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Strong maximum principle for a sublinear elliptic problem at resonance

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Abstract. We examine the semilinear resonant problem

$$-\Delta u = \lambda_1 u + \lambda g(u) \text{ in } \Omega, \quad u \geq 0 \text{ in } \Omega, \ \ u_{|\partial\Omega} = 0,$$

where $\Omega \subset \mathbb{R}^N$ is a smooth, bounded domain, λ_1 is the first eigenvalue of $-\Delta$ in Ω, $\lambda > 0$. Inspired by a previous result in literature involving power-type nonlinearities, we consider here a generic sublinear term g and single out conditions to ensure: the existence of solutions for all $\lambda > 0$; the validity of the strong maximum principle for sufficiently small λ. The proof rests upon variational arguments.

Keywords: resonant problem, existence, maximum principle.

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded domain of class C^2, and let λ_1 be the first eigenvalue of $-\Delta$ in Ω with Dirichlet boundary conditions. The issue of the existence of solutions of the problem

$$\begin{cases} -\Delta u = \lambda_1 u + u^{s-1} - \mu u^{r-1} & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

$s \in (1, 2)$, $r \in (1, s)$, and $\mu > 0$, has been the subject of study of the recent [3]. As a distinctive feature, the right-hand side term $f(t) := \lambda_1 t + t^{s-1} - \mu t^{r-1}$ in (1.1) is not locally Lipschitz near 0, and moreover satisfies the sign property

$$f^{-1}((-\infty, 0]) \supseteq (0, a], \quad \text{for some } a > 0.$$

As a result, from the celebrated paper [13] (see also [8]), it is known that the strong maximum principle may fail to be valid in this context. By adopting minimax and perturbation

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techniques, the author of [3] showed instead that such a principle does hold as long as the perturbation parameter is chosen sufficiently large. More precisely, the main results in [3] state that problem (1.1) has non-zero solutions for the entire positive range of μ; positive solutions for μ large enough.

The fact that, after a rescaling, (1.1) can be turned into the problem

$$\begin{cases} -\Delta u = \lambda_1 u + \lambda(u^{s-1} - u^{r-1}) & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

for a suitable $\lambda > 0$, raises the natural question whether, as explicitly expressed in [3, Remark 2.4], the same results mentioned above continue to hold when the powers in (1.2) are replaced by a generic nonlinear term g. And, if it is so, it would be interesting of course to identify some "minimal" structure conditions on g for the validity of such results. In the present paper we address these questions and consider the problem

$$\begin{cases} -\Delta u = \lambda_1 u + \lambda g(u) & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where $g : [0, +\infty) \to \mathbb{R}$ is continuous, $g(0) = 0$, and obeys the following conditions:

(g_1) there exists $q \in (1, 2)$ such that $k_1 := \sup_{t>0} \frac{|g(t)|}{1 + t^{q-1}} < +\infty$;

(g_2) $\lim_{t \to 0^+} \frac{g(t)}{t} = -\infty$;

(g_3) $\liminf_{t \to +\infty} G(t) > 0$;

(g_4) $\lim_{t \to +\infty} (g(t)t - 2G(t)) = -\infty$,

where, as usual,

$$G(t) := \int_0^t g(s)ds, \quad \text{for all } t \geq 0.$$

Problems like (P_λ) are being investigated since Landesman and Lazer's pioneering work [9], in which sufficient conditions, based on the interaction between the nonlinearity and the spectrum of the linear operator, were given for them to have a solution. Noteworthy contributions following that work can be found in [2, 5, 12] and also in [6, 7, 10, 11, 14] (see the related references as well) in which several classes of elliptic problems at resonance are investigated via variational and topological methods.

