon}} < L < L_{\varepsilon_n} \quad \text{for } n \in \mathbb{N}.$$

So there exists $\bar{\alpha} \in (0, \rho_{L,\bar{\varepsilon}})$ such that

$$\frac{\partial}{\partial\alpha}\lambda_L(\bar{\alpha},\bar{\varepsilon}) < 0 \quad \text{and} \quad \frac{\partial}{\partial\alpha}\lambda_L(\alpha,\varepsilon_n) > 0 \quad \text{for } 0 < \alpha < \rho_{L,\varepsilon_n} \text{ and } n \in \mathbb{N}. \tag{2.56}$$

Since $f(\beta_\varepsilon) = 0$, and by implicit function theorem, β_ε is a strictly decreasing and continuous function of $\varepsilon > 0$. So we see that $\bar{\alpha} < \rho_{L,\bar{\varepsilon}} \le \beta_{\bar{\varepsilon}} < \beta_{\varepsilon_n}$ for $n \in \mathbb{N}$. It implies that $0 < \bar{\alpha} < \rho_{L,\varepsilon_n}$ for $n \in \mathbb{N}$. By (2.56), we observe that

$$0 > \frac{\partial}{\partial\alpha}\lambda_L(\bar{\alpha},\bar{\varepsilon}) = \lim_{n\to\infty}\frac{\partial}{\partial\alpha}\lambda_L(\bar{\alpha},\varepsilon_n) \ge 0,$$

which is a contradiction.

So by Cases 1 and 2, we see that $\limsup_{\varepsilon\to\bar{\varepsilon}} L_\varepsilon \le L_{\bar{\varepsilon}} \le \liminf_{\varepsilon\to\bar{\varepsilon}} L_\varepsilon$. It follows that $L_{\bar{\varepsilon}} = \lim_{a\to\bar{a}} L_\varepsilon$. Thus L_ε is a continuous function on $(0,\varepsilon_0)$.

The proof is complete. □

Lemma 2.15. *Consider* (1.1) *with* $0 < \varepsilon < \varepsilon_0$. *Then there exists* $\tilde{L}_\varepsilon > L_\varepsilon$ *such that* $\lambda_L(\alpha)$ *has exactly one local maximum and exactly one local minimum on* $(0,\rho_{L,\varepsilon})$ *for* $L > \tilde{L}_\varepsilon$.

Proof. Let $\lambda_* \in (0,\kappa_\varepsilon)$ be given. By Lemma 2.10, then

$$T'_\lambda(\alpha)\begin{cases} > 0 & \text{for } \alpha \in (0,\alpha_M(\lambda)) \cup (\alpha_m(\lambda),\beta_\varepsilon), \\ = 0 & \text{for } \alpha = \alpha_M(\lambda) \text{ or } \alpha = \alpha_m(\lambda), \\ < 0 & \text{for } \alpha \in (\alpha_M(\lambda),\alpha_m(\lambda)), \end{cases} \quad \text{if } 0 < \lambda \le \lambda^*. \tag{2.57}$$

Let $\tilde{L}_\varepsilon \equiv T_{\lambda^*}(\alpha_M(\lambda^*))$. We divide this proof into the next three steps.

Step 1. We prove that $\tilde{L}_\varepsilon > L_\varepsilon$. Let $L \ge \tilde{L}_\varepsilon$ and

$$\alpha_1 \in \left(\alpha_M(\lambda^*), \min\left\{\alpha_m(\lambda^*), \frac{5}{12\varepsilon}\right\}\right). \tag{2.58}$$

By (2.57) and (2.58), we see that

$$\lim_{\lambda\to 0^+} T_\lambda(\alpha) = \infty > L \ge T_{\lambda^*}(\alpha_M(\lambda^*)) > T_{\lambda^*}(\alpha_1).$$

So by Lemma 2.1(iii) and continuity of $T_\lambda(\alpha)$ with respect to λ, there exists $\lambda_* \in (0,\lambda^*)$ such that $L = T_{\lambda_*}(\alpha_1)$. Clearly, $\lambda_* = \lambda_L(\alpha_1)$ by Lemma 2.12(i). Then by (2.57), (2.58) and Lemma 2.3, we observe that

$$\sqrt{\lambda_*}T'_{\lambda_L(\alpha_1)}(\alpha_1) = \sqrt{\lambda_*}T'_{\lambda_*}(\alpha_1) < \sqrt{\lambda^*}T'_{\lambda^*}(\alpha_1) < 0.$$

So by Lemma 2.12(iii), we obtain that $\lambda'_L(\alpha_1) < 0$. It implies that $L > L_\varepsilon$ by Lemma 2.14. Thus $\tilde{L}_\varepsilon > L_\varepsilon$.

