



# Fixed-time and state-dependent time discontinuities in the theory of Stieltjes differential equations

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**Abstract.** In the present paper, we are concerned with a very general problem, namely the Stieltjes differential Cauchy problem involving state-dependent discontinuities.

Given that the theory of Stieltjes differential equations covers the framework of impulsive problems with fixed-time impulses, in the present work we generalize this setting by allowing the occurrence of fixed-time impulses, as well as the occurrence of state-dependent impulses.

Along with an existence result obtained under an overarching set of assumptions involving Stieltjes integrals, it is showed that a least and a greatest solution can be found.

**Keywords:** Stieltjes differential equation, state-dependent impulsive equation, Stieltjes integral, extremal solution.

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## 1 Introduction

The important role played by the theory of initial value impulsive differential problems in describing the evolution of many processes in the real life is well-known [1, 15, 27]. The most encountered framework in literature is that of impulsive equations with impulses occurring at fixed times [1, 5].

The more general setting of state-dependent time discrete perturbations is (despite its wide applicability, e.g. [6, 12, 24]) far less studied, due to its complexity – see [2, 4, 10] or [25] and the references therein. To give just an idea, fixed point results are not applicable since the continuity of Nemytskii operator cannot be checked, while the control of the number and position of the state-dependent impulse moments requires strong specific assumptions.

At the same time, the theory of differential equations with Stieltjes derivative – see [19] (called Stieltjes differential equations, e.g. [11, 17]), which has been shown to be generally equivalent to the theory of measure differential equations (see [8, 9, 21]) covers a wide variety

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of real life processes. For instance, it allows the occurrence of stationary intervals (where the derivator  $g$  is constant) coupled with moments with abrupt changes in the state (where  $g$  has discontinuities).

We have in mind the possibility to allow both behaviours: stationary intervals coupled with pre-established moments with abrupt changes and also with state-dependent time impulses.

We thus focus on Stieltjes first-order Cauchy differential problem with impulses depending on the state

$$\begin{cases} x'_g(t) = f(t, x(t)), \mu_g\text{-a.e. } t \in [0, 1] \setminus (A_x \setminus A) \\ \Delta^+ x(t) = x(t+) - x(t) = I_i(x(t)), \text{ if } t \in A_x^i \setminus A, \text{ for } i = 1, \dots, k \\ x(0) = x_0 \end{cases} \quad (1.1)$$

where  $g : [0, 1] \rightarrow \mathbf{R}$  is a left-continuous nondecreasing function which induces the Stieltjes measure  $\mu_g$ ,  $B \subset \mathbf{R}$  is a closed set containing  $x_0$ ,  $f : [0, 1] \times B \rightarrow \mathbf{R}$  is the function describing the rate of change of the unknown function, while  $I_i : \mathbf{R} \rightarrow \mathbf{R}$ ,  $i = 1, \dots, k$  give the jumps at the points where the barriers  $\gamma_i : \mathbf{R} \rightarrow [0, 1]$ ,  $i = 1, \dots, k$  are reached.

By  $A$ ,  $A_x^i$  and  $A_x$  ( $x$  being a real valued function on  $[0, 1]$ ) one denotes the sets

$A =$  the set of points of discontinuity of  $g$ ,

$A_x^i = \{t \in [0, 1] : t = \gamma_i(x(t))\}$  for every  $i = 1, \dots, k$ ,

respectively

$$A_x = \bigcup_{i=1}^k A_x^i.$$

To avoid ambiguity at the common points of  $A_x^i$  and  $A_x^j$  (with  $i \neq j$ ), respectively of  $A_x^i$  and  $A$ , we impose the conditions *H4).iii)*, respectively *H4).iv)* below.

Using of the Stieltjes derivative  $x'_g$  with respect to a left-continuous nondecreasing map  $g$  enables the presence of dead times (intervals where the process is stationary – corresponding to intervals where  $g$  is constant) as well as of fixed-time discrete perturbations (at the discontinuities of  $g$ ).

In the particular case where  $g(t) = t$  for every  $t \in [0, 1]$ , the existence of solutions for this problem has been provided e.g. in [2, 10, 13] or [25]. However, even in this specific case, basic properties of the set of solutions are difficult to be proved (we refer to [13] or [33] for a detailed discussion).

The very wide framework of Stieltjes differential problems (which already covers many classical cases, such as ordinary differential and difference equations, impulsive equations, time-scales dynamic equations) with state-dependent discontinuities is studied here for the first time, as far as the author knows.

More precisely, we first present an existence result inspired by [22] (available for measure differential equations without allowing state dependent discontinuities, in particular for impulsive problems with fixed time impulse moments) by taking the advantage of the method used in [10] for state-dependent impulsive equations with  $g(t) = t$ .

Finally, we prove, using a nice result for measure differential problems without variable time impulses in [22], that a least and a greatest solution can be found. Note that, by a different method and different hypotheses, the existence of extremal solutions has been obtained in [13] when  $g(t) = t$  under assumptions involving that each barrier is hit only once.

## 2 Notions and preliminary facts

A function  $u : [0, 1] \rightarrow \mathbf{R}$  is said to be regulated if for every  $t \in [0, 1)$  there exists the limit  $u(t+)$  and for every  $s \in (0, 1]$  there exists the limit  $u(s-)$ . The set of discontinuity points of a regulated function is at most countable and the bounded variation or continuous functions are, without any doubt, regulated. The space  $G([0, 1], \mathbf{R})$  of regulated functions  $u : [0, 1] \rightarrow \mathbf{R}$  is a Banach space with respect to the sup-norm. By  $G_-([0, 1], \mathbf{R})$  we denote its subspace consisting in left-continuous functions.

Given a left-continuous nondecreasing function  $g : [0, 1] \rightarrow \mathbf{R}$ , the measurability with respect to (in short, w.r.t.) the  $\sigma$ -algebra defined by  $g$  will be called  $g$ -measurability,  $\mu_g$  denotes the Stieltjes measure generated by  $g$  and the Lebesgue–Stieltjes (shortly, LS-) integrability w.r.t.  $g$  means the abstract Lebesgue integrability w.r.t. the Stieltjes measure  $\mu_g$ . It is well known that if  $f$  is LS-integrable w.r.t.  $g$ , the primitive  $\int_0^\cdot f(s)dg(s) = \int_{[0, \cdot)} f(s)dg(s)$  is a  $g$ -absolutely continuous function in the following sense (see [31], [11] or [19]): a function  $u : [0, 1] \rightarrow \mathbf{R}$  is  $g$ -absolutely continuous if for every  $\varepsilon > 0$  there is  $\delta_\varepsilon > 0$  such that

$$\sum_{j=1}^m |u(t_j'') - u(t_j')| < \varepsilon$$

for any set  $\{(t_j', t_j'')\}_{j=1}^m$  of non-overlapping subintervals of  $[0, 1]$  with  $\sum_{j=1}^m (g(t_j'') - g(t_j')) < \delta_\varepsilon$ .

