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Weak center for a class of Λ–Ω differential systems

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Abstract. In this paper, we give the necessary and sufficient conditions for a class of higher degree polynomial systems to have a weak center. As corollaries, we prove the correctness of the two conjectures about the weak center problem for the Λ–Ω differential systems.

Keywords: weak center, Λ–Ω system, composition center, center conditions.

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1 Introduction

Consider differential system of the form

$$\begin{cases} x' = -y + P, \\ y' = x + Q, \end{cases} \tag{1.1}$$

where $P = \sum_{k=2}^{m} P_k(x,y)$ and $Q = \sum_{k=2}^{m} Q_k(x,y)$, P_k and Q_k are homogeneous polynomials in x and y of degree k . The equilibrium point $O(0,0)$ is a center if there exists an open neighborhood U of O where all the orbits contained in U/O are periodic. The center-focus problem asks about the conditions on the coefficients of P and Q under which the origin of system (1.1) is a center. The study of the centers of analytical or polynomial differential system (1.1) has a long history. The first works are due to Poincaré [13] and Dulac [8], and continued by Liapunov [9] and many others. Unfortunately, the center-focus problem has been solved only for quadratic system and some special cubic system and others [2, 6, 7, 12]. Up to now, very little is known about the center conditions for polynomial differential system with arbitrary degree m $(m > 2)$.

A center of (1.1) is called a **weak center** if the Poincaré–Liapunov first integral can be written as $H = \frac{1}{2}(x^2 + y^2)(1 + h.o.t.)$. By literature [10, 11] we know that a center of a polynomial differential system (1.1) is a weak center if and only if it can be written as

$$\begin{cases} x' = -y(1+\Lambda) + x\Omega, \\ y' = x(1+\Lambda) + y\Omega, \end{cases} \tag{1.2}$$

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where $\Lambda = \Lambda(x,y)$ and $\Omega = \Omega(x,y)$ are polynomials of degree at most $m-1$ such that $\Lambda(0,0) = \Omega(0,0) = 0$. The weak centers contain the uniform isochronous centers and the holomorphic isochronous centers [10], they also contain the class of centers studied by Alwash and Lloyd [5], but they do not coincide with all classes of isochronous centers [10].

The class of differential system (1.2) is called the **Λ–Ω system**. The reason of called such system in this way is due to the fact that a subclass of these systems already appears in physics [11].

In [11] the authors put forward such conjectures:

Conjecture 1.1. *The polynomial differential system of degree m*

$$\begin{cases} x' = -y(1+\mu(a_2x - a_1y)) + x((a_1x + a_2y) + \Phi_{m-1}), \\ y' = x(1+\mu(a_2x - a_1y)) + y((a_1x + a_2y) + \Phi_{m-1}), \end{cases} \tag{1.3}$$

where $(\mu + m - 2)(a_1^2 + a_2^2) \neq 0$ and $\Phi_{m-1} = \Phi_{m-1}(x,y)$ is a homogeneous polynomial of degree $m-1$, has a weak center at the origin if and only if system (1.3) after a linear change of variables $(x,y) \to (X,Y)$ is invariant under the transformations $(X,Y,t) \to (-X,Y,-t)$.

Conjecture 1.2. *The polynomial differential system of degree m*

$$\begin{cases} x' = -y(1+a_1x+a_2y) + x\Phi_{m-1}, \\ y' = x(1+a_1x+a_2y) + y\Phi_{m-1} \end{cases} \tag{1.4}$$

has a weak center at the origin if and only if system (1.4) after a linear change of variables $(x,y) \to (X,Y)$ is invariant under the transformations $(X,Y,t) \to (-X,Y,-t)$.

The authors of [11] have used Poincaré–Liapunov first integral and Reeb inverse integrating factor to prove that Conjecture 1.1 and Conjecture 1.2 are correct when $m = 2,3,4,5,6$. They remarked that the only difficulty for proving Conjectures 1.1 and 1.2 for the Λ–Ω system of degree m with $m > 6$ is the huge number of computations for obtaining the conditions that characterize the centers.

In this paper we will research the weak center problem of the Λ–Ω system

$$\begin{cases} x' = -y(1+\mu(a_2x - a_1y)) + x(\nu(a_1x + a_2y) + \Lambda_{m-1} + \Omega_{2m-1}), \\ y' = x(1+\mu(a_2x - a_1y)) + y(\nu(a_1x + a_2y) + \Lambda_{m-1} + \Omega_{2m-1}), \end{cases} \tag{1.5}$$

in which $m > 2$ and $(\mu^2 + \nu^2)(\mu + \nu(m-2))(a_1^2 + a_2^2) \neq 0$, $\Lambda_{m-1} = \Lambda_{m-1}(x,y), \Omega_{2m-1} = \Omega_{2m-1}(x,y)$ are respectively homogeneous polynomials of degree $m-1$ and $2m-1$. In the section 3 we will see that by suitable transformation this system can be transformed into

$$\begin{cases} x' = -y(1-\mu y) + x(\nu x + \Phi_{m-1} + \Psi_{2m-1}), \\ y' = x(1-\mu y) + y(\nu x + \Phi_{m-1} + \Psi_{2m-1}). \end{cases} \tag{1.6}$$

In the following we use a method different from Llibre [11] and more simply, without huge number of computation, to prove that for system (1.6), under several restrictive conditions, it has a weak center at the origin if and only if

$$\int_0^{2\pi} \sin^i\theta \, \Phi_{m-1}(\cos\theta, \sin\theta) d\theta = 0 \qquad (i = 0,1,2,\ldots,m-1) \tag{1.7}$$

and

$$\int_0^{2\pi} \sin^j\theta\,\Psi_{2m-1}(\cos\theta,\sin\theta)d\theta = 0 \qquad (j = 0,1,2,\ldots,2m-1). \tag{1.8}$$

As corollaries, we also show that for arbitrary $m\,(> 2)$, Conjecture 1.1 with $\mu = 1$ and Conjecture 1.2 are correct; When $\mu \neq 1$ under several restrictive conditions Conjecture 1.1 is correct, too.