Step 2. We prove that $\lambda_L(\alpha)$ has exactly one local maximum in $(0,\rho_{L,\varepsilon})$ for $L > \tilde{L}_\varepsilon$. Let $L > \tilde{L}_\varepsilon$ be given. By Lemmas 2.2(i) and 2.12(iii), we see that $\lambda'_L(\alpha) > 0$ for $0 < \alpha \le 1$. Since

$L > \tilde{L}_\varepsilon$, and by Lemma 2.14, $\lambda_L(\alpha)$ has at least one local maximum in $(0, \rho_{L,\varepsilon})$. Assume that $\lambda_L(\alpha)$ has two local maximums at α_M^1 and α_M^2 $(> \alpha_M^1)$. Then $\lambda_L(\alpha)$ has a local minimum at $\alpha_m \in (\alpha_M^1, \alpha_M^2)$. Without loss of generality, we assume that $\lambda_L(\alpha_M^1) > \lambda_L(\alpha_m)$. For the sake of convenience, we let

$$\lambda_1 = \lambda_L(\alpha_M^1), \qquad \lambda_2 = \lambda_L(\alpha_M^2) \quad \text{and} \quad \lambda_3 = \lambda_L(\alpha_m).$$

So by Lemma 2.13, we see that $T_{\lambda_1}(\alpha_M^1)$ and $T_{\lambda_2}(\alpha_M^2)$ are local maximum values and $T_{\lambda_3}(\alpha_m)$ is a local minimum value. In addition, we note that

$$T_{\lambda_1}(\alpha_M^1) = T_{\lambda_L(\alpha_M^1)}(\alpha_M^1) = L > \tilde{L}_\varepsilon = T_{\lambda^*}(\alpha_M(\lambda^*)). \tag{2.59}$$

Assume that $\lambda_1 \geq \lambda^*$. By Lemma 2.1(iii) and (2.59), we observe that $T_{\lambda^*}(\alpha_M^1) \geq T_{\lambda_1}(\alpha_M^1) > T_{\lambda^*}(\alpha_M(\lambda^*))$. It implies that

$$\alpha_m(\lambda^*) < \alpha_M^1 \quad \text{and} \quad T'_{\lambda^*}(\alpha_M^1) > 0. \tag{2.60}$$

By Lemma 2.2(ii), we have $\alpha_M^1 < \alpha_M^2 < \frac{5}{12\varepsilon}$. So by Lemma 2.3 and (2.60), we observe hat

$$0 < \sqrt{\lambda^*} T'_{\lambda^*}(\alpha_M^1) \leq \sqrt{\lambda_1} T'_{\lambda_1}(\alpha_M^1) = 0,$$

which is a contradiction. So $\lambda_1 < \lambda^*$. Similarly, we obtain that $\lambda_2 < \lambda^*$. So by (2.57) and Lemma 2.10, we see that

$$\alpha_M(\lambda_1) = \alpha_M^1 < \alpha_m = \alpha_m(\lambda_3) < \alpha_M^2 = \alpha_M(\lambda_2),$$

which is a contradiction by Lemma 2.11. Thus $\lambda_L(\alpha)$ has exactly one local maximum in $(0, \rho_{L,\varepsilon})$.

Step 3. We prove Lemma 2.15. Since $\lambda'_L(\alpha) > 0$ for $0 < \alpha \leq 1$, and by Lemma 2.12(ii) and Step 2, we see that $\lambda_L(\alpha)$ has exactly one local maximum and one local minimum on $(0, \rho_{L,\varepsilon})$ for $L > \tilde{L}_\varepsilon$.

The proof is complete. □

3 Proof of the main result

Proof of Theorem 1.3. (I) The statement (i) follows immediately by Lemma 2.12(i)(ii).

(II) Assume that $\varepsilon \geq \varepsilon_0$. By Theorem 1.2 and (2.4), we obtain that $\bar{T}'(\alpha) \geq 0$ for $0 < \alpha < \beta_\varepsilon$. So by Lemmas 2.1(ii) and 2.3, we see that

$$0 \leq \bar{T}'(\alpha) = \lim_{\lambda \to 0^+} \sqrt{\lambda} T'_\lambda(\alpha) < \sqrt{\lambda} T'_\lambda(\alpha) \quad \text{for } 0 < \alpha \leq \frac{5}{12\varepsilon} \text{ and } \lambda > 0. \tag{3.1}$$

Since $T'_\lambda(\frac{5}{12\varepsilon}) > 0$ for $\lambda > 0$, and by Lemma 2.2(ii), we further see that

$$T'_\lambda(\alpha) > 0 \quad \text{for } \frac{5}{12\varepsilon} < \alpha < \beta_\varepsilon \text{ and } \lambda > 0. \tag{3.2}$$

So by (3.1), (3.2) and Lemma 2.12(iii), we obtain that

$$\lambda'_L(\alpha) > 0 \quad \text{for } 0 < \alpha < \rho_{L,\varepsilon} \text{ and } \lambda > 0.$$

Then the statement (ii) holds.