We shall also use the theory of Kurzweil–Stieltjes integral (we refer the reader to [14, 23, 30], see also [28, 29]) motivated by the fact that it is easy to handle (by integral sums), it fits well with the setting of regulated functions (i.e. it covers the situation where both the integrand and the integrator possess discontinuities) and, moreover, it can integrate functions that are not absolutely integrable.

Below are listed the basic properties of KS-integrals.

**Definition 2.1.** A function  $f : [0, 1] \rightarrow \mathbf{R}$  is Kurzweil–Stieltjes integrable with respect to  $g : [0, 1] \rightarrow \mathbf{R}$  (or KS-integrable w.r.t.  $g$ ) if there exists  $\int_0^1 f(s)dg(s) \in \mathbf{R}$  such that, for every  $\varepsilon > 0$ , there is a positive function  $\delta_\varepsilon : [0, 1] \rightarrow \mathbf{R}$  with

$$\left| \sum_{i=1}^p f(\xi_i)(g(t_i) - g(t_{i-1})) - \int_0^1 f(s)dg(s) \right| < \varepsilon$$

for every  $\delta_\varepsilon$ -fine partition  $\{([t_{i-1}, t_i], \xi_i) : i = 1, \dots, p\}$  of  $[0, 1]$ . This means that  $[t_{i-1}, t_i] \subset ]\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)[$ , for all  $i = 1, \dots, p$ .

The function  $t \mapsto \int_0^t f(s)dg(s)$  is called the KS-primitive of  $f$  w.r.t.  $g$ .

**Proposition 2.2** ([23]). *Let  $f : [0, 1] \rightarrow \mathbf{R}$  be Kurzweil–Stieltjes integrable w.r.t.  $g : [0, 1] \rightarrow \mathbf{R}$ . If  $g$  is regulated, then so is the primitive  $h : [0, 1] \rightarrow \mathbf{R}$ ,  $h(t) = \int_0^t f(s)dg(s)$  and for every  $t \in [0, 1]$ ,*

$$h(t+) - h(t) = f(t) [g(t+) - g(t)] \quad \text{and} \quad h(t) - h(t-) = f(t) [g(t) - g(t-)].$$

*Therefore,  $h$  is left-continuous, respectively right-continuous at the points where  $g$  has the same property.*

Note that the Lebesgue–Stieltjes integrability of a function  $f$  implies the Kurzweil–Stieltjes integrability and in the framework of a left-continuous nondecreasing function  $g$ , as a consequence of [23, Theorem 6.11.3] (see also [26, Theorem 8.1]), if  $t \in [0, 1]$  then

$$\int_0^t f(s)dg(s) = \int_{[0, t]} f(s)d\mu_g(s) - f(t)(g(t+) - g(t)) = \int_{[0, t)} f(s)d\mu_g(s).$$

In order to recall more properties of the primitive, we need the notion of (Stieltjes) derivative of a function with respect to another function, given in [19] (see also [31]).

**Definition 2.3.** Let  $g : [0, 1] \rightarrow \mathbf{R}$  be nondecreasing and left-continuous. The derivative of  $f : [0, 1] \rightarrow \mathbf{R}$  with respect to  $g$  (or the  $g$ -derivative) at the point  $t \in [0, 1]$  is

$$f'_g(t) = \lim_{t' \rightarrow t} \frac{f(t') - f(t)}{g(t') - g(t)} \quad \text{if } g \text{ is continuous at } t,$$

$$f'_g(t) = \lim_{t' \rightarrow t^+} \frac{f(t') - f(t)}{g(t') - g(t)} \quad \text{if } g \text{ is discontinuous at } t,$$

if the limit exists.

The  $g$ -derivative has found meaningful applications in solving real-world problems where periods of time where no activity occurs and instants with abrupt changes are both involved, such as [11], [18] or [20].

Remark that if  $t$  is a discontinuity point of  $g$ , then

$$f'_g(t) = \frac{f(t+) - f(t)}{g(t+) - g(t)}.$$

There is a set where Definition 2.3 does not work, more precisely,

$$C_g = \{t \in [0, 1] : g \text{ is constant on } (t - \varepsilon, t + \varepsilon) \text{ for some } \varepsilon > 0\}$$

but we must take into account that  $\mu_g(C_g) = 0$  [19] and, when studying differential equations, the equation has to be satisfied  $\mu_g$ -almost everywhere.

The connection between Stieltjes integrals and the Stieltjes derivative is given by Fundamental Theorems of Calculus [19, Theorems 5.4, 6.2, 6.5].

For Lebesgue–Stieltjes integrals, it is contained in [19, Theorem 5.4], we give the entire statement below.

**Theorem 2.4.** Let  $g : [0, 1] \rightarrow \mathbf{R}$  be a nondecreasing left-continuous function. Then  $f : [0, 1] \rightarrow \mathbf{R}$  is  $g$ -absolutely continuous if and only if it is  $g$ -differentiable  $\mu_g$ -a.e.,  $f'_g$  is Lebesgue–Stieltjes integrable w.r.t  $g$  and

$$f(t') = f(t'') + \int_{[t'', t']} f'_g(s) d\mu_g(s), \quad \text{for every } 0 \leq t'' < t' \leq 1.$$

### 3 Main results

We are concerned with the Stieltjes initial value differential problem with state-dependent discontinuities

$$\begin{cases} x'_g(t) = f(t, x(t)), \mu_g - a.e. \ t \in [0, 1] \setminus (A_x \setminus A) \\ \Delta^+ x(t) = x(t+) - x(t) = I_i(x(t)), \text{ if } t \in A_x^i \setminus A, \text{ for } i = 1, \dots, k \end{cases}$$

where  $B \subset \mathbf{R}$  is closed,  $x_0 \in B$ ,  $f : [0, 1] \times B \rightarrow \mathbf{R}$  and for each  $i = 1, \dots, k$ ,  $I_i : B \rightarrow \mathbf{R}$  describes the jumps at the points where the barrier  $\gamma_i : \mathbf{R} \rightarrow [0, 1]$  is reached. Recall that  $A$  is the set of discontinuity points of the left-continuous nondecreasing function  $g : [0, 1] \rightarrow \mathbf{R}$  continuous at 0,  $A_x^i$  is the set of points where the function  $x : [0, 1] \rightarrow \mathbf{R}$  hits the barrier  $\gamma_i$ , i.e.  $\tau \in A_x^i$  if  $t = \gamma_i(x(t))$  and  $A_x$  is the union of these  $A_x^i$ .