Coming back to (P_λ), our approach develops along the same line of reasoning as [3]. We prove initially that (P_λ) has at least a non-zero solution for all $\lambda > 0$. This is accomplished by considering a sequence of problems near resonance whose solutions are shown to converge to a solution of the original problem. In this regard, assumption (g_4) comes into play to prove the boundedness of the sequence of approximating solutions. Then, by exploiting the classical decomposition of $H_0^1(\Omega)$ into the first eigenspace and its orthogonal complement, we show that, for sufficiently small λ, the set of solutions to (P_λ) is contained in the interior of the positive cone of $C_0^1(\overline{\Omega})$. It still remains an open question to investigate the uniqueness of positive solutions to (P_λ) (in the one-dimensional case and for power-nonlinearities it has instead been established in [4]), as well as the existence of non-zero solutions compactly supported in Ω, in the spirit of [8].

Our main results, Theorems 2.3 and 2.4, are stated and proved in the coming section. Before going on, we arrange some notation and the variational framework for (P_λ). We set

$$\|u\| := \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}}, \quad \text{for all } u \in H_0^1(\Omega),$$

and denote by $\|\cdot\|_p$, $p \in [1, +\infty]$, the classical L^p-norm on Ω. We also set

$$c_p := \sup_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\|u\|_p}{\|u\|}$$

for each $p \geq 1$, with $p \leq \frac{2N}{N-2}$ if $N \geq 3$, and denote by ϕ_1 the positive eigenfunction associated with λ_1 and normalized with respect to $\|\cdot\|_\infty$. We recall that the first two eigenvalues λ_1, λ_2 of $-\Delta$ in Ω admit the variational characterization

$$\lambda_1 = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\|u\|^2}{\|u\|_2^2}, \quad \lambda_2 = \inf_{u \in \operatorname{span}\{\phi_1\}^\perp \setminus \{0\}} \frac{\|u\|^2}{\|u\|_2^2}.$$

Given a set $E \subset \mathbb{R}^N$, its Lebesgue measure will be denoted by the symbol $|E|$. Throughout this paper, the symbols $C, C_1, C_2, \ldots$ represent generic positive constants whose exact value may change from occurence to occurrence.

For all $\lambda > 0$, we denote by $I_\lambda : H_0^1(\Omega) \to \mathbb{R}$ the energy functional associated with (P_λ),

$$I_\lambda(u) := \frac{1}{2}\|u\|^2 - \frac{\lambda_1}{2}\|u_+\|_2^2 - \lambda \int_\Omega G(u_+) dx, \quad \text{for all } u \in H_0^1(\Omega),$$

where $u_+ = \max\{u, 0\}$. By a weak solution to (P_λ) we mean any $u \in C^0(\overline{\Omega}) \cap H_0^1(\Omega)$ verifying

$$\int_\Omega (\nabla u \nabla v - \lambda_1 u v - \lambda g(u) v)\, dx = 0, \quad \text{for all } v \in H_0^1(\Omega).$$

2 Results

As already mentioned, we start by considering a sequence of approximating problems.

Lemma 2.1. *For each $\lambda > 0$, there exists $\bar{n} \in \mathbb{N}$ such that the problem*

$$\begin{cases} -\Delta u = \left(\lambda_1 - \dfrac{1}{n}\right) u + \lambda g(u) & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_n}$$

admits a non-zero weak solution u_n, with positive energy, for all $n \geq \bar{n}$.

Proof. Fix $\lambda > 0$ and let $n \in \mathbb{N}$ with $n > \frac{1}{\lambda_1}$. Let us first show that the energy functional $I_n : H_0^1(\Omega) \to \mathbb{R}$ corresponding to (P_n),

$$I_n(u) := I_\lambda(u) + \frac{1}{2n} \|u_+\|_2^2 = \frac{1}{2} \|u\|^2 - \frac{1}{2}\left(\lambda_1 - \frac{1}{n}\right) \|u_+\|_2^2 - \lambda \int_\Omega G(u_+)dx, \tag{2.1}$$

for all $u \in H_0^1(\Omega)$, has the mountain pass geometry for sufficiently large $n \in \mathbb{N}$.