(III) Assume that $0 < \varepsilon < \varepsilon_0$. By Lemma 2.14, there exists a continuous function $L_\varepsilon \in (0, \infty)$ of ε such that

$$\Lambda_\varepsilon = \left\{ L > 0 : \lambda_L'(\alpha) < 0 \ \text{ for some } \alpha \in (0, \rho_{L,\varepsilon}) \right\} = (L_\varepsilon, \infty).$$

So by Lemma 2.12(i), the bifurcation curve S_L is monotone increasing if $0 < L \leq L_\varepsilon$, and is S-like shaped if $L > L_\varepsilon$. In addition, by Lemma 2.15, there exists $\tilde{L}_\varepsilon > L_\varepsilon$ such that $\lambda_L(\alpha)$ has one local maximum and one local minimum on $(0, \rho_{L,\varepsilon})$ for $L > \tilde{L}_\varepsilon$. So by Lemma 2.12(i), the bifurcation curve S_L is S-shaped if $L > \tilde{L}_\varepsilon$. Next, we divide into the next two steps to prove that $\lim_{\varepsilon \to 0^+} L_\varepsilon \in (0, \infty)$ and $\lim_{\varepsilon \to \varepsilon_0^-} L_\varepsilon = \infty$.

Step 1. We prove that $\lim_{\varepsilon \to \varepsilon_0^-} L_\varepsilon = \infty$. Assume that $\lim_{\varepsilon \to \varepsilon_0^-} L_\varepsilon < \infty$. Let $L > \lim_{\varepsilon \to \varepsilon_0^-} L_\varepsilon$. For the sake of convenience, we let

$$\lambda_L(\alpha, \varepsilon) = \lambda_L(\alpha), \qquad T_\lambda(\alpha, \varepsilon) = T_\lambda(\alpha) \quad \text{and} \quad \bar{T}(\alpha, \varepsilon) = \bar{T}(\alpha).$$

Since $L > \lim_{\varepsilon \to \varepsilon_0^-} L_\varepsilon$, there exists $\delta > 0$ such that $L > L_\varepsilon$ for $\varepsilon \in (\varepsilon_0 - \delta, \varepsilon_0)$. So for $\varepsilon \in (\varepsilon_0 - \delta, \varepsilon_0)$, by Lemmas 2.2(ii) and 2.14, there exists $\alpha_\varepsilon \in \left[1, \frac{5}{12\varepsilon}\right]$ such that $\frac{\partial}{\partial \alpha} \lambda_L(\alpha_\varepsilon, \varepsilon) < 0$. Without loss of generality, we assume that $\lim_{\varepsilon \to \varepsilon_0^+} \alpha_\varepsilon = \alpha_0 \in \left[1, \frac{5}{12\varepsilon}\right]$. By Theorem 1.2 and (2.4), we see that $\bar{T}'(\alpha_0, \varepsilon_0) \geq 0$. So by Lemma 2.3, we further see that

$$0 \leq \bar{T}'(\alpha_0, \varepsilon_0) = \lim_{\lambda \to 0^+} \sqrt{\lambda} T_\lambda'(\alpha_0, \varepsilon_0) < \sqrt{\lambda} T_\lambda'(\alpha_0, \varepsilon_0) \quad \text{for } \lambda > 0.$$

Then by Lemma 2.12(iii), we obtain that $\frac{\partial}{\partial \alpha} \lambda_L(\alpha_0, \varepsilon_0) > 0$. It follows that

$$0 \geq \lim_{\varepsilon \to \varepsilon_0^+} \frac{\partial}{\partial \alpha} \lambda_L(\alpha_\varepsilon, \varepsilon) = \frac{\partial}{\partial \alpha} \lambda_L(\alpha_0, \varepsilon_0) > 0,$$

which is a contradiction. So $\lim_{\varepsilon \to \varepsilon_0^-} L_\varepsilon = \infty$.

Step 2. We prove that $\lim_{\varepsilon \to 0^+} L_\varepsilon \in (0, \infty)$. Notice that as $\varepsilon \to 0^+$, the cubic polynomial $f(u)$ reduces to the quadratic polynomial $u^2 + u + 1$. So we consider the equation

$$\begin{cases} -\left(\dfrac{u'(x)}{\sqrt{1 - (u'(x))^2}} \right)' = \lambda (u^2 + u + 1), \quad -L < x < L, \\ u(-L) = u(L) = 0. \end{cases} \tag{3.3}$$

Since $u^2 + u + 1$ satisfies all hypotheses of [7, Theorem 3.2], there exists $L_0 > 0$ such that the bifurcation curve S_L of (3.3) is S-like shaped for $L > L_0$, monotone increasing for $0 < L \leq L_0$, and has no vertical tangent lines for $0 < L < L_0$. Thus we have the following assertions (i)–(iii):

(i) if $L > L_0$, then $\lambda_L'(\alpha, 0) < 0$ for some $\alpha > 0$.

(ii) if $L = L_0$, then $\lambda_L'(\alpha, 0) \geq 0$ for $\alpha > 0$.

(iii) if $0 < L < L_0$, then $\lambda_L'(\alpha, 0) > 0$ for $\alpha > 0$.

By a similar argument as in the proof of Lemma 2.14, we can prove that L_ε is a continuous function of $\varepsilon \in [0, \varepsilon_0)$. Thus $\lim_{\varepsilon \to 0^+} L_\varepsilon = L_0 \in (0, \infty)$.