### 3.1 Existence result

#### Definition 3.1.

- i) A function  $x : [0, a] \rightarrow \mathbf{R}$  ( $a \in (0, 1]$ ) is called an integral solution of the state-dependent impulsive Stieltjes differential problem (1.1) on  $[0, a]$  if it is a solution of the impulsive integral equation

$$x(t) = x_0 + \int_0^t f(s, x(s)) dg(s) + \sum_{i=1}^k \sum_{\tau \in (A_x^i \setminus A) \cap [0, t)} I_i(x(\tau)), \quad t \in [0, a]. \quad (3.1)$$

- ii) (e.g. [10]) We say that a function  $x : [0, a] \rightarrow \mathbf{R}$  ( $a \in (0, 1]$ ) is a  $g$ -Carathéodory solution of the state-dependent impulsive Stieltjes differential problem (1.1) on  $[0, a]$  if:

- it is  $g$ -absolutely continuous and  $x'_g(t) = f(t, x(t))$ ,  $\mu_g$ -a.e. on  $[0, a] \setminus (A_x \setminus A)$ ;
- for each  $i = 1, \dots, k$ , at every  $t \in (A_x^i \setminus A) \cap [0, a]$ ,  $x$  is left-continuous, it has finite right limit and  $x(t+) = x(t) + I_i(x(t))$ ;
- $x(0) = x_0$ .

Consider  $B \subset \mathbf{R}$  a compact set.

We shall impose the following hypotheses on  $f : [0, 1] \times B \rightarrow \mathbf{R}$ :

- H1) For each  $x \in B$ , the map  $f(\cdot, x)$  is Kurzweil–Stieltjes integrable w.r.t.  $g$  on  $[0, 1]$ ;
- H2) One can find a non-decreasing function  $h : [0, 1] \rightarrow \mathbf{R}$  and a function  $M : [0, 1] \rightarrow \mathbf{R}$  KS-integrable w.r.t.  $h$  such that for every  $x \in G_-([0, 1], B)$ ,

$$\left| \int_u^v f(t, x(t)) dg(t) \right| \leq \int_u^v M(t) dh(t), \quad \text{for all } 0 \leq u \leq v \leq 1;$$

- H3) For any  $t \in [0, 1]$ ,  $f(t, \cdot)$  is continuous on  $B$ ;

**Remark 3.2.** Using [22, Lemma 3.1], from the preceding assumptions it follows that for each  $x \in G_-([0, 1], B)$ , the map  $f(\cdot, x(\cdot))$  is Kurzweil–Stieltjes integrable w.r.t.  $g$ .

The assumptions on the barriers  $\gamma_i : \mathbf{R} \rightarrow [0, 1]$  (known as transversality assumptions) and on the jumps  $I_i : \mathbf{R} \rightarrow \mathbf{R}$ ,  $i = 1, \dots, k$  are described below:

- H4) i) The maps  $\gamma_i$ ,  $i = 1, \dots, k$  are strictly monotone and continuous;
- ii)  $\gamma_i^{-1}(0) \neq x_0$  for all  $i$  and  $\gamma_i^{-1}(t) \neq \gamma_j^{-1}(t) + I_j(\gamma_j^{-1}(t))$  for all  $i, j = 1, \dots, k$ ,  $t \in [0, 1]$ ;
- iii) if  $\gamma_i(x) = \gamma_j(x)$  for some  $x \in B$  and  $i \neq j$  then  $I_i(x) = I_j(x)$ ;
- iv) whenever  $\tau \in A_x^i \cap A$  for some  $x \in G_-([0, 1], B)$ ,

$$I_i(x(\tau)) = f(\tau, x(\tau)) \cdot \Delta^+ g(\tau);$$

- H5) There is a positive integer  $\tilde{M}$  such that each integral solution of (1.1) on any subinterval of  $[0, 1]$  hits the barriers at at most  $\tilde{M}$  points.

We make the convention that, whenever a solution hits the intersection of two barriers, the moment is counted only once.

**Remark 3.3.** The last part of Condition H4) means that for every  $\tau \in A$  satisfying  $x(\tau) = \gamma_i^{-1}(\tau)$  for some  $x \in G_-([0, 1], B)$  and some  $i \in \{1, \dots, k\}$ ,

$$I_i(\gamma_i^{-1}(\tau)) = f(\tau, \gamma_i^{-1}(\tau)) \cdot \Delta^+ g(\tau).$$

Condition H5) is presented in a very general form, but we stress that it is ensured by the hypotheses imposed in other works on state-dependent impulsive differential problems when  $g(t) = t$ .

For instance, in [10] it is assumed that the distance between any two consecutive points where a solution hits the barriers is bigger than some constant, see (3.4) in Theorem 3.1. Also, in [2] there are a fixed number of barriers which are hit at most once by any solution, while in [25] there is only one barrier hit exactly once by any solution, see [25, Lemma 5.1].

By combining the hypotheses imposed for integral measure driven equations in [22] with the method used in the framework of state dependent impulsive equations in [10], we can prove an existence result for the state dependent impulsive Stieltjes differential problem (1.1):

**Theorem 3.4.** *Let  $f : [0, 1] \times B \rightarrow \mathbf{R}$  satisfy the hypotheses H1)–H3) and the barriers and jumps satisfy H4), H5). Suppose that*

$$\left\{ x \in \mathbf{R}; |x - x_0| \leq \int_0^1 M(s) dh(s) + K_1 + \dots + K_{\tilde{M}} \right\} \subset B, \quad (*)$$

where

$$K_1 = \max_{i=1}^k \sup_{|x-x_0| \leq \int_0^1 M(s) dh(s)} |I_i(x)|,$$

$$K_{n+1} = \max_{i=1}^k \sup_{|x-x_0| \leq \int_0^1 M(s) dh(s) + K_1 + \dots + K_n} |I_i(x)|, \quad \forall n \geq 1.$$

Then the problem (1.1) admits integral solutions on  $[0, 1]$ .

*Proof.* Consider at the beginning the measure-driven integral equation

$$x(t) = x_0 + \int_0^t f(s, x(s)) dg(s), \quad t \in [0, 1].$$

Since our assumption on  $B$  implies that

$$\left\{ x \in \mathbf{R} : |x - x_0| \leq \int_0^1 M(t) dh(t) \right\} \subset B,$$

by [22, Theorem 3.2], one can find an integral solution  $x_1$  on  $[0, 1]$ . By usual properties of Kurzweil–Stieltjes integrals,  $x_1$  is left-continuous on  $[0, 1]$  and continuous at any point where  $g$  is continuous (thus, it is continuous at 0).