Fix $k \in (2, 2^*)$ and set

$$M := \frac{k}{2} \sup_{t>0} \frac{\lambda_1 t^2 + 2\lambda G(t)}{t^k}.$$

By (g_1) and (g_2) one has $0 < M < +\infty$ and $\frac{\lambda_1}{2} t^2 + \lambda G(t) \le \frac{M}{k} t^k$, for all $t \ge 0$. Then, defining

$$R := (M c_k^k)^{\frac{1}{2-k}},$$

we easily obtain

$$\begin{aligned} \inf_{u \in S_R} I_n(u) &\ge \inf_{\|u\|=R} \left(\frac{1}{2}\|u\|^2 - \frac{M}{k}\|u\|_k^k\right) \\ &\ge \inf_{u \in S_R} \left(\frac{1}{2}\|u\|^2 - \frac{M c_k^k}{k}\|u\|^k\right) \\ &= \left(\frac{1}{2} - \frac{1}{k}\right) R^2 > 0, \end{aligned} \tag{2.2}$$

for any $n \in \mathbb{N}$, where $S_R := \{u \in H_0^1(\Omega) : \|u\| = R\}$.

Now, let us show that there exist $u_1 \in H_0^1(\Omega)$, with $\|u_1\| > R$, and $\bar{n} \in \mathbb{N}$, such that $I_n(u_1) < 0$ for all $n \ge \bar{n}$. Owing to (g_3), there exist $L, b > 0$ such that

$$G(t) > L, \quad \text{for all } t \ge b.$$

If we denote by

$$E_\gamma := \{x \in \Omega : \phi_1(x) < \gamma\},$$

with $\gamma > 0$, then there exists $\gamma_1 > 0$ such that

$$L > \frac{k_1(bq + b^q)|E_\gamma|}{q(|\Omega| - |E_\gamma|)}, \quad \text{for all } \gamma \in (0, \gamma_1). \tag{2.3}$$

Fix $\bar{\gamma} \in \mathbb{R}$ satisfying

$$0 < \bar{\gamma} < \min\left\{\gamma_1, \frac{b}{R}\right\}.$$

Since the function $\psi(t) := q\bar{\gamma}t + \bar{\gamma}^q t^q$ is continuous in $(0, +\infty)$ and $\psi\left(\frac{b}{\bar{\gamma}}\right) = bq + b^q$, thanks to (2.3), there exists $\bar{t} > \frac{b}{\bar{\gamma}}$ such that

$$L > \frac{k_1(q\bar{\gamma}\bar{t} + \bar{\gamma}^q \bar{t}^q)|E_{\bar{\gamma}}|}{q(|\Omega| - |E_{\bar{\gamma}}|)}. \tag{2.4}$$

With the aid of (g_1) and (2.4) we then obtain

$$\begin{aligned}\int_\Omega G(\bar{t}\phi_1)dx &= \int_{E_{\bar\gamma}} G(\bar{t}\phi_1)dx + \int_{\{\phi_1\geq\bar\gamma\}} G(\bar{t}\phi_1)dx \\ &\geq -k_1\int_{E_{\bar\gamma}}\left(\bar{t}\phi_1+\frac{(\bar{t}\phi_1)^q}{q}\right)dx + \int_{\{\phi_1\geq\bar\gamma\}} G(\bar{t}\phi_1)dx \\ &\geq -k_1\left(\bar{t}\bar\gamma+\frac{\bar{t}^q\bar\gamma^q}{q}\right)|E_{\bar\gamma}| + L(|\Omega|-|E_{\bar\gamma}|) \\ &> 0.\end{aligned}$$

As a result, there exists $\bar{n}\in\mathbb{N}$, with $\bar{n}>\frac{1}{\lambda_1}$, such that

$$I_n(\bar{t}\phi_1) = \frac{\bar{t}^2}{2n}\|\phi_1\|_2^2 - \lambda\int_\Omega G(\bar{t}\phi_1)dx < 0$$

for all $n\geq\bar{n}$. Therefore, the functional I_n satisfies the geometric conditions required by the mountain pass theorem for all $n\geq\bar{n}$.