The proof is complete. □

4 Appendix

In this section, we prove assertion (2.31). Let $\bar{\varepsilon} = \sqrt{\frac{31}{1000}}$ (≈ 0.176). By Lemma 2.5(iii), we have $\hat{\varepsilon} < \bar{\varepsilon}$. To prove (2.31), it is sufficient to prove that

$$N_2(\alpha, u) < 0 \quad \text{for} \quad 0 < u < \alpha, 1.7 \leq \alpha \leq \frac{1}{3\varepsilon} \quad \text{and} \quad 0.07 \leq \varepsilon \leq \bar{\varepsilon}\ (\approx 0.176). \tag{4.1}$$

Let $\alpha \in [1.7, \frac{1}{3\varepsilon}]$ be given and $N_2(u) = N_2(\alpha, u)$. It is easy to compute that

$$\begin{aligned} N_2'(u) = & -\frac{1}{2}\varepsilon^2 u^7 + \frac{49}{24}\varepsilon u^6 + \left(\frac{21}{4}\varepsilon - \frac{2}{3}\right) u^5 + \left(\frac{125}{8}\varepsilon - \frac{25}{12}\right) u^4 + \left(\frac{1}{2}\varepsilon^2\alpha^4 - \frac{7}{6}\varepsilon\alpha^3 \right. \\ & \left. -\frac{7}{2}\varepsilon\alpha^2 - \frac{25}{2}\varepsilon\alpha - \frac{20}{3}\right) u^3 + \left(-\frac{7}{8}\varepsilon\alpha^4 + \frac{2}{3}\alpha^3 + \frac{5}{4}\alpha^2 + 5\alpha + \frac{3}{4}\right) u^2 \\ & + \left(-\frac{7}{4}\varepsilon\alpha^4 + \frac{5}{6}\alpha^3 - \frac{1}{2}\alpha + 4\right) u - \frac{25}{8}\varepsilon\alpha^4 + \frac{5}{3}\alpha^3 - \frac{1}{4}\alpha^2 - 4\alpha, \end{aligned}$$

$$\begin{aligned} N_2''(u) = & -\frac{7}{2}\varepsilon^2 u^6 + \frac{49}{4}\varepsilon u^5 + \left(\frac{105}{4}\varepsilon - \frac{10}{3}\right) u^4 + \left(\frac{125}{2}\varepsilon - \frac{25}{3}\right) u^3 + \left(\frac{3}{2}\varepsilon^2\alpha^4 - \frac{7}{2}\varepsilon\alpha^3 \right. \\ & \left. -\frac{21}{2}\varepsilon\alpha^2 - \frac{75}{2}\varepsilon\alpha - 20\right) u^2 + \left(-\frac{7}{4}\varepsilon\alpha^4 + \frac{4}{3}\alpha^3 + \frac{5}{2}\alpha^2 + 10\alpha + \frac{3}{2}\right) u \\ & -\frac{7}{4}\varepsilon\alpha^4 + \frac{5}{6}\alpha^3 - \frac{1}{2}\alpha + 4, \end{aligned}$$

$$\begin{aligned} N_2'''(u) = & -21\varepsilon^2 u^5 + \frac{245}{4}\varepsilon u^4 + \left(105\varepsilon - \frac{40}{3}\right) u^3 + \left(\frac{375}{2}\varepsilon - 25\right) u^2 + (3\varepsilon^2\alpha^4 - 7\varepsilon\alpha^3 \\ & -21\varepsilon\alpha^2 - 75\varepsilon\alpha - 40)u - \frac{7}{4}\varepsilon\alpha^4 + \frac{4}{3}\alpha^3 + \frac{5}{2}\alpha^2 + 10\alpha + \frac{3}{2}, \end{aligned}$$

$$\begin{aligned} N_2^{(4)}(u) = & -105\varepsilon^2 u^4 + 245\varepsilon u^3 + (315\varepsilon - 40) u^2 + (375\varepsilon - 50) u + 3\varepsilon^2\alpha^4 - 7\varepsilon\alpha^3 \\ & -21\varepsilon\alpha^2 - 75\varepsilon\alpha - 40, \end{aligned}$$

$$N_2^{(5)}(u) = -420\varepsilon^2 u^3 + 735\varepsilon u^2 + (630\varepsilon - 80) u + 375\varepsilon - 50,$$

$$N_2^{(6)}(u) = -1260\varepsilon^2 u^2 + 1470\varepsilon u + 630\varepsilon - 80.$$

Then we divide the proof into the next four steps.