2 Several lemmas

In polar coordinates, the system (1.1) becomes

$$\frac{dr}{d\theta} = \frac{\sum_{k=2}^m A_k(\theta)r^k}{1+\sum_{k=2}^m B_k(\theta)r^{k-1}},$$

where

$$A_k(\theta) = \cos\theta P_k(\cos\theta,\sin\theta) + \sin\theta Q_k(\cos\theta,\sin\theta),$$
$$B_k(\theta) = \cos\theta Q_k(\cos\theta,\sin\theta) - \sin\theta P_k(\cos\theta,\sin\theta).$$

By [3,4], the **composition condition** is satisfied if there exists a trigonometric polynomial $u(\theta)$ such that

$$A_k(\theta) = u'(\theta)\sum a_{kj}u^j(\theta), \qquad B_k(\theta) = \sum b_{kj}u^j(\theta) \qquad (k = 2,3,\ldots,m), \tag{2.1}$$

where a_{kj}, b_{kj} are real numbers.

Lemma 2.1 ([4]). *If the conditions* (2.1) *are satisfied, then the origin point of* (1.1) *is a center and this center is called* **composition center**.

Lemma 2.2 ([14]). *If*

$$P_n = \sum_{i+j=n} p_{ij}\cos^i\theta\sin^j\theta, \qquad p_{ij} \in R,$$

$$\hat{P}_1 = p_{10}\sin\theta - p_{01}\cos\theta, \qquad p_{10}^2 + p_{01}^2 \neq 0$$

and

$$\int_0^{2\pi} \hat{P}_1^k P_n d\theta = 0 \qquad (k = 0,1,2,\ldots,n),$$

then

$$P_n = P_1\sum_{i=1}^n \lambda_i \hat{P}_1^{i-1},$$

where λ_i $(i = 1,2,\ldots,n)$ *are real numbers.*

Lemma 2.3. *Let* $\Phi_{m-1}(x,y) = \sum_{i+j=m-1}\phi_{ij}x^iy^j$ $(\phi_{ij} \in R)$. *If relation* (1.7) *holds, then*

$$\Phi_{m-1}(\cos\theta,\sin\theta) = \cos\theta\sum_{i=1}^{m-1}\lambda_i\sin^{i-1}\theta,$$

where λ_i $(i = 1,2,\ldots,m-2)$ *are real numbers and*

$$\lambda_{m-1} = \sum_{i=0}^{[\frac{m-2}{2}]}(-1)^i\phi_{2i+1\;m-2-2i}. \tag{2.2}$$

Proof. In Lemma 2.2, taking $P_1 = \cos\theta, \hat{P}_1 = \sin\theta$ we get

$$\Phi_{m-1}(\cos\theta, \sin\theta) = \cos\theta \sum_{i=1}^{m-1} \lambda_i \sin^{i-1}\theta,$$

thus

$$\Phi_{m-1}(x,y) = \sum_{i+j=2n} \phi_{ij} x^i y^j = x \sum_{i=1}^{n} \lambda_{2i} y^{2i-1}(x^2+y^2)^{n-i}, \qquad m-1 = 2n;$$

$$\Phi_{m-1}(x,y) = \sum_{i+j=2n+1} \phi_{ij} x^i y^j = x \sum_{i=0}^{n} \lambda_{2i+1} y^{2i}(x^2+y^2)^{n-i}, \qquad m-1 = 2n+1.$$

Equating the corresponding coefficients of the same power of x, y, we obtain

$$\lambda_{m-1} = \sum_{i=0}^{n-1} (-1)^i \phi_{2i+1\,2(n-i)-1}, \qquad m-1 = 2n;$$

$$\lambda_{m-1} = \sum_{i=0}^{n} (-1)^i \phi_{2i+1\,2(n-i)}, \qquad m-1 = 2n+1.$$

Therefore, the conclusion of the present lemma is valid. □

By this lemma, it is easy to deduce the following conclusion.

Lemma 2.4. *Let $\Phi_{m-1}(x,y)$ be a homogeneous polynomial of degree $m-1$. Then it can be written as*

$$\Phi_{m-1}(x,y) = x\check{\Phi}(x^2+y^2, y)$$

if and only if the relation (1.7) holds. Where $\check{\Phi}$ is a polynomial on x^2+y^2 and y.

3 Main results

As $a_1^2 + a_2^2 \neq 0$, taking the linear change:

$$X = a_1 x + a_2 y, \qquad Y = -a_2 x + a_1 y, \tag{3.1}$$

the system (1.5) becomes

$$\begin{cases} X' = -Y(1-\mu Y) + X(\nu X + \Phi_{m-1} + \Psi_{2m-1}), \\ Y' = X(1-\mu Y) + Y(\nu X + \Phi_{m-1} + \Psi_{2m-1}), \end{cases}$$

where $\Phi_{m-1} = \Lambda_{m-1}\left(\frac{a_1X-a_2Y}{a_1^2+a_2^2}, \frac{a_1Y+a_2X}{a_1^2+a_2^2}\right)$, $\Psi_{2m-1} = \Omega_{2m-1}\left(\frac{a_1X-a_2Y}{a_1^2+a_2^2}, \frac{a_1Y+a_2X}{a_1^2+a_2^2}\right)$, and they are respectively homogeneous polynomials of degree $m-1$ and $2m-1$.

Obviously, if $\Phi_{m-1} = X\check{\Phi}_{m-1}(X^2+Y^2, Y)$, $\Psi_{2m-1} = X\check{\Psi}_{2m-1}(X^2+Y^2, Y)$, then the Λ–Ω system (1.5) after a linear change of variables $(x,y) \to (X,Y)$ is invariant under the transformations $(X,Y,t) \to (-X,Y,-t)$. By Lemma 2.4, in order to find the necessary and sufficient conditions for (1.5) to have a weak center, only need to seek the conditions under which the identities (1.7) and (1.8) are valid.

Case A. If $\nu \neq 0$, applying the transformation $X = \frac{1}{\nu}x$, $Y = \frac{1}{\nu}y$, we get

$$\begin{cases} x' = -y(1-\hat{\mu}y) + x(x + \hat{\Phi}_{m-1} + \hat{\Psi}_{2m-1}), \\ y' = x(1-\hat{\mu}y) + y(x + \hat{\Phi}_{m-1} + \hat{\Psi}_{2m-1}), \end{cases}$$

where $\hat{\mu} = \frac{\mu}{\nu}$, $\hat{\Phi}_{m-1} = \frac{1}{\nu^{m-1}}\Phi_{m-1}(x,y)$, $\hat{\Psi}_{2m-1} = \frac{1}{\nu^{2m-1}}\Psi_{2m-1}(x,y)$. Thus, if the identities (1.7) and (1.8) are valid, then replacing Φ_{m-1} and Ψ_{2m-1} by $\hat{\Phi}_{m-1}$ and $\hat{\Psi}_{2m-1}$ respectively, these identities also hold.