Define then  $r_{i,1} : [0, 1] \rightarrow \mathbf{R}$  by

$$r_{i,1}(t) = \gamma_i(x_1(t)) - t.$$

Due to H4).ii),  $r_{i,1}(0) \neq 0$  for all  $i = 1, \dots, k$  and, since  $r_{i,1}$  is continuous at 0, we might encounter the following situations:

- if  $r_{i,1}(t) \neq 0$  for all  $i = 1, \dots, k$  and  $t \in (0, 1] \setminus A$ , then  $x_1$  is a solution of (1.1) on  $[0, 1]$ .

- if  $r_{i,1}(t) = 0$  for some  $i \in \{1, \dots, k\}$  and  $t \in (0, 1] \setminus A$ , then let  $t_1$  be a continuity point of  $g$  (i.e.  $t \notin A$ ) such that  $r_{i,1}(t_1) = 0$  for some  $i_1$  and  $r_{i,1}(t) \neq 0$  on  $[0, t_1] \setminus A$  for all  $i$ .

Consider in what follows the measure-driven integral problem

$$x(t) = x_1(t_1) + I_{i_1}(x_1(t_1)) + \int_{t_1}^t f(s, x(s))dg(s), \quad t \in [t_1, 1].$$

The assumption made on  $B$  brings us to

$$\left\{ x \in \mathbf{R} : |x - (x_1(t_1) + I_{i_1}(x_1(t_1)))| \leq \int_{t_1}^1 M(s)dh(s) \right\} \subset B$$

since for each such  $x$

$$\begin{aligned} |x - x_0| &\leq |x - (x_1(t_1) + I_{i_1}(x_1(t_1)))| + |x_1(t_1) - x_0| + |I_{i_1}(x_1(t_1))| \\ &\leq \int_{t_1}^1 M(s)dh(s) + \int_0^{t_1} M(s)dh(s) + \sup_{|x-x_0| \leq \int_0^1 M(s)dh(s)} |I_{i_1}(x)| \\ &\leq \int_0^1 M(s)dh(s) + K_1 \end{aligned}$$

and so,  $x \in B$  and the inclusion is proved.

We can thus apply [22, Theorem 3.2] once again and one can find an integral solution  $x_2$  on  $[t_1, 1]$ ; it is left-continuous on  $[t_1, 1]$  and continuous at any point where  $g$  is continuous (in particular, at  $t_1$ ). As above, define  $r_{i,2} : [t_1, 1] \rightarrow \mathbf{R}$  by  $r_{i,2}(t) = \gamma_i(x_2(t)) - t$  which is continuous at  $t_1$ .

Besides, by H4).ii), for all  $i$

$$\gamma_i^{-1}(t_1) \neq \gamma_i^{-1}(t_1) + I_{i_1}(\gamma_i^{-1}(t_1)) = x_2(t_1),$$

so  $r_{i,2}(t_1) \neq 0$ , whence we might have the following situations:

- if  $r_{i,2}(t) \neq 0$  for all  $i = 1, \dots, k$  and  $t \in (t_1, 1] \setminus A$ , then a solution of (1.1) on  $[0, 1]$  can be found if we take  $x_1$  on  $[0, t_1]$  and  $x_2$  on  $(t_1, 1]$ ;
- if  $r_{i,2}(t) = 0$  for some  $i \in \{1, \dots, k\}$  and  $t \in (t_1, 1] \setminus A$ , then let  $t_2 > t_1$  be a continuity point of  $g$  chosen such that  $r_{i,2}(t_2) = 0$  for some  $i_2$  and  $r_{i,2}(t) \neq 0$  on  $[t_1, t_2] \setminus A$  for all  $i$ .

Let us next look at the measure-driven Cauchy problem

$$x(t) = x_2(t_2) + I_{i_2}(x_2(t_2)) + \int_{t_2}^t f(s, x(s))dg(s), \quad t \in [t_2, 1].$$

It is not difficult to see that

$$\left\{ x \in \mathbf{R} : |x - (x_2(t_2) + I_{i_2}(x_2(t_2)))| \leq \int_{t_2}^1 M(s)dh(s) \right\} \subset B$$

as for each  $x$  in this set, as before,

$$\begin{aligned} |x - x_0| &\leq |x - (x_2(t_2) + I_{i_2}(x_2(t_2)))| + |x_2(t_2) - x_1(t_1)| + |x_1(t_1) - x_0| + |I_{i_2}(x_2(t_2))| \\ &\leq \int_{t_2}^1 M(s)dh(s) + \int_{t_1}^{t_2} M(s)dh(s) + |I_{i_1}(x_1(t_1))| + \int_0^{t_1} M(s)dh(s) + |I_{i_2}(x_2(t_2))| \\ &\leq \int_0^{t_1} M(s)dh(s) + \sup_{|x-x_0| \leq \int_0^1 M(s)dh(s)} |I_{i_1}(x)| + \sup_{|x-x_0| \leq \int_0^1 M(s)dh(s) + K_1} |I_{i_2}(x)| \\ &\leq \int_0^1 M(s)dh(s) + K_1 + K_2 \end{aligned}$$



and so,  $x \in B$ .

We can thus apply [22, Theorem 3.2] and one can continue the process and we claim that it will be finished after less than  $\tilde{M} + 1$  steps (otherwise, hypothesis H5) would be contradicted).  $\square$

Under stronger assumptions on  $f$  and keeping the hypothesis on the barriers, one can obtain the existence of  $g$ -Carathéodory solutions for the impulsive measure differential problem (1.1).

**Theorem 3.5.** *Let  $f : [0, 1] \times B \rightarrow \mathbf{R}$  satisfy the hypotheses*

*H1') For each  $x \in G_-([0, 1], B)$ , the map  $f(\cdot, x(\cdot))$  is  $g$ -measurable on  $[0, 1]$ ;*

*H2') One can find a function  $M : [0, 1] \rightarrow \mathbf{R}$  Lebesgue–Stieltjes-integrable w.r.t.  $g$  such that for every  $x \in B$ ,*

$$|f(t, x)| \leq M(t), \quad \text{for } \mu_g\text{-a.e. } t \in [0, 1];$$

*together with H3) and the barriers and jumps satisfy H4), H5).*

*Suppose that*

$$\left\{ x \in \mathbf{R}; |x - x_0| \leq \int_0^1 M(s) dg(s) + K_1 + \cdots + K_{\tilde{M}} \right\} \subset B, \quad (**)$$

*where*

$$K_1 = \max_{i=1}^k \sup_{|x-x_0| \leq \int_0^1 M(s) dg(s)} |I_i(x)|,$$

$$K_{n+1} = \max_{i=1}^k \sup_{|x-x_0| \leq \int_0^1 M(s) dg(s) + K_1 + \cdots + K_n} |I_i(x)|, \quad \forall n \geq 1.$$

*Then the problem (1.1) admits  $g$ -Carathéodory solutions on  $[0, 1]$ .*

*Proof.* We follow the same lines as in the previous result. Consider first the measure-driven Cauchy problem

$$\begin{cases} x'_g(t) = f(t, x(t)), & \mu_g\text{-a.e. } t \in [0, 1], \\ x(0) = x_0. \end{cases}$$

By the Peano existence result [11, Theorem 7.5], one can find a  $g$ -Carathéodory solution  $x_1$  on  $[0, 1]$ .