Step 1. We prove that, for $0.07 \leq \varepsilon \leq \bar{\varepsilon}$,

$$N_2''(0) = -\frac{7}{4}\varepsilon\alpha^4 + \frac{5}{6}\alpha^3 - \frac{1}{2}\alpha + 4 > 0. \tag{4.2}$$

It is easy to see that

$$1.7 \leq \alpha \leq \frac{1}{3\varepsilon} \leq \frac{1}{3(0.07)} = \frac{100}{21} \quad \text{for } 0.07 \leq \varepsilon \leq \bar{\varepsilon}. \tag{4.3}$$

Since $\varepsilon \leq \frac{1}{3\alpha}$, and by (4.3), we observe that

$$N_2''(0) \geq -\frac{7}{4}\left(\frac{1}{3\alpha}\right)\alpha^4 + \frac{5}{6}\alpha^3 - \frac{1}{2}\alpha + 4 = \frac{1}{4}\left(\alpha^3 - 2\alpha + 16\right)$$
$$> \frac{1}{4}\left[(1.7)^3 - 2\left(\frac{100}{21}\right) + 16\right] = \frac{239173}{84000} > 0.$$

Step 2. We prove that, for $0.07 \leq \varepsilon \leq \bar{\varepsilon}$,

$$N_2''(\alpha) = -2\alpha^6\varepsilon^2 + \alpha^3\left(7\alpha^2 + 14\alpha + 25\right)\varepsilon - 2\alpha^4 - 5\alpha^3 - 10\alpha^2 + \alpha + 4 < 0. \tag{4.4}$$

Clearly,

$$\left\{(\alpha,\varepsilon) : 1.7 \leq \alpha \leq \frac{1}{3\varepsilon} \text{ and } 0.07 \leq \varepsilon \leq \bar{\varepsilon}\right\} = \Omega_1 \cup \Omega_2,$$

where

$$\Omega_1 \equiv \left\{(\alpha,\varepsilon) : 1.7 \leq \alpha \leq \frac{1}{3\bar{\varepsilon}} \text{ and } 0.07 \leq \varepsilon \leq \bar{\varepsilon}\right\}, \tag{4.5}$$

$$\Omega_2 \equiv \left\{(\alpha,\varepsilon) : \frac{1}{3\bar{\varepsilon}} \leq \alpha \leq \frac{1}{3\varepsilon} \text{ and } 0.07 \leq \varepsilon \leq \bar{\varepsilon}\right\}, \tag{4.6}$$

see Figure 4.1. So we consider the following two cases.

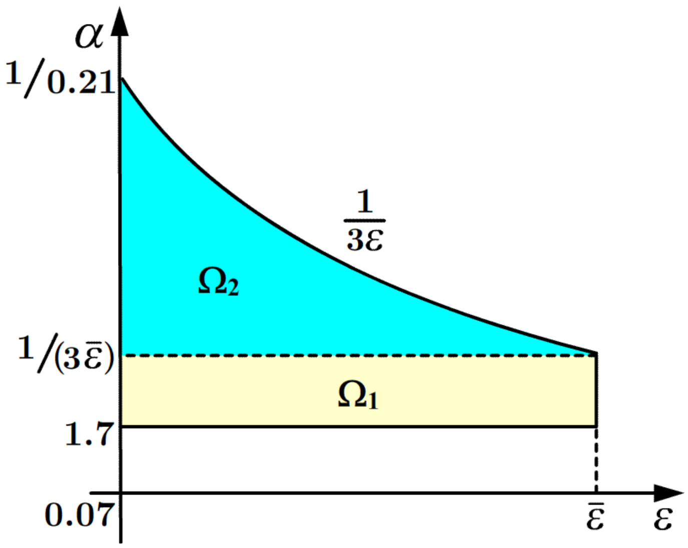

Figure 4.1: The sets Ω_1 and Ω_2.0

Case 1. Assume that $(\alpha,\varepsilon) \in \Omega_1$. It implies that

$$1.7 \leq \alpha \leq \frac{1}{3\bar{\varepsilon}} \ (\approx 1.893) < 1.9. \tag{4.7}$$

So we observe that

$$\frac{\partial}{\partial\varepsilon}N_2''(\alpha) = -4\varepsilon\alpha^6 + 7\alpha^5 + 14\alpha^4 + 25\alpha^3 > -4\bar{\varepsilon}\alpha^6 + 7\alpha^5 + 14\alpha^4 + 25\alpha^3$$
$$> -4\bar{\varepsilon}(1.9)^6 + 7(1.7)^5 + 14(1.7)^4 + 25(1.7)^3$$
$$= \frac{33914439}{10^5} - \frac{47045881}{25 \times 10^6}\sqrt{310} \ (\approx 306.01) > 0.$$

Then by (4.7),

$$\begin{aligned} N_2''(\alpha) <\ & N_2''(\alpha)\big|_{\varepsilon=\bar{\varepsilon}} \\ =\ & -\frac{31}{500}\alpha^6 + \frac{7}{10}\sqrt{\frac{31}{10}}\alpha^5 + \left(\frac{7}{5}\sqrt{\frac{31}{10}} - 2\right)\alpha^4 + \left(\frac{5}{2}\sqrt{\frac{31}{10}} - 5\right)\alpha^3 \\ & - 10\alpha^2 + \alpha + 4 \\ <\ & 0, \end{aligned}$$

see Figure 4.2(i).