Moreover, by (g_1) and Sobolev embeddings, one has

$$\begin{aligned}I_n(u) &\geq \frac{1}{2n\lambda_1}\|u\|^2 - \lambda k_1\left(\int_\Omega |u|dx + \frac{1}{q}\int_\Omega |u|^q dx\right) \\ &\geq \frac{1}{2n\lambda_1}\|u\|^2 - \lambda c_1 k_1\|u\| - \frac{\lambda c_q k_1}{q}\|u\|^q,\end{aligned}$$

and thus $I_n(u)\to+\infty$ as $\|u\|\to+\infty$. This fact, in addition to standard arguments (see for instance Example 38.25 of [15]), ensures that I_n satisfies the Palais–Smale condition. Then, by invoking the classical mountain pass theorem, I_n admits a critical point $u_n\in H_0^1(\Omega)\setminus\{0\}$ for all $n\geq\bar{n}$, and, by (2.2), one also has

$$I_n(u_n) = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_n(\gamma(t)) \geq \left(\frac{1}{2}-\frac{1}{k}\right)R^2, \tag{2.5}$$

where $\Gamma := \{\gamma\in C^0([0,1],H_0^1(\Omega)) : \gamma(0)=0,\ \gamma(1)=u_1\}$. This concludes the proof. ☐

Lemma 2.2. *Let $\lambda>0$, $\bar{n}\in\mathbb{N}$ and let u_n, with $n\geq\bar{n}$, be as in Lemma 2.1. Then, the sequence $\{u_n\}_{n\geq\bar{n}}$ is bounded in $H_0^1(\Omega)$.*

Proof. Let $n\in\mathbb{N}$, $n\geq\bar{n}$. By standard regularity theory, $u_n\in C^{1,\alpha}(\overline{\Omega})$, for some $\alpha\in(0,1)$. For any $n\in\mathbb{N}$, $n\geq\bar{n}$ there exist, uniquely determined, $t_n\in\mathbb{R}$ and $w_n\in\operatorname{span}\{\phi_1\}^\perp$ such that

$$u_n = t_n\phi_1 + w_n.$$

It is straightforward to verify that $w_n\in C^{1,\alpha}(\overline{\Omega})$ is a weak solution to

$$\begin{cases} -\Delta u = \left(\lambda_1-\frac{1}{n}\right)u + \lambda g(t_n\phi_1+u) - \frac{t_n}{n}\phi_1 & \text{in } \Omega \\ u\geq 0 & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{2.6}$$

and therefore, also by (g_1), one has

$$\begin{aligned}\|w_n\|^2 &\le \left(\frac{\lambda_1-\frac{1}{n}}{\lambda_2}\right)\|w_n\|^2+\lambda\int_\Omega g(t_n\phi_1+w_n)w_n dx\\ &\le \left(\frac{\lambda_1-\frac{1}{n}}{\lambda_2}\right)\|w_n\|^2+\lambda k_1\|w_n\|_1+\lambda k_1 t_n^{q-1}\|\phi_1\|_\infty^{q-1}\|w_n\|_1+\lambda k_1\|w_n\|_q^q.\end{aligned} \tag{2.7}$$

From (2.7), it follows that

$$\|w_n\|\le C\left((1+t_n^{q-1})+\|w_n\|^{q-1}\right), \tag{2.8}$$

for some $C>0$. We claim that the sequence $\{t_n\}_{n\ge\bar{n}}$ is bounded in $\mathbb{R}$. Arguing by contradiction, assume that, up to a subsequence, $t_n\to+\infty$ as $n\to+\infty$. Without loss of generality, we can assume that $t_n\ge 1$ for all $n\ge\bar{n}$ and, since