Define then  $r_{i,1} : [0, 1] \rightarrow \mathbf{R}$  as before and we can fall into one of the following situations:

- if  $r_{i,1}(t) \neq 0$  for all  $i = 1, \dots, k$  and  $t \in (0, 1] \setminus A$ , then  $x_1$  is a  $g$ -Carathéodory solution of (1.1) on  $[0, 1]$ ;
- if  $r_{i,1}(t) = 0$  for some  $i \in \{1, \dots, k\}$  and  $t \in (0, 1] \setminus A$ , then let  $t_1 \in (0, 1] \setminus A$  be chosen such that  $r_{i_1,1}(t_1) = 0$  for some  $i_1$  and  $r_{i,1}(t) \neq 0$  on  $[0, t_1] \setminus A$  for all  $i$ .

Consider then the measure-driven Cauchy problem

$$\begin{cases} x'_g(t) = f(t, x(t)), & \mu_g\text{-a.e. } t \in [t_1, 1], \\ x(t_1) = x_1(t_1) + I_{i_1}(x_1(t_1)). \end{cases}$$

We can again apply [11, Theorem 7.5] in order to get a  $g$ -Carathéodory solution on  $[t_1, 1]$  and so on.  $\square$



**Remark 3.6.** We could have obtained the previous result by applying Theorem 3.4 and remarking that the assumptions  $H1'$ ,  $H2'$  together with the Fundamental Theorem of Calculus imply that any integral solution of our problem is a g-Carathéodory solution.

### 3.2 Existence of extremal solutions

Using the existence of extremal solutions for measure differential equations ([22, Theorem 4.4]), we get the existence of extremal solutions for measure differential equations with state-dependent impulses.

We need several additional assumptions.

H6) One of the following sets of conditions holds:

- a)  $x_0 > \gamma_i^{-1}(0)$  for each  $i$ , together with
- i)  $\gamma_i^{-1}(t) < x + f(t, x)\Delta^+g(t)$  for every  $i = 1, \dots, k, t \in A$  whenever  $\gamma_i^{-1}(t) < x$ ;
  - ii)  $\gamma_i^{-1}(t) < \gamma_j^{-1}(t) + I_j(\gamma_j^{-1}(t))$  for all  $i, j = 1, \dots, k, t \in [0, 1]$

or

- b)  $x_0 < \gamma_i^{-1}(0)$  for each  $i$ , together with
- i)  $\gamma_i^{-1}(t) > x + f(t, x)\Delta^+g(t)$  for every  $i = 1, \dots, k, t \in A$  whenever  $\gamma_i^{-1}(t) > x$ ;
  - ii)  $\gamma_i^{-1}(t) > \gamma_j^{-1}(t) + I_j(\gamma_j^{-1}(t))$  for all  $i, j = 1, \dots, k, t \in [0, 1]$ .

**Remark 3.7.** In the first case, when  $i = j$  one gets  $I_i(\gamma_i^{-1}(t)) > 0$  and, obviously, in the second case,  $I_i(\gamma_i^{-1}(t)) < 0$ .

H7) For every  $x, y \in B$  with  $x \leq y$ ,

$$x + f(t, x) \cdot \Delta^+g(t) \leq y + f(t, y) \cdot \Delta^+g(t), \quad \forall t \in A$$

together with

$$\gamma_i^{-1}(t) + I_i(\gamma_i^{-1}(t)) \leq \gamma_j^{-1}(t) + I_j(\gamma_j^{-1}(t)) \quad \text{whenever } \gamma_i^{-1}(t) \leq \gamma_j^{-1}(t)$$

for some  $t \in [0, 1], i, j \in \{1, \dots, k\}$ .

**Definition 3.8.** A solution  $y : [0, 1] \rightarrow \mathbf{R}$  is said to be the least (resp. greatest) solution of (1.1) if for any other solution  $x : [0, 1] \rightarrow \mathbf{R}$ ,

$$y(t) \leq x(t) \quad \text{for every } t \in [0, 1],$$

respectively

$$y(t) \geq x(t) \quad \text{for every } t \in [0, 1].$$

**Theorem 3.9.** Let the hypotheses H1)–H7) and (\*) be satisfied. Then the problem (1.1) admits a greatest integral solution and a least integral solution on  $[0, 1]$ .

*Proof.* We proceed as in the proof of Theorem 3.4, with convenient adjustments, in order to get the existence of a least solution.

Thus, consider in the first place the measure-driven integral equation

$$x(t) = x_0 + \int_0^t f(s, x(s))dg(s), \quad t \in [0, 1].$$

Since all the hypotheses of [22, Theorem 4.4] are satisfied, one can find a least solution  $y_1$  on  $[0, 1]$  (left-continuous everywhere and continuous at the continuity points of  $g$ , such as 0).

Let  $r_{i,1} : [0, 1] \rightarrow \mathbf{R}$  be defined by

$$r_{i,1}(t) = \gamma_i(y_1(t)) - t.$$

Due to  $H4).ii)$ ,  $r_{i,1}(0) \neq 0$  for all  $i = 1, \dots, k$  and since  $r_{i,1}$  is continuous at 0, the following situations are possible:

- if  $r_{i,1}(t) \neq 0$  for all  $i = 1, \dots, k$  and all  $t \in (0, 1] \setminus A$ , let  $y_-$  be  $y_1$  on  $[0, 1]$ .
- if  $r_{i,1}(t) = 0$  for some  $i \in \{1, \dots, k\}$  and  $t \in (0, 1] \setminus A$ , then let  $t_1$  be a continuity point of  $g$  such that  $r_{i_1,1}(t_1) = 0$  (i.e.  $y_1(t_1) = \gamma_{i_1}^{-1}(t_1)$ ) for some  $i_1$  and  $r_{i,1}(t) \neq 0$  on  $[0, t_1) \setminus A$  for all  $i$ .