Case 2. Assume that $(\alpha, \varepsilon) \in \Omega_2$. It implies that

$$(\alpha, \varepsilon) \in \Omega_2 = \left\{ (\alpha, \varepsilon) : \frac{1}{3\bar{\varepsilon}} \leq \alpha \leq \frac{1}{0.21} \text{ and } 0 < \varepsilon < \frac{1}{3\alpha} \right\}.$$

Then we observe that

$$\begin{aligned} \frac{\partial}{\partial \varepsilon} N_2''(\alpha) =\ & -4\alpha^6\varepsilon + \alpha^3\left(7\alpha^2 + 14\alpha + 25\right) > -4\alpha^6\left(\frac{1}{3\alpha}\right) + \alpha^3\left(7\alpha^2 + 14\alpha + 25\right) \\ =\ & \frac{1}{3}\left(17\alpha^2 + 75 + 42\alpha\right)\alpha^3 > 0. \end{aligned} \tag{4.8}$$

Since

$$1.8 < (1.89 \approx)\frac{1}{3\bar{\varepsilon}} \leq \alpha \leq \frac{1}{0.21} < 5, \tag{4.9}$$

and by (4.8), we observe that

$$N_2''(\alpha) < N_2''(\alpha)\big|_{\varepsilon=\frac{1}{3\alpha}} = \frac{1}{9}\left(\alpha^2 - 3\right)\left(\alpha^2 - 3\alpha - 12\right) < 0,$$

see Figure 4.2(ii).

Thus (4.4) holds by Cases 1–2.

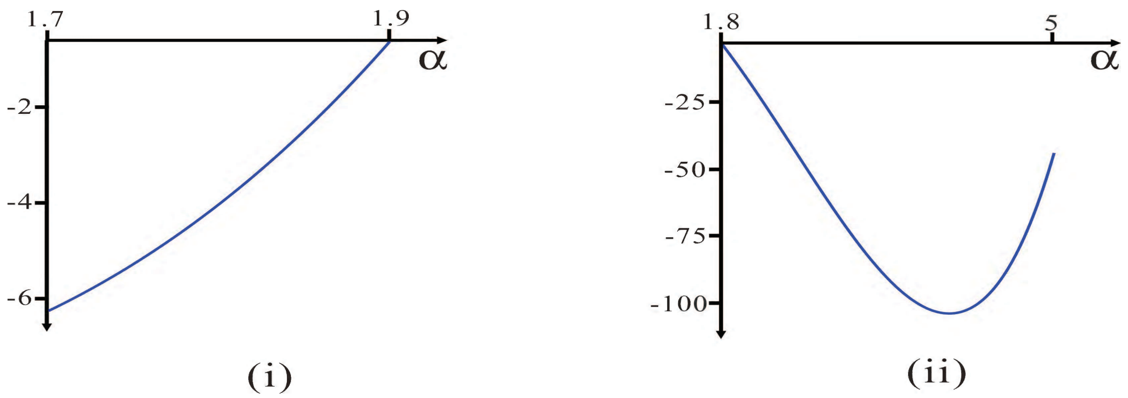

Figure 4.2: (i) The graph of $-\frac{31}{500}\alpha^6 + \frac{7}{10}\sqrt{\frac{31}{10}}\alpha^5 + \left(\frac{7}{5}\sqrt{\frac{31}{10}} - 2\right)\alpha^4 + \left(\frac{5}{2}\sqrt{\frac{31}{10}} - 5\right)\alpha^3$ $-10\alpha^2 + \alpha + 4$ on $[1.7, 9]$. (ii) The graph of $\left(\alpha^2 - 3\right)\left(\alpha^2 - 3\alpha - 12\right)$ on $[1.8, 5]$.

Step 3. We prove that, for $0.07 \leq \varepsilon \leq \bar{\varepsilon}$,

$$\begin{aligned} & N_2''(u) \text{ is strictly increasing, or strictly increasing-decreasing,} \\ & \text{or strictly increasing-decreasing-increasing on } (0, \alpha). \end{aligned} \tag{4.10}$$

Clearly, $N_2^{(6)}(u)$ is a quadratic polynomial of u with negative leading coefficient. Since, for $\varepsilon > 0$,

$$N_2^{(6)}(0) = 630\varepsilon - 80 \begin{cases} < 0 & \text{if } 0.07 \leq \varepsilon < \frac{8}{63}, \\ \geq 0 & \text{if } \frac{8}{63} \leq \varepsilon \leq \bar{\varepsilon}, \end{cases} \quad \text{and} \quad N_2^{(6)}\left(\frac{1}{3\varepsilon}\right) = 90(7\varepsilon + 3) > 0,$$

we see that

$$\begin{cases} N_2^{(5)}(u) \text{ is strictly decreasing-increasing on } (0,\alpha) \text{ if } 0.07 \leq \varepsilon < \frac{8}{63}, \\ N_2^{(5)}(u) \text{ is strictly increasing on } (0,\alpha) \text{ if } (0.126 \approx) \ \frac{8}{63} \leq \varepsilon \leq \bar{\varepsilon}. \end{cases} \tag{4.11}$$