Case 1. $\nu \neq 0$, $\hat{\mu} = 1$.

Consider the Λ–Ω system

$$\begin{cases} x' = -y(1-y) + x(x + \Phi_{m-1} + \Psi_{2m-1}), \\ y' = x(1-y) + y(x + \Phi_{m-1} + \Psi_{2m-1}). \end{cases} \tag{3.2}$$

Theorem 3.1. *Suppose that*

$$\prod_{m-1 \leq k \leq 2m-3} L_k \neq 0; \tag{3.3}$$

$$L_{2m-2} + \frac{m(2m-1)}{2(m-1)^2}\lambda_{m-1}^2 \neq 0; \tag{3.4}$$

$$L_{2m-1} + \left(2d_1 + e_1\frac{m(m+1)}{(m-1)^2}\right)\lambda_{m-1}^2 \neq 0, \tag{3.5}$$

where λ_{m-1} is expressed by (2.2),

$$\begin{gathered} L_k := e_k + \sum_{i=0}^{k-m+1} \frac{k+1-2i}{m-1+i} d_i e_{k-m+1-i}\lambda_{m-1} \qquad (k = m-1, m, \ldots, 2m-1), \\ d_k = (m-1)\frac{(m+k-1)^{k-1}}{k!}, \qquad e_k = (2m-1)\frac{(2m+k-1)^{k-1}}{k!} \\ (k = 1,2,3,\ldots), \quad d_0 = 1,\ e_0 = 1. \end{gathered} \tag{3.6}$$

Then the origin point of (3.2) is a center if and only if (1.7) and (1.8) hold.

Moreover, this center is a composition center and weak center.

Proof. In polar coordinates, the system (3.2) can be written as

$$\frac{dr}{d\theta} = \frac{r^2\cos\theta + \Phi_{m-1}r^m + \Psi_{2m-1}r^{2m}}{1 - r\sin\theta},$$

where $\Phi_{m-1} = \Phi_{m-1}(\cos\theta, \sin\theta), \Psi_{2m-1} = \Psi_{2m-1}(\cos\theta, \sin\theta)$.

Taking $\rho = \frac{r}{e^{r\sin\theta}}$, the above equation becomes

$$\frac{d\rho}{d\theta} = \rho^m e^{(m-1)r\sin\theta}\Phi_{m-1} + \rho^{2m}e^{(2m-1)r\sin\theta}\Psi_{2m-1}. \tag{3.7}$$

Now we recall the Langrange–Bürman formula [1]. If real or complex w and z satisfy that $w = \frac{z}{\phi(z)}$, where $\phi(0) = 1$ and $\phi(z)$ is analytic at $z = 0$, then in a neighborhood of $w = 0$, the analytic function $F(z)$ can be expressed as a power series:

$$F(z) = F(0) + \sum_{n=1}^{\infty} \frac{w^n}{n!} \frac{d^{n-1}(F'(x)\phi^n(x))}{dx^{n-1}}\bigg|_{x=0},$$

which is analytic at $w = 0$.

Applying the Langrange–Bürman formula we have

$$e^{(m-1)r\sin\theta} = 1 + (m-1)\sum_{n=1}^{\infty}\frac{(m+n-1)^{n-1}}{n!}\rho^n \sin^n\theta,$$
$$e^{(2m-1)r\sin\theta} = 1 + (2m-1)\sum_{n=1}^{\infty}\frac{(2m+n-1)^{n-1}}{n!}\rho^n \sin^n\theta.$$

Thus the equation (3.7) can be written as

$$\frac{d\rho}{d\theta} = \Phi_{m-1}\sum_{n=0}^{\infty} d_n\rho^{m+n}\sin^n\theta + \Psi_{2m-1}\sum_{n=0}^{\infty} e_n\rho^{2m+n}\sin^n\theta, \tag{3.8}$$

where d_n, e_n $(n = 0, 1, 2, \dots)$ are expressed by (3.6).

Therefore, the system (3.2) has a center at $(0, 0)$ if and only if all the solutions $\rho(\theta)$ of equation (3.8) near $\rho = 0$ are periodic [2].

Let $\rho(\theta, c)$ be the solution of (3.8) such that $\rho(0, c) = c$ $(0 < c \ll 1)$. We write

$$\rho(\theta, c) = c\sum_{n=0}^{\infty} a_n(\theta)c^n,$$

where $a_0(0) = 1$ and $a_n(0) = 0$ for $n \geq 1$. The origin point of (3.2) is a center if and only if $\rho(\theta + 2\pi, c) = \rho(\theta, c)$, i.e., $a_0(2\pi) = 1$, $a_n(2\pi) = 0$ $(n = 1, 2, 3, \dots)$ [5].

Substituting $\rho(\theta, c)$ into (3.8) we obtain

$$c\sum_{i=0}^{\infty} a_i'(\theta)c^n = \Phi_{m-1}\sum_{n=0}^{\infty} d_n\sin^n\theta\left(c\sum_{i=0}^{\infty}a_i(\theta)c^i\right)^{m+n} + \Psi_{2m-1}\sum_{n=0}^{\infty} e_n\sin^n\theta\left(c\sum_{i=0}^{\infty}a_i(\theta)c^i\right)^{2m+n}. \tag{3.9}$$

Equating the corresponding coefficients of c^n of (3.9) yields

$$a_0(\theta) = 1,\ a_i(\theta) = 0, \qquad (i = 1, 2, \dots, m-2).$$

Rewriting

$$\rho = c(1 + c^{m-1}h), \qquad h = \sum_{i=0}^{\infty} h_i(\theta)c^i, h_i(0) = 0, \qquad (i = 0, 1, 2 \dots).$$

Substituting it into (3.8) we get

$$\sum_{k=0}^{\infty} h_k'(\theta)c^k = \Phi_{m-1}\sum_{k=0}^{\infty} d_k c^k\sin^k\theta\sum_{j=0}^{m+k} C_{m+k}^j h^j c^{(m-1)j} \tag{3.10}$$
$$+ \Psi_{2m-1}\sum_{k=0}^{\infty} e_k c^{m+k}\sin^k\theta\sum_{j=0}^{2m+k} C_{2m+k}^j h^j c^{(m-1)j}, \quad h_k(0) = 0 \quad (k = 0, 1, 2, \dots).$$