Consider in what follows the measure-driven integral problem:

$$x(t) = \gamma_{i_1}^{-1}(t_1) + I_{i_1}(\gamma_{i_1}^{-1}(t_1)) + \int_{t_1}^t f(s, x(s)) dg(s), \quad t \in [t_1, 1].$$

The assumption we made on  $B$  implies that we can apply [22, Theorem 4.4] once again in order to find a least solution on  $[t_1, 1]$ , denoted by  $y_2$ .

As above, define  $r_{i,2} : [t_1, 1] \rightarrow \mathbf{R}$  by  $r_{i,2}(t) = \gamma_i(y_2(t)) - t$  and since it is continuous at  $t_1$  and  $r_{i,2}(t_1) \neq 0$ , we might have the following situations:

- if  $r_{i,2}(t) \neq 0$  for all  $i = 1, \dots, k$  and  $t \in (t_1, 1] \setminus A$ , then we construct the solution  $y_-$  of (1.1) on  $[0, 1]$  taking  $y_1$  on  $[0, t_1]$  and  $y_2$  on  $(t_1, 1]$ .
- if  $r_{i,2}(t) = 0$  for some  $i \in \{1, \dots, k\}$  and  $t \in (t_1, 1] \setminus A$ , then let  $t_2 > t_1$  be a continuity point of  $g$  chosen such that  $r_{i_2,2}(t_2) = 0$  for some  $i_2$  and  $r_{i,2}(t) \neq 0$  on  $[t_1, t_2) \setminus A$  for all  $i$ .

Let us next look at the problem:

$$x(t) = \gamma_{i_2}^{-1}(t_2) + I_{i_2}(\gamma_{i_2}^{-1}(t_2)) + \int_{t_2}^t f(s, x(s)) dg(s), \quad t \in [t_2, 1]$$

for each  $i \in \{1, \dots, k\}$ .

The hypothesis on  $B$  implies that, by [22, Theorem 4.4], we can find, for each of these problems, a least solution on  $[t_2, 1]$ , denoted by  $y_3$  and one can continue the process, which will be finished after less than  $\tilde{M} + 1$  steps (otherwise, hypothesis  $H5$ ) would be contradicted).

Let us see that the solution constructed in this way, namely  $y_-$ , is a least solution of (1.1) on  $[0, 1]$ . Suppose that  $H6).a)$  is satisfied (the case  $b)$  can be analyzed in a similar way).

Let  $x$  be an arbitrary solution of (1.1) on  $[0, 1]$ . We first show that  $y_-(t) \leq x(t)$  for every  $t \in [0, t_1]$ .

i) If  $(A_x \setminus A) \cap [0, t_1] = \emptyset$ , then  $y_-(t) \leq x(t)$  for every  $t \in [0, t_1]$ .

ii) If there are points in  $(A_x \setminus A) \cap [0, t_1]$ , let us focus on the first one since their number is finite and for all such points the discussion can be led in the same way; let  $\tau_1 \in (A_x \setminus A) \cap [0, t_1]$  be the first point where  $x$  hits some barrier  $\gamma_{i_0}$ . Then the following situations can be encountered:

ii.a) none of the discontinuity points of  $g$  lies in between 0 and  $\tau_1$ ; in this case, since  $y_-(0) = x_0 > \gamma_{i_0}^{-1}(0)$  and  $y_-(\tau_1) \leq x(\tau_1) = \gamma_{i_0}^{-1}(\tau_1)$  (as  $y_-$  is the least solution of the measure integral equation on  $[0, t_1]$ ), by the continuity of  $\gamma_{i_0}^{-1}$  and  $y_-$  on  $(0, \tau_1)$ , it would follow that  $y_-$  hits the barrier  $\gamma_{i_0}$  on  $(0, \tau_1]$ , contradiction with the choice of  $t_1$ .

ii.b) if there are discontinuity points of  $g$  lying in between 0 and  $\tau_1$ , we can fall into one of the three cases below:

- this is a finite subset of  $A$ ,  $\{\tilde{t}_i, i = 1, \dots, k\}$ ; then for each  $i$ ,  $x(\tilde{t}_i), y_-(\tilde{t}_i) > \gamma_{i_0}^{-1}(\tilde{t}_i)$  since otherwise the graphs of  $x, y_-$  would hit the barrier  $\gamma_{i_0}$  before  $\tau_1$  and this is not possible.

By H6).a).i), for each  $i = 1, \dots, k$ ,

$$\gamma_{i_0}^{-1}(\tilde{t}_i) < x(\tilde{t}_i+) \quad \text{and} \quad \gamma_{i_0}^{-1}(\tilde{t}_i) < y_-(\tilde{t}_i+),$$

whence, due to the fact that  $y_-(\tau_1) \leq x(\tau_1) = \gamma_{i_0}^{-1}(\tau_1)$  (since  $y_-$  is the least solution of the measure integral equation on  $[0, t_1]$ ),  $y_-$  would hit the barrier  $\gamma_{i_0}$  on  $(t_k, \tau_1]$  which again is impossible.

- this is a countable set  $\{\tilde{t}_i, i \in \mathbb{N}\}$  accumulating towards  $\bar{t} < \tau_1$ ; then, as before, at each such point

$$\gamma_{i_0}^{-1}(\tilde{t}_i) < x(\tilde{t}_i) \quad \text{and} \quad \gamma_{i_0}^{-1}(\tilde{t}_i) < y_-(\tilde{t}_i),$$

which, taking into account the left continuity of  $x, y_-$  and the continuity of  $\gamma_{i_0}^{-1}$ , imply

$$\gamma_{i_0}^{-1}(\bar{t}) \leq x(\bar{t}) \quad \text{and} \quad \gamma_{i_0}^{-1}(\bar{t}) \leq y_-(\bar{t}).$$

But equality is impossible as this would mean that  $x$ , respectively  $y_-$  would hit some barrier at  $\bar{t}$ , therefore

$$\gamma_{i_0}^{-1}(\bar{t}) < x(\bar{t}) \quad \text{and} \quad \gamma_{i_0}^{-1}(\bar{t}) < y_-(\bar{t})$$

and thus, by H6).a).i),

$$\gamma_{i_0}^{-1}(\bar{t}) < x(\bar{t}+) \quad \text{and} \quad \gamma_{i_0}^{-1}(\bar{t}) < y_-(\bar{t}+).$$

Again it would imply that  $y_-$  hits the barrier  $\gamma_{i_0}$  on  $(\bar{t}, \tau_1]$  which cannot happen.