In addition, we compute and find that

$$N_2^{(5)}(0) = 375\varepsilon - 50 \begin{cases} < 0 & \text{for } 0.07 \leq \varepsilon < \frac{2}{15} \ (\approx 0.133), \\ \geq 0 & \text{for } \frac{2}{15} \leq \varepsilon \leq \bar{\varepsilon}, \end{cases} \tag{4.12}$$

$$N_2^{(5)}(1.7) = -\frac{103173}{50}\varepsilon^2 + \frac{71403}{20}\varepsilon - 186 > 0 \quad \text{for } 0.07 \leq \varepsilon \leq \bar{\varepsilon}. \tag{4.13}$$

Since $0 < u < \alpha$ and $1.7 \leq \alpha \leq \frac{1}{3\varepsilon}$, and by (4.11)–(4.13), we obtain that

$$N_2^{(4)}(u) \text{ is either strictly decreasing-increasing, or strictly increasing on } (0,\alpha). \tag{4.14}$$

Since $1.7 \leq \alpha \leq \frac{1}{3\varepsilon}$ and $0.07 \leq \varepsilon \leq \bar{\varepsilon}$, we compute and find that

$$\begin{aligned} N_2^{(4)}(0) &= 3\varepsilon^2\alpha^4 - 7\varepsilon\alpha^3 - 21\varepsilon\alpha^2 - 75\varepsilon\alpha - 40 \\ &< 3\varepsilon^2\left(\frac{1}{3\varepsilon}\right)^4 - 7\varepsilon(1.7)^3 - 21\varepsilon(1.7)^2 - 75\varepsilon(1.7) - 40 \\ &= \frac{1}{27000\varepsilon^2}\left(-6009687\varepsilon^3 - 1080000\varepsilon^2 + 1000\right) \\ &< \frac{1}{27000\varepsilon^2}\left[-6009687(0.07)^3 - 1080000(0.07)^2 + 1000\right] \\ &= -\frac{6353322641}{27 \times 10^9\varepsilon^2} < 0. \end{aligned} \tag{4.15}$$

So by (4.14) and (4.15), we obtain that

$$N_2'''(u) \text{ is either strictly decreasing, or strictly decreasing-increasing on } (0,\alpha). \tag{4.16}$$

Since $0.07 \leq \varepsilon \leq \bar{\varepsilon}$, and by (4.3), we see that

$$\begin{aligned} N_2'''(0) &= -\frac{7}{4}\varepsilon\alpha^4 + \frac{4}{3}\alpha^3 + \frac{5}{2}\alpha^2 + 10\alpha + \frac{3}{2} \geq -\frac{7}{4}\bar{\varepsilon}\alpha^4 + \frac{4}{3}\alpha^3 + \frac{5}{2}\alpha^2 + 10\alpha + \frac{3}{2} \\ &= \frac{1}{12}\left(-\frac{21}{10}\sqrt{\frac{31}{10}}\alpha^4 + 16\alpha^3 + 30\alpha^2 + 120\alpha + 18\right) > 0, \end{aligned} \tag{4.17}$$

see Figure 4.3. Then by (4.16) and (4.17), we obtain (4.10).

Step 4. We prove (4.1). By Steps 1–2 and (4.10), we obtain that

$$N_2'(u) \text{ is strictly increasing-decreasing on } (0,\alpha). \tag{4.18}$$

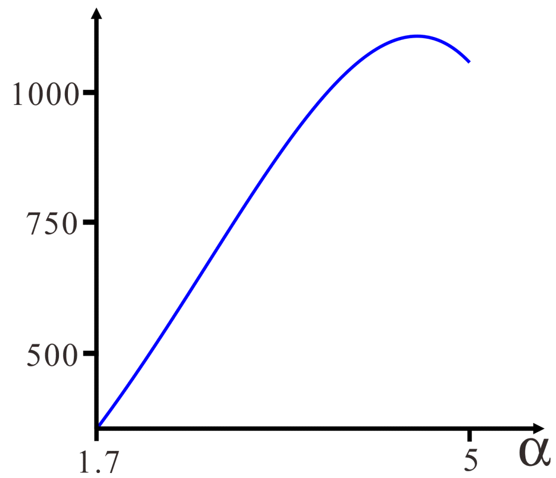

Figure 4.3: The graph of $-\frac{21}{10}\sqrt{\frac{31}{10}}\alpha^4+16\alpha^3+30\alpha^2+120\alpha+18$ on $[1.7,5]$.