In the following we denote

$$g_k = d_k\overline{\sin^k\theta\,\Phi_{m-1}}, \qquad \beta_k = e_k\overline{\sin^k\theta\,\Psi_{2m-1}}, \qquad (k = 0, 1, 2, \dots), \tag{3.11}$$

where

$$\overline{\sin^k\theta\,\Phi_{m-1}} = \int_0^\theta \sin^k\theta\Phi_{m-1}d\theta, \qquad \overline{\sin^k\theta\,\Psi_{2m-1}} = \int_0^\theta \sin^k\theta\Psi_{2m-1}d\theta.$$

Equating the corresponding coefficients of c^k of the equation (3.10) we obtain

$$h_k' = d_k \sin^k\theta\,\Phi_{m-1}, \qquad h_k(0) = 0 \qquad (k = 0,1,2,\ldots,m-2),$$

$$h_{m-1}' = \Phi_{m-1}C_m^1 h_0 + \Phi_{m-1}d_{m-1}\sin^{m-1}\theta, \qquad h_{m-1}(0) = 0,$$

solving these equations we get

$$h_k(\theta) = g_k, \qquad (k = 0,1,2,\ldots,m-2),$$

$$h_{m-1}(\theta) = g_{m-1} + \alpha_0, \qquad \alpha_0 = \frac{m}{2}\bar{\Phi}_{m-1}^2.$$

As $d_k \neq 0$ $(k = 0,1,2\ldots.)$, from $h_k(2\pi) = 0$ $(k = 0,1,2,\ldots,m-1)$ follow that

$$\int_0^{2\pi} \sin^k\theta\,\Phi_{m-1}d\theta = 0 \qquad (k = 0,1,2,\ldots,m-1),$$

i.e., the condition (1.7) is a necessary condition for $\rho = 0$ to be a center. By Lemma 2.3 which implies that

$$\Phi_{m-1} = \cos\theta\sum_{k=1}^{m-1}\lambda_k\sin^{k-1}\theta, \qquad \bar{\Phi}_{m-1} = \int_0^\theta \Phi_{m-1}d\theta = \sum_{k=1}^{m-1}\frac{\lambda_k}{k}\sin^k\theta, \tag{3.12}$$

where λ_k $(k = 1,2,\ldots,m-1)$ are real numbers and λ_{m-1} is expressed by (2.2).

Applying (3.12) we get

$$\int_0^{2\pi}\sin^k\theta\,\Phi_{m-1}d\theta = 0, \qquad g_k = g_k(\sin\theta), \qquad g_k(2\pi) = 0 \qquad (k = 0,1,2,\ldots.). \tag{3.13}$$

Equating the corresponding coefficients of c^{m-1+k} of the equation (3.10) we obtain

$$h_{m-1+k}' = \Phi_{m-1}\sum_{i=0}^{k} d_i\sin^i\theta C_{m+i}^1 h_{k-i} + d_{m-1+k}\sin^{m-1+k}\theta\Phi_{m-1} + e_{k-1}\sin^{k-1}\theta\Psi_{2m-1},$$

$$h_{m-1+k}(0) = 0 \qquad (k = 1,2,\ldots,m-2),$$

solving these equations we get

$$h_{m-1+k}(\theta) = g_{m-1+k} + \alpha_k + \beta_{k-1} \qquad (k = 1,2,\ldots,m-2),$$

where g_{m-1+k} and β_{k-1} are expressed by (3.11), α_k is the solution of the following equation

$$\alpha_k' = \Phi_{m-1}\sum_{i=0}^{k} d_i d_{k-i}\sin^i\theta C_{m+i}^1\overline{\sin^{k-i}\theta\Phi_{m-1}},\ \alpha_k(0) = 0. \tag{3.14}$$

By this we get: when $k = 2n$,

$$\begin{aligned}\alpha_k = {} & \sum_{i=0}^{n-1} d_i d_{k-i}\left(C_{m+i}^1\overline{\sin^i\theta\Phi_{m-1}}\,\overline{\sin^{k-i}\theta\Phi_{m-1}} + (C_{m+k-i}^1 - C_{m+i}^1)\overline{\overline{\sin^i\theta\Phi_{m-1}}\sin^{k-i}\theta\Phi_{m-1}}\right)\\ & + \frac{1}{2}d_n^2 C_{m+n}^1\overline{\sin^n\theta\Phi_{m-1}}^2;\end{aligned} \tag{3.15}$$

when $k = 2n+1$,

$$\alpha_k = \sum_{i=0}^{n} d_i d_{k-i} \left(C^1_{m+i} \overline{\sin^i \theta \Phi_{m-1}}\, \overline{\sin^{k-i} \theta \Phi_{m-1}} + (C^1_{m+k-i} - C^1_{m+i}) \overline{\overline{\sin^i \theta \Phi_{m-1}} \sin^{k-i} \theta \Phi_{m-1}} \right). \tag{3.16}$$

By (3.13) we see that $\alpha_k = \alpha_k(\sin\theta)$, $\alpha_k(2\pi) = 0$ $(k = 0, 1, 2, 3, \dots)$. Then from

$$h_{m-1+k}(2\pi) = g_{m-1+k}(2\pi) + \alpha_k(2\pi) + \beta_{k-1}(2\pi) = 0 \qquad (k = 1, 2, \dots, m-2)$$

imply that

$$\beta_k(2\pi) = 0 \qquad (k = 0, 1, 2, \dots, m-3),$$

in view of $e_k \neq 0$ $(k = 0, 1, 2 \dots)$, so

$$\int_0^{2\pi} \sin^k \theta \Psi_{2m-1} d\theta = 0 \qquad (k = 0, 1, 2 \dots, m-3). \tag{3.17}$$