- this is a countable set  $\{\tilde{t}_i, i \in \mathbb{N}\}$  accumulating towards  $\tau_1$ , in which case, as before,

$$\gamma_{i_0}^{-1}(\tau_1) \leq x(\tau_1) \quad \text{and} \quad \gamma_{i_0}^{-1}(\tau_1) \leq y_-(\tau_1)$$

and since  $y_-(\tau_1) \leq x(\tau_1) = \gamma_{i_0}^{-1}(\tau_1)$ ,  $y_-$  would hit the barrier before  $t_1$  and this is a contradiction.

Let us now check that  $y_-(t) \leq x(t)$  for every  $t \in [t_1, t_2]$  (on the next intervals the same discussion is to be carried out).

Hypothesis H6).a).ii) implies that

$$y_-(t_1+) > \gamma_i^{-1}(t_1) \quad \text{for each } i = 1, \dots, k \quad (3.2)$$

and from the preceding step we know that  $y_-(t_1) \leq x(t_1)$  which, by H7), implies that

$$y_-(t_1+) \leq x(t_1+). \quad (3.3)$$

If in (3.3) one has equality, then the proof on  $[0, t_1]$  has to be repeated in order to get the assertion on  $[t_1, t_2]$ . If strict inequality holds, then some modifications are necessary but we take the same steps as on  $[0, t_1]$  in order to prove that  $y_-(t) \leq x(t)$  for every  $t \in [t_1, t_2]$ . Thus:

i) If  $(A_x \setminus A) \cap [t_1, t_2] = \emptyset$ , then suppose there is  $\bar{t} \in (t_1, t_2]$  such that  $y_-(\bar{t}) > x(\bar{t})$ . We could be in the following situations:

i.a)  $[t_1, \bar{t}] \cap A = \emptyset$ , in which case  $x$  and  $y_-$  are continuous on  $(t_1, \bar{t}]$ ,  $y_-(t_1+) < x(t_1+)$  is valid and so there is a point  $\tilde{t}$  in this interval where the two trajectories intersect; then the solution defined by

$$\begin{cases} y_-(t), & \text{for } t \in (t_1, \tilde{t}], \\ x(t), & \text{for } t \in (\tilde{t}, \bar{t}] \end{cases}$$

would contradict the definition of  $y_-$  on  $(t_1, t_2]$  as being the least solution of the measure integral equation.

i.b)  $[t_1, \bar{t}] \cap A \neq \emptyset$ , in which case we might have:

- this is a finite subset of  $A$ ,  $\{\tilde{t}_i, i = 1, \dots, k\}$ ; then, since  $y_-(t_1+) < x(t_1+)$ , for each  $i$ ,  $y_-(\tilde{t}_i) \leq x(\tilde{t}_i)$  since otherwise, as in i.a), the fact that  $y_-$  is a least solution of the measure integral equation would be disobeyed.

So, by H7),  $y_-(\tilde{t}_k+) \leq x(\tilde{t}_k+)$  and, as  $y_-(\bar{t}) > x(\bar{t})$ , as in i.a), the fact that  $y_-$  is the least solution of the measure integral equation on  $[t_1, t_2]$  is contradicted.

- this is a countable set  $\{\tilde{t}_i, i \in \mathbb{N}\}$  accumulating towards  $\bar{t} < \bar{t}$ ; then, as before, at each such point  $y_-(\tilde{t}_i) \leq x(\tilde{t}_i)$  which, taking into account the left continuity of  $x, y_-$ , imply

$$y_-(\bar{t}) \leq x(\bar{t})$$

and thus

$$y_-(\bar{t}+) \leq x(\bar{t}+).$$

Again it would follow that  $y_-$  is not a least solution on  $(\bar{t}, \bar{t}]$ .

- this is a countable set  $\{\tilde{t}_i, i \in \mathbb{N}\}$  accumulating towards  $\bar{t}$ , in which case, as before,  $y_-(\bar{t}) \leq x(\bar{t})$  - contradiction.

ii) If there are points in  $(A_x \setminus A) \cap [t_1, t_2]$ , let us focus only on the first one  $\tau_1 \in (A_x \setminus A) \cap [t_1, t_2]$  where  $x$  hits some barrier  $\gamma_{i_0}$ . Then the following situations can be encountered:

ii.a) none of the discontinuity points of  $g$  lies in between  $t_1$  and  $\tau_1$ ; in this case, let us put together (3.2) and the fact that  $y_-(\tau_1) \leq x(\tau_1) = \gamma_{i_0}^{-1}(\tau_1)$  (since otherwise, together with (3.3) and the continuity of  $y_-$  and  $x$  it would be contradicted, as before, the choice of  $y_-$  as the least solution of the measure integral equation on  $[t_1, t_2]$ ). By the continuity of  $\gamma_{i_0}^{-1}$  and  $y_-$  on  $(t_1, \tau_1)$ , it would then follow that  $y_-$  hits the barrier  $\gamma_{i_0}$  on  $(t_1, \tau_1]$ , contradiction with the choice of  $t_2$ .

ii.b) if there are discontinuity points of  $g$  lying in between  $t_1$  and  $\tau_1$ , we can fall again into one of the three cases below:

- this is a finite subset of  $A$ ,  $\{\tilde{t}_i, i = 1, \dots, k\}$ ; then for each  $i$ ,  $x(\tilde{t}_i), y_-(\tilde{t}_i) > \gamma_{i_0}^{-1}(\tilde{t}_i)$  since otherwise the graphs of  $x, y_-$  would hit the barrier  $\gamma_{i_0}$  on  $(t_1, \tau_1)$  and this is not possible.

By H6).a).i), for each  $i = 1, \dots, k$ ,

$$\gamma_{i_0}^{-1}(\tilde{t}_i) < x(\tilde{t}_i+) \quad \text{and} \quad \gamma_{i_0}^{-1}(\tilde{t}_i) < y_-(\tilde{t}_i+),$$

whence, due to the fact that  $y_-(\tau_1) \leq x(\tau_1) = \gamma_{i_0}^{-1}(\tau_1)$  (otherwise, as before, the fact that  $y_-$  is the least solution of the measure integral equation on  $[t_1, t_2)$  would be contradicted),  $y_-$  would hit the barrier  $\gamma_{i_0}$  on  $(t_k, \tau_1]$  which again is impossible.