Since $N_2'(\alpha)=0$ for $1.7\leq\alpha\leq\frac{1}{3\varepsilon}$, and by (4.18), we obtain that

$$N_2(u) \text{ is either strictly increasing, or strictly decreasing-increasing on } (0,\alpha). \tag{4.19}$$

We assert that

$$N_2(0)=-\frac{1}{16}\varepsilon^2\alpha^8+\frac{7}{24}\varepsilon\alpha^7+\frac{7}{8}\varepsilon\alpha^6+\frac{25}{8}\varepsilon\alpha^5-\frac{1}{9}\alpha^6-\frac{5}{12}\alpha^5-\frac{5}{3}\alpha^4+\frac{1}{4}\alpha^3+2\alpha^2\leq 0. \tag{4.20}$$

Since $N_2(\alpha)=0$, and by (4.19) and (4.20), we see that (4.1) holds. Next, we prove assertion (4.20). Since $1.7\leq\alpha\leq\frac{1}{3\varepsilon}$ and $0.07\leq\varepsilon\leq\bar{\varepsilon}$, we compute and find that

$$\begin{aligned}\frac{\partial}{\partial\varepsilon}N_2(0)&=\left(-\frac{1}{8}\varepsilon\alpha^3+\frac{7}{24}\alpha^2+\frac{7}{8}\alpha+\frac{25}{8}\right)\alpha^5\\&\geq\left[-\frac{1}{8}\varepsilon\left(\frac{1}{3\varepsilon}\right)^3+\frac{7}{24}(1.7)^2+\frac{7}{8}(1.7)+\frac{25}{8}\right]\alpha^5\\&=\frac{117837\varepsilon^2-100}{21600\varepsilon^2}\alpha^5\geq\frac{117837(0.07)^2-100}{21600\varepsilon^2}\alpha^5\\&=\frac{4774013}{216\times10^6\varepsilon^2}\alpha^5>0.\end{aligned} \tag{4.21}$$

Recall the sets Ω_1 and Ω_2 defined by (4.5) and (4.6) respectively, see Figure 4.1. Then we consider the following two cases.

Case 1. Assume that $(\alpha,\varepsilon)\in\Omega_1$. By (4.7) and (4.21), we see that

$$N_2(0)\leq N_2(0)|_{\varepsilon=\bar{\varepsilon}}=Q_1(\alpha)<0 \quad \text{for } 0.07\leq\varepsilon\leq\bar{\varepsilon},$$

where

$$\begin{aligned}Q_1(\alpha)\equiv&-\frac{31}{16000}\alpha^8+\frac{7}{240}\sqrt{\frac{31}{10}}\alpha^7+\left(\frac{7}{80}\sqrt{\frac{31}{10}}-\frac{1}{9}\right)\alpha^6\\&+\left(\frac{5}{16}\sqrt{\frac{31}{10}}-\frac{5}{12}\right)\alpha^5-\frac{5}{3}\alpha^4+\frac{1}{4}\alpha^3+2\alpha^2,\end{aligned}$$

see Figure 4.4(i).

Case 2. Assume that $(\alpha, \varepsilon) \in \Omega_2$. By (4.9) and (4.21), we see that

$$N_2(0) \leq N_2(0)|_{\varepsilon=\frac{1}{3\alpha}} = Q_2(\alpha) < 0 \quad \text{for } \frac{1}{3\bar{\varepsilon}} \leq \alpha \leq \frac{1}{0.21},$$

where

$$Q_2(\alpha) \equiv -\frac{1}{48}\alpha^6 - \frac{1}{8}\alpha^5 - \frac{5}{8}\alpha^4 + \frac{1}{4}\alpha^3 + 2\alpha^2,$$

see Figure 4.4(ii).

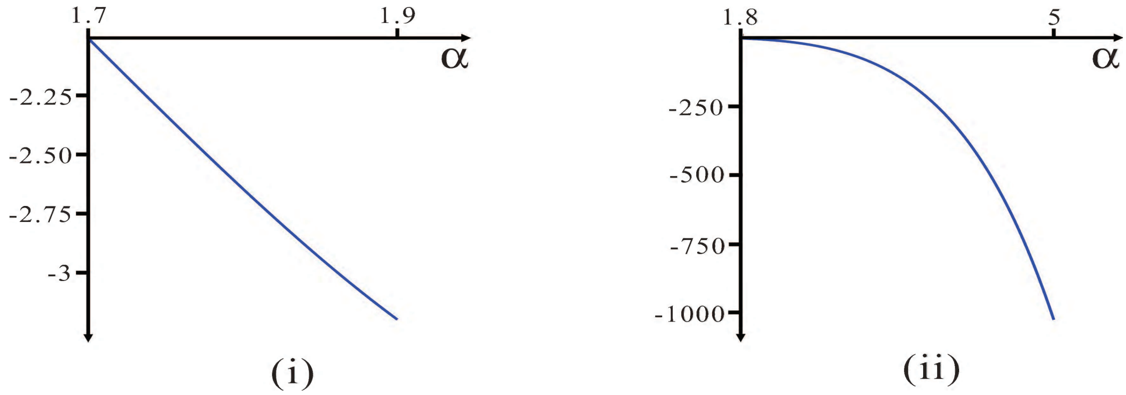

Figure 4.4: (i) The graph of $Q_1(\alpha)$ on $[1.7, 1.9]$. (ii) The graph of $Q_2(\alpha)$ on $[1.8, 5]$.

Thus, by Cases 1 and 2, assertion (4.20) holds. The proof is complete.

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