Equating the corresponding coefficients of c^{2m-2} of the equation (3.10) we get

$$h'_{2m-2} = \Phi_{m-1} \sum_{i=0}^{m-1} d_i \sin^i \theta C^1_{m+i} h_{m-1-i} + \Phi_{m-1}(C^1_m \alpha_0 + C^2_m h_0^2)$$
$$+ d_{2m-2} \sin^{2m-2} \theta \Phi_{m-1} + e_{m-2} \sin^{m-2} \theta \Psi_{2m-1}, \qquad h_{2m-2}(0) = 0,$$

by this we get

$$h_{2m-2}(\theta) = g_{2m-2} + \alpha_{m-1} + \beta_{m-2} + \delta_0,$$

where

$$\delta_0 = \frac{m(2m-1)}{6} \bar{\Phi}^3_{m-1}.$$

α_{m-1} is a solution of (3.14) with $k = m-1$ and $\alpha_{m-1} = \alpha_{m-1}(\sin\theta)$. Thus, using (3.12) and (3.13), from $h_{2m-2}(2\pi) = 0$ follows that $\beta_{m-2}(2\pi) = 0$, i.e.,

$$\int_0^{2\pi} \sin^{m-2} \theta\, \Psi_{2m-1} d\theta = 0. \tag{3.18}$$

Equating the corresponding coefficients of c^{2m-2+k} of the equation (3.10) we obtain

$$h'_{2m-2+k} = \Phi_{m-1} \sum_{i=0}^{m-1+k} d_i \sin^i \theta C^1_{m+i} h_{m-1+k-i} + \Phi_{m-1} \sum_{i=0}^{k} d_i \sin^i \theta C^2_{m+i} \sum_{j+l=k-i} h_j h_l$$
$$+ d_{2m-2+k} \sin^{2m-2+k} \theta \Phi_{m-1} + e_{m-2+k} \sin^{m-2+k} \theta \Psi_{2m-1}$$
$$+ \Psi_{2m-1} \sum_{i=0}^{k-1} e_i \sin^i \theta C^1_{2m+i} h_{k-1-i},$$
$$h_{2m-2+k}(0) = 0 \qquad (k = 1, 2, \dots, m-2),$$

solving these equations we get

$$h_{2m-2+k} = g_{2m-2+k} + \alpha_{k+m-1} + \beta_{k+m-2} + \delta_k + \varepsilon_{k-1} \qquad (k = 1, 2, \dots, m-2),$$

where α_{k+m-1} is a solution of (3.14), δ_k and ε_{k-1} are the solutions of the following equations, respectively,

$$\delta'_k = \Phi_{m-1} \left(\sum_{i=0}^{k} d_i \sin^i \theta C^1_{m+i} \alpha_{k-i} + \sum_{i=0}^{k} C^2_{m+i} d_i \sin^i \theta \sum_{j+l=k-i} h_l h_j \right),$$

$$\varepsilon'_{k-1} = \Phi_{m-1}\sum_{i=0}^{k-1} C^1_{m+i} d_i \sin^i\theta\beta_{k-1-i} + \Psi_{2m-1}\sum_{i=0}^{k-1} e_i \sin^i\theta C^1_{2m+i} g_{k-1-i}. \tag{3.19}$$

By (3.12) and (3.13) we see that $\delta_k = \delta_k(\sin\theta)$ and $\delta_k(2\pi) = 0$.

Solving (3.19) we get

$$\varepsilon_{k-1} = \sum_{i=0}^{k-1} d_i e_{k-1-i}\Big(C^1_{m+i}\overline{\sin^i\theta\Phi_{m-1}}\,\overline{\sin^{k-1-i}\theta\Psi_{2m-1}} + (C^1_{2m+k-1-i} - C^1_{m+i})\overline{\overline{\sin^i\theta\Phi_{m-1}}\sin^{k-1-i}\theta\Psi_{2m-1}}\Big). \tag{3.20}$$

Therefore, from $h_{2m-2+k}(2\pi) = 0$ $(k = 1, 2, \ldots, m-2)$ implies that

$$\beta_{k+m-2}(2\pi) + \varepsilon_{k-1}(2\pi) = 0 \qquad (k = 1, 2, \ldots, m-2),$$

simplifying this relation by using (3.17) and (3.18), (3.20) and (3.12) we get

$$\left(e_k + \sum_{i=0}^{k-m+1}\frac{k+1-2i}{m-1+i} d_i e_{k-m+1-i}\lambda_{m-1}\right)\int_0^{2\pi}\sin^k\theta\Psi_{2m-1}d\theta = L_k\int_0^{2\pi}\sin^k\theta\Psi_{2m-1}d\theta = 0,$$

$$(k = m-1, m, \ldots, 2m-4).$$

By the hypothesis (3.3), $L_k \neq 0$, so

$$\int_0^{2\pi}\sin^k\theta\Psi_{2m-1}d\theta = 0 \qquad (k = m-1, m, \ldots, 2m-4). \tag{3.21}$$

Equating the corresponding coefficients of c^{3m-3} of the equation (3.10) we obtain

$$h_{3m-3} = g_{3m-3} + \alpha_{2m-2} + \beta_{2m-3} + \delta_{m-1} + \varepsilon_{m-2},$$

where α_{2m-2} is a solution of (3.14) with $k = 2m-2$ and $\alpha_{2m-2} = \alpha_{2m-2}(\sin\theta)$, ε_{m-2} is expressed by (3.20) with $k = m-1$, δ_{m-1} is a solution of the following equation

$$\delta'_{m-1} = \Phi_{m-1}\left(\sum_{i=0}^{m-1} d_i\sin^i\theta C^1_{m+i}\alpha_{m-1-i} + \sum_{i=0}^{m-1} C^2_{m+i} d_i\sin^i\theta\sum_{j+l=m-1-i} g_l g_j + C^1_m\delta_0 + 2C^2_m h_0\alpha_0 + C^3_m h_0^3\right).$$