• this is a countable set  $\{\tilde{t}_i, i \in \mathbb{N}\}$  accumulating towards  $\bar{t} < \tau_1$ ; then, as before, at each such point

$$\gamma_{i_0}^{-1}(\tilde{t}_i) < x(\tilde{t}_i) \quad \text{and} \quad \gamma_{i_0}^{-1}(\tilde{t}_i) < y_-(\tilde{t}_i),$$

which, taking into account the left continuity of  $x, y_-$ , imply

$$\gamma_{i_0}^{-1}(\bar{t}) \leq x(\bar{t}) \quad \text{and} \quad \gamma_{i_0}^{-1}(\bar{t}) \leq y_-(\bar{t}).$$

Equality is not possible (because it would mean that  $x, y_-$  hit the barrier at  $\bar{t}$ ), so

$$\gamma_{i_0}^{-1}(\bar{t}) < x(\bar{t}) \quad \text{and} \quad \gamma_{i_0}^{-1}(\bar{t}) < y_-(\bar{t})$$

and thus

$$\gamma_{i_0}^{-1}(\bar{t}) < x(\bar{t}+) \quad \text{and} \quad \gamma_{i_0}^{-1}(\bar{t}) < y_-(\bar{t}+).$$

Again it would follow that  $y_-$  would hit the barrier  $\gamma_{i_0}$  on  $(\bar{t}, \tau_1]$  which cannot happen.

• this is a countable set  $\{\tilde{t}_i, i \in \mathbb{N}\}$  accumulating towards  $\tau_1$ , in which case, as before,

$$\gamma_{i_0}^{-1}(\tau_1) \leq x(\tau_1) \quad \text{and} \quad \gamma_{i_0}^{-1}(\tau_1) \leq y_-(\tau_1)$$

and since  $y_-(\tau_1) \leq x(\tau_1) = \gamma_{i_0}^{-1}(\tau_1)$ ,  $y_-$  would hit the barrier before  $t_2$  and this is a contradiction.  $\square$

**Corollary 3.10.** *If H1'), H2') are imposed, then there exist a least and a greatest g-Carathéodory solutions of (1.1).*

**Remark 3.11.** The present work provides the existence of extremal solutions for a large class of differential problems, namely Stieltjes differential equations involving fixed time and state-dependent time impulses. Let us note once again that, even in the particular case where the Stieltjes derivative is the usual derivative, the allowance of state-dependent time impulses leads to really complex situations.

The very general discussion developed here could be applied to study real life problems where the results available in literature for measure (Stieltjes) differential equations or for classical impulsive ODEs fail.

The wide applicability of our results can be seen by looking at the following example, which represents a generalization of a problem in [11], describing the evaporating water in an open top cylindrical tank.

**Example 3.12.** Suppose that the initial level of the water in the tank is  $x_0$  and that the water level decreases due to evaporation. If  $x(t)$  denotes the water height at time  $t > 0$ , then the model adopted in [11] (which takes into account that the level remains constant during the

nights, while during the days the evaporation speed is maximum at middays) states that the evolution of  $x$  can be described by the Stieltjes differential problem

$$x'_g(t) = f(t, x(t)), \quad t \in [0, T] \quad \text{and} \quad x(0) = x_0.$$

Note that in [11] the map  $f$  is supposed to be linear in  $x$ , but the nonlinear framework is realistic as well. The nondecreasing left-continuous function  $g$  can be chosen conveniently [11, page 20], for instance if we want to refill the tank every morning with an amount of water depending on to the level before refilling, then one may set

$$g(t) = \int_0^t \max(\sin(\pi s), 0) ds + \max\{k \in \mathbb{N} : 2k \leq t\}$$

and

$$f(2k, x(2k)) = \Delta^+ x(2k) = \lambda_k x(2k), \quad \lambda_k > 0$$

(the intervals  $[2k, 2k+1), k \in \mathbb{N}$  correspond to day times and, obviously, the intervals  $[2k+1, 2k+2), k \in \mathbb{N}$  to night times).

In other words, the moments  $2k, k \in \mathbb{N}$  are fixed-time impulsive moments with  $\Delta^+ g(2k) = 1, \forall k$  and so far, the problem can be solved through the theory of Stieltjes differential equations.

Suppose now that we want to add an amount of water (equal to  $I(x(t))$ ) whenever a state-dependent condition is satisfied, such as  $x(t) = \beta(t)$ , where  $\beta$  is a decreasing function measuring the water level in a huge second tank where the level water decreases due to evaporation, without adding or removing any quantity and without stationary intervals.

In this case, the theory in [11] cannot be applied due to the occurrence of state-dependent impulses. At the same time, nor the studies developed for state-dependent impulsive problems ([10], [2] or [25]) apply since the involved derivative is the Stieltjes derivative (not the usual derivative).

The announced problem can be investigated by applying our results for

$$\begin{cases} x'_g(t) = f(t, x(t)), \quad \mu_g\text{-a.e. } t \in [0, T] \setminus (A_x \setminus A) \\ \Delta^+ x(t) = x(t+) - x(t) = I(x(t)), \quad \text{if } t \in A_x \setminus A \\ x(0) = x_0 \end{cases}$$

where  $A = \{2k : k \in \mathbb{N}\} \cap [0, T]$  and, for some function  $x \in G_-([0, T], \mathbf{R})$ ,  $A_x = \{t \in [0, T] : x(t) = \beta(t)\}$ ; we thus face the occurrence of only one barrier  $\gamma_1 = \beta^{-1}$ .

Theorem 3.9 yields the existence of a least  $g$ -Carathéodory solution and of a greatest  $g$ -Carathéodory solution provided  $f$  and  $I$  satisfy the following conditions:

- a) for each  $x \in G_-([0, T], B)$ , the map  $f(\cdot, x(\cdot))$  is  $g$ -measurable;
- b) one can find a function  $M : [0, T] \rightarrow \mathbf{R}$  Lebesgue–Stieltjes-integrable w.r.t.  $g$  such that for every  $x \in B$ ,

$$|f(t, x)| \leq M(t), \quad \text{for } \mu_g\text{-a.e. } t \in [0, T]$$

such that (\*\*) is valid (with  $T$  instead of 1);

- c)  $f$  is continuous with respect to its second argument;
- d)  $\beta : [0, 1] \rightarrow \mathbf{R}$  is strictly monotone and continuous and whenever  $\tau \in A_x \cap A$  for some  $x \in G_-([0, T], B)$ ,

$$I(\beta(\tau)) = f(\tau, \beta(\tau));$$

- e) there is a positive integer  $\tilde{M}$  such that each integral solution of (1.1) on any subinterval of  $[0, T]$  hits the barrier at at most  $\tilde{M}$  points;
- f)  $x_0 > \beta(0)$ ,  $I(\beta(t)) > 0$  for every  $t \in [0, T]$  and  $\beta(t) < x + f(t, x)$  for every  $t \in A$ ,  $x \in B$  with  $\beta(t) < x$ ;
- g) for every  $x, y \in B$  with  $x \leq y$ ,

$$x + f(t, x) \leq y + f(t, y), \quad \forall t \in A.$$

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