By (3.12) and (3.13) we see that $g_k = g_k(\sin\theta)$ $(k = 0, 1, 2, \ldots, m-1)$ and $\delta_{m-1} = \delta_{m-1}(\sin\theta)$. Thus, from $h_{3m-3}(2\pi) = 0$ follows that

$$\beta_{2m-3}(2\pi) + \varepsilon_{m-2}(2\pi) = 0,$$

simplifying this relation by using (3.17) and (3.18) and (3.21), (3.20) and (3.12) we get

$$L_{2m-3}\int_0^{2\pi}\sin^{2m-3}\theta\Psi_{2m-1}d\theta = 0,$$

as $L_{2m-3} \neq 0$,

$$\int_0^{2\pi}\sin^{2m-3}\theta\Psi_{2m-1}d\theta = 0. \tag{3.22}$$

Equating the corresponding coefficients of c^{3m-2} of the equation (3.10) we obtain

$$h_{3m-2}(\theta) = g_{3m-2} + \alpha_{2m-1} + \beta_{2m-2} + \delta_m + \varepsilon_{m-1} + \eta_0,$$

where α_{2m-1} is a solution of (3.14) with $k = 2m-1$, ε_{m-1} is a solution of (3.19) with $k = m$, δ_m is a solution of the following equation

$$\delta'_m = \Phi_{m-1}\left(\sum_{i=0}^{m} d_i \sin^i\theta C^1_{m+i}\alpha_{m-i} + \sum_{i=0}^{m} d_i \sin^i\theta C^2_{m+i} \sum_{j+l=m-i} g_j g_l\right)$$
$$+ \Phi_{m-1}(C^1_m\delta_1 + d_1\sin\theta C^1_{m+1}\delta_0 + C^2_m(2h_0\alpha_1 + 2h_1\alpha_0) + 3C^3_m h_0^2 h_1 + d_1\sin\theta C^3_{m+1}h_0^3),$$

by (3.12) and (3.13) we see that $\delta_m = \delta_m(\sin\theta)$. η_0 is a solution of equation

$$\eta'_0 = \Phi_{m-1}(C^1_m\varepsilon_0 + 2C^2_m h_0\beta_0) + \Psi_{2m-1}(C^2_{2m}h_0^2 + C^1_{2m}\alpha_0),$$

solving this we get

$$\eta_0 = \frac{1}{2}m(m-1)\bar{\Phi}^2_{m-1}\bar{\Psi}_{2m-1} + m^2\bar{\Phi}_{m-1}\overline{\bar{\Phi}_{m-1}\Psi_{2m-1}} + \frac{2m^2-m}{2}\overline{\bar{\Phi}^2_{m-1}\Psi_{2m-1}}. \tag{3.23}$$

Thus, from $h_{3m-2}(2\pi) = 0$ follows that

$$\beta_{2m-2}(2\pi) + \varepsilon_{m-1}(2\pi) + \eta_0(2\pi) = 0,$$

calculating this relation by using (3.17) and (3.18) and (3.20)–(3.23) and (3.12) we get

$$\left(L_{2m-2} + \frac{2m^2-m}{2(m-1)^2}\lambda^2_{m-1}\right)\int_0^{2\pi} \sin^{2m-2}\Psi_{2m-1}d\theta = 0,$$

in view of the condition (3.4) we have

$$\int_0^{2\pi} \sin^{2m-2}\theta\Psi_{2m-1}d\theta = 0. \tag{3.24}$$

Equating the corresponding coefficients of c^{3m-1} of the equation (3.10) we obtain

$$h_{3m-1}(\theta) = g_{3m-1} + \alpha_{2m} + \beta_{2m-1} + \delta_{m+1} + \varepsilon_m + \eta_1, \tag{3.25}$$

where g_{3m-1}, α_{2m}, β_{2m-1} and ε_m are the same as above, δ_{m+1} is a solution of the equation

$$\delta'_{m+1} = \Phi_{m-1}\Bigg(\sum_{i=0}^{m+1} d_i\sin^i\theta C^1_{m+i}\alpha_{m+1-i} + \sum_{i=0}^{m+1} d_i\sin^i\theta C^2_{m+i}\sum_{j+l=m+1-i} g_j g_l$$
$$+ \sum_{i=0}^{2} d_i\sin^i\theta\left(C^1_{m+i}\delta_{2-i} + C^2_{m+i}\sum_{l+j=2-i} g_j\alpha_i + C^3_{m+i}\sum_{l+j+k=2-i} g_l g_j g_k\right)\Bigg).$$

By (3.12) and (3.13) and $\alpha_k = \alpha_k(\sin\theta)$ $(k = 0,1,2\dots)$, $\delta_i = \delta_i(\sin\theta)$ $(i = 0,1,2)$, which imply that $\delta_{m+1} = \delta_{m+1}(\sin\theta)$. η_1 is a solution of the following equation

$$\eta'_1 = \Phi_{m-1}(C^1_m\varepsilon_1 + d_1\sin\theta C^1_{m+1}\varepsilon_0 + C^2_m(2h_0\beta_1 + 2h_1\beta_0) + d_1\sin\theta C^2_{m+1}2h_0\beta_0)$$
$$+ \Psi_{2m+1}(C^2_{2m}2h_0h_1 + e_1\sin\theta C^2_{2m+1}h_0^2 + C^1_{2m}(\alpha_1 + \beta_0) + e_1\sin\theta C^1_{2m+1}\alpha_0),$$

solving this equation we get

$$\begin{aligned}\eta_1 = m\bar{\Psi}^2_{2m-1} + e_1\Big(\Big(m^2 - \frac{m}{2}\Big)\bar{\Phi}^2_{m-1}\overline{\sin\theta\Psi_{2m-1}} + m(m+1)\bar{\Phi}_{m-1}\overline{\bar{\Phi}_{m-1}\sin\theta\Psi_{2m-1}}\\ + m(m+1)\overline{\sin\theta\Psi_{2m-1}\bar{\Phi}^2_{m-1}}\Big)\\ + d_1\Big(2m^2\bar{\Phi}_{m-1}\,\bar{\Psi}_{2m-1}\,\overline{\sin\theta\Phi_{m-1}} + m(m-1)\bar{\Phi}_{m-1}\,\overline{\overline{\sin\theta\Phi_{m-1}}\Psi_{2m-1}}\\ + m(m+1)\overline{\sin\theta\Phi_{m-1}}\,\overline{\bar{\Phi}_{m-1}\Psi_{2m-1}}\\ + 2m\bar{\Psi}_{2m-1}\overline{\bar{\Phi}_{m-1}\Psi_{m-1}\sin\theta} + 2(m^2-m)\overline{\bar{\Phi}_{m-1}\Psi_{2m-1}\overline{\sin\theta\Phi_{m-1}}}\Big).\end{aligned} \tag{3.26}$$

By (3.25) we see that if $h_{3m-1}(2\pi) = 0$, then

$$\beta_{2m-1}(2\pi) + \varepsilon_m(2\pi) + \eta_1(2\pi) = 0,$$

simplifying this equation by using (3.17) and (3.18) and (3.20)–(3.24), (3.26) and (3.12) we get

$$\left(L_{2m-1} + \left(2d_1 + e_1\frac{m(m+1)}{(m-1)^2}\right)\lambda^2_{m-1}\right)\int_0^{2\pi}\sin^{2m-1}\theta\Psi_{2m-1}d\theta = 0,$$

by the hypothesis (3.5) we obtain

$$\int_0^{2\pi}\sin^{2m-1}\theta\Psi_{2m-1}d\theta = 0.$$

In summary, under the conditions (3.3)–(3.5), the (1.7) and (1.8) are the necessary conditions for $\rho = 0$ to be a center of (3.2). Therefore, the necessity has been proved. On the other hand, by Lemma 2.1 and Lemma 2.3, if the conditions (1.7) and (1.8) are satisfied, then $\rho = 0$ is a center of equation (3.2), this means that the sufficiency is proved. By Lemma 2.3 this center is a composition center, by Lemma 2.4 this center is a weak center. □

Corollary 3.2. *For arbitrary m (> 2), if $\mu = 1$, then the origin point of* (1.3) *is a center if and only if* (1.7) *is satisfied.*

Proof. Under the linear change of variables (3.1) the system (1.3) becomes

$$\begin{cases} x' = -y(1-y) + x(x + \Phi_{m-1}),\\ y' = x(1-y) + y(x+\Phi_{m-1}),\end{cases}$$

which in polar coordinates becomes

$$\frac{dr}{d\theta} = \frac{r^2\cos\theta + \Phi_{m-1}r^m}{1 - r\sin\theta}.$$

Taking $\rho = \frac{r}{e^{r\sin\theta}}$ we get

$$\frac{d\rho}{d\theta} = \Phi_{m-1}\rho^m\sum_{n=0}^{\infty} d_n\rho^n\sin^n\theta,$$

where $d_0 = 1$, $d_n = \frac{1}{n!}(m-1)(m+n-1)^{n-1}$, $(n = 1, 2, 3, \dots)$. Similar to Theorem 3.1, it can be deduced that the solution ρ of this equation such that $\rho(0) = c$ $(0 < |c| \ll 1)$ is

$$\rho = c + c^m\sum_{k=0}^{m-2} c^k d_k\overline{\sin^k\theta\Phi_{m-1}} + c^{2m-1}\left(d_{m-1}\overline{\sin^{m-1}\theta\Phi_{m-1}} + \frac{m}{2}\bar{\Phi}^2_{m-1}\right) + o(c^{2m-1}).$$

As $d_n \neq 0$ $(n = 0, 1, 2\dots.)$, from $\rho(2\pi) = c$ it follows that the condition (1.7) is satisfied. Using Lemma 2.3 and Lemma 2.4, the conclusion of the present corollary is valid. □

Remark 3.3. By Corollary 3.2, when $\mu = 1$, Conjecture 1.1 is correct for arbitrary $m > 2$.

Case 2. $\nu \neq 0$, $\hat{\mu} \neq 1$.

Consider Λ–Ω system

$$\begin{cases} x' = -y(1-\hat{\mu}y) + x(x + \Phi_{m-1} + \Psi_{2m-1}), \\ y' = x(1-\hat{\mu}y) + y(x + \Phi_{m-1} + \Psi_{2m-1}). \end{cases} \tag{3.27}$$

Theorem 3.4. *Suppose that*

$$\prod_{1\le n\le m-1} \tilde{d}_n \neq 0; \qquad \prod_{m-1\le k\le 2m-3} \tilde{L}_k \neq 0;$$

$$\tilde{L}_{2m-2} + \frac{m(2m-1)}{2(m-1)^2}\lambda_{m-1}^2 \neq 0;$$

$$\tilde{L}_{2m-1} + \left(2\tilde{d}_1 + \tilde{e}_1\frac{m(m+1)}{(m-1)^2}\right)\lambda_{m-1}^2 \neq 0,$$

where λ_{m-1} is expressed by (2.2),

$$\tilde{L}_k := \tilde{e}_k + \sum_{i=0}^{k-m+1} \frac{k+1-2i}{m-1+i}\tilde{d}_i\tilde{e}_{k-m+1-i}\lambda_{m-1}, \qquad (k = m-1, m, \dots, 2m-1),$$

$$\tilde{d}_n = \frac{\tilde{d}_1}{n!}\prod_{0\le r\le n-2}(\sigma - r(1-\hat{\mu}))$$
$$(n = 2,3,\dots),\ \tilde{d}_0 = 1,\ \tilde{d}_1 = m + \hat{\mu} - 2,\ \sigma = n + m + 2\hat{\mu} - 3; \tag{3.28}$$

$$\tilde{e}_n = \frac{\tilde{e}_1}{n!}\prod_{0\le r\le n-2}(\epsilon - r(1-\hat{\mu}))$$
$$(n = 2,3,\dots),\ \tilde{e}_0 = 1,\ \tilde{e}_1 = 2m + \hat{\mu} - 2,\ \epsilon = n + 2m + 2\hat{\mu} - 3. \tag{3.29}$$

Then the origin point of (3.27) is a center if and only if (1.7) and (1.8) hold.

Moreover, this center is a composition center and weak center.

Proof. In polar coordinates, the system (3.27) becomes

$$\frac{dr}{d\theta} = \frac{r^2\cos\theta + \Phi_{m-1}r^m + \Psi_{2m-1}r^{2m}}{1 - \hat{\mu}r\sin\theta}, \tag{3.30}$$

where $\Phi_{m-1} = \Phi_{m-1}(\cos\theta, \sin\theta), \Psi_{2m-1} = \Psi_{2m-1}(\cos\theta, \sin\theta)$.

Taking

$$\rho = \frac{r}{(1 + (1-\hat{\mu})r\sin\theta)^{\frac{1}{1-\hat{\mu}}}},$$

the equation (3.30) can be written as

$$\frac{d\rho}{d\theta} = \rho^m\Phi_{m-1}(1 + (1-\hat{\mu})r\sin\theta)^{\frac{m+\hat{\mu}-2}{1-\hat{\mu}}} + \rho^{2m}\Psi_{2m-1}(1 + (1-\hat{\mu})r\sin\theta)^{\frac{2m+\hat{\mu}-2}{1-\hat{\mu}}}. \tag{3.31}$$

Applying the Langrange–Bürman formula we have

$$(1+(1-\hat{\mu})r\sin\theta)^{\frac{m+\hat{\mu}-2}{1-\hat{\mu}}} = \sum_{n=0}^{\infty} \tilde{d}_n\rho^n \sin^n\theta;$$
$$(1+(1-\hat{\mu})r\sin\theta)^{\frac{2m+\hat{\mu}-2}{1-\hat{\mu}}} = \sum_{n=0}^{\infty} \tilde{e}_n\rho^n \sin^n\theta,$$

where $\tilde{d}_n$, $\tilde{e}_n$ are expressed by (3.28), (3.29), respectively.

Substituting them into (3.31) we get

$$\frac{d\rho}{d\theta} = \Phi_{m-1}\rho^m \sum_{n=0}^{\infty} \tilde{d}_n\rho^n \sin^n\theta + \rho^{2m}\Psi_{2m-1} \sum_{n=0}^{\infty} \tilde{e}_n\rho^n \sin^n\theta. \tag{3.32}$$

Comparing the equations (3.8) and (3.32), we see that they have the same form, only with different coefficients. Similar to Theorem 3.1, the present theorem can be derived. □

Remark 3.5. When $\hat{\mu} = 0$, from Theorem 3.4 implies the Theorem 3.1 of [15].

Corollary 3.6. *If $\mu \neq 1$ and $\hat{d}_n = \tilde{d}_n|_{\hat{\mu}=\mu} \neq 0$ $(n = 1, 2, \ldots, m-1)$ $(m > 2)$, where $\tilde{d}_n$ $(n = 1, 2, \ldots, m-1)$ is expressed by (3.28). Then the origin point of (1.3) is a center if and only if (1.7) is satisfied.*

Proof. Similar to Theorem 3.4, when $\Psi_{2m-1} = 0$, the equation (1.3) can be transformed as following

$$\frac{d\rho}{d\theta} = \Phi_{m-1}\rho^m \sum_{n=0}^{\infty} \hat{d}_n\rho^n \sin^n\theta. \tag{3.33}$$

Similar to Theorem 3.1, we get that the solution of (3.33) such that $\rho(0) = c$ $(0 < |c| \ll 1)$ is

$$\rho = c + c^m \sum_{k=0}^{m-2} c^k \hat{d}_k \overline{\sin^k\theta\Phi_{m-1}} + c^{2m-1}\left(\hat{d}_{m-1}\overline{\sin^{m-1}\theta\Phi_{m-1}} + \frac{m}{2}\bar{\Phi}^2_{m-1}\right) + o(c^{2m-1}).$$

As $\hat{d}_n = \tilde{d}_n|_{\hat{\mu}=\mu} \neq 0\,(n = 1, 2 \ldots, m-1)$, $\tilde{d}_0 = 1$, from $\rho(2\pi) = c$ follows that the condition (1.7) is satisfied. Using Lemma 2.4, the conclusion of the present corollary is valid. □

Remark 3.7. By Corollary 3.6, if $\mu \neq 1$, Conjecture 1.1 is valid when $\prod_{1\le n\le m-1} \hat{d}_n \neq 0$, $(m > 2)$.

Case B. $\nu = 0$, $\mu \neq 0$.

Consider Λ–Ω system

$$\begin{cases} x' = -y(1-\mu y) + x(\Phi_{m-1} + \Psi_{2m-1}), \\ y' = x(1-\mu y) + y(\Phi_{m-1} + \Psi_{2m-1}). \end{cases} \tag{3.34}$$

Theorem 3.8. *Suppose that*

$$\prod_{m-1\le k\le 2m-3} \hat{L}_k \neq 0;$$
$$\hat{L}_{2m-2} + \frac{m(2m-1)}{2(m-1)^2}\lambda^2_{m-1} \neq 0;$$
$$\hat{L}_{2m-1} + \mu\left(2 + \frac{m(m+1)}{(m-1)^2}\right)\lambda^2_{m-1} \neq 0,$$

where λ_{m-1} is expressed by (2.2), $\hat{L}_k := \mu^k + \mu^{1-m+k}\sum_{i=0}^{k-m+1} \frac{k+1-2i}{m-1+i}\lambda_{m-1}$, $(k = m-1, m, \ldots, 2m-1)$. Then the origin point of (3.34) is a center if and only if (1.7) and (1.8) hold.

Moreover, this center is a composition center and weak center.

Proof. In polar coordinates, the system (3.34) becomes

$$\frac{dr}{d\theta} = \Phi_{m-1} \sum_{n=0}^{\infty} \mu^n r^{m+n} \sin^n \theta + \Psi_{2m-1} \sum_{n=0}^{\infty} \mu^n r^{2m+n} \sin^n \theta, \tag{3.35}$$

where $\Phi_{m-1} = \Phi_{m-1}(\cos\theta, \sin\theta), \Psi_{2m-1} = \Psi_{2m-1}(\cos\theta, \sin\theta)$.

Obviously, the equation (3.35) has the same form as (3.8), in Theorem 3.1 taking $d_k = e_k = \mu^k$ $(k = 0, 1, 2, \dots)$, the present theorem can be derived directly. □

Corollary 3.9. *For arbitrary $m > 2$, the origin point of (1.4) is a center if and only if (1.7) is satisfied.*

Proof. Under the linear changes of variables (3.1) the system (1.4) becomes

$$\begin{cases} x' = -y(1-y) + x\Phi_{m-1}, \\ y' = x(1-y) + y\Phi_{m-1}. \end{cases} \tag{3.36}$$

In polar coordinates (3.36) can be written as

$$\frac{dr}{d\theta} = \Phi_{m-1} \sum_{n=0}^{\infty} r^{m+n} \sin^n \theta. \tag{3.37}$$

Similar to Theorem 3.1, we get that the solution of (3.37) such that $r(0) = c$ $(0 < |c| \ll 1)$ is

$$r = c + c^m \sum_{i=0}^{m-2} \overline{\sin^k \theta \Phi_{m-1}} + c^{2m-1} \left(\overline{\sin^{m-1} \theta \Phi_{m-1}} + \frac{m}{2} \bar{\Phi}^2_{m-1} \right) + o(c^{2m-1}).$$

Obviously, from $r(2\pi) = c$ follows that the condition (1.7) is satisfied. Using Lemma 2.4 the conclusion of the present corollary is correct. □

Remark 3.10. By Corollary 3.9, Conjecture 1.2 is valid for $m > 2$.

Remark 3.11. In the case of $\mu = \nu = 0$, $m = 2$ the center problem of system (1.5) has been discussed by [14].

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