$$y^{q-1}\le C_1+\frac{1}{2C}y\le C_1t_n^{q-1}+\frac{1}{2C}y,\quad \text{for all } y>0,$$

from (2.8) we deduce

$$\|w_n\|\le 2Ct_n^{q-1}+C\|w_n\|^{q-1}\le 2Ct_n^{q-1}+CC_1t_n^{q-1}+\frac{1}{2}\|w_n\|,$$

and then

$$\|w_n\|\le C_2t_n^{q-1}.$$

Therefore, fixing $p>\max\left\{\frac{N}{2},\frac{q}{q-1}\right\}$, we obtain

$$\begin{aligned}\|w_n\|_\infty &\le C_3\left(\|w_n\|_p+\|g(t_n\phi_1+w_n)\|_p+\frac{t_n}{n}\|\phi_1\|_p\right)\\ &\le C_4\left(\|w_n\|_\infty^{\frac{p-1}{p}}\|w_n\|_1^{\frac{1}{p}}+1+t_n^{q-1}+\|w_n\|_\infty^{q-1-\frac{q}{p}}\|w_n\|_q^{\frac{q}{p}}+\frac{t_n}{n}\right)\\ &\le C_5\left(\|w_n\|_\infty^{\frac{p-1}{p}}t_n^{\frac{q-1}{p}}+t_n^{q-1}+\|w_n\|_\infty^{q-1-\frac{q}{p}}t_n^{\frac{q(q-1)}{p}}+\frac{t_n}{n}\right).\end{aligned}$$

Dividing the first and the last side of the previous inequality by t_n and bearing in mind that $y^m\le 1+y$, for all $m\in[0,1]$ and $y>0$, we get

$$\begin{aligned}\left\|\frac{w_n}{t_n}\right\|_\infty &\le C_5\left(\left\|\frac{w_n}{t_n}\right\|_\infty^{\frac{p-1}{p}}t_n^{\frac{q-2}{p}}+t_n^{q-2}+\left\|\frac{w_n}{t_n}\right\|_\infty^{q-1-\frac{q}{p}}t_n^{(q-2)\left(1+\frac{q}{p}\right)}+\frac{1}{n}\right)\\ &\le C_5\left(t_n^{q-2}+\left(t_n^{\frac{q-2}{p}}+t_n^{(q-2)\left(1+\frac{q}{p}\right)}\right)\left(1+\left\|\frac{w_n}{t_n}\right\|_\infty\right)+\frac{1}{n}\right)\\ &\le C_5\left(t_n^{\frac{q-2}{p}}+2t_n^{\frac{q-2}{p}}\left(1+\left\|\frac{w_n}{t_n}\right\|_\infty\right)+\frac{1}{n}\right).\end{aligned}$$

It follows that

$$\left(1-2C_5t_n^{\frac{q-2}{p}}\right)\left\|\frac{w_n}{t_n}\right\|_\infty\le 3C_5t_n^{\frac{q-2}{p}}+\frac{C_5}{n},$$

and, as a consequence,

$$\lim_{n\to+\infty}\left\|\frac{w_n}{t_n}\right\|_\infty=0,$$

i.e.,

$$\frac{u_n}{t_n} \to \phi_1 \quad \text{uniformly in } \overline{\Omega}.$$

So, fixing $\gamma \in (0, \|\phi_1\|_\infty)$, we can find $E \subset \Omega$, with $|E| > 0$, and $\tilde{n} \in \mathbb{N}$, $\tilde{n} \geq \bar{n}$, such that

$$u_n(x) \geq \gamma t_n, \quad \text{for all } n \geq \tilde{n} \text{ and } x \in E.$$

At this point, set

$$\delta := \sup_{t>0}(g(t)t - 2G(t)) \in [0, +\infty),$$

and let $\bar{t} > 0$ such that

$$g(t)t - 2G(t) \leq -\frac{(\delta+1)|\Omega|}{|E|}, \quad \text{for all } t \geq \bar{t},$$

and $n^* \geq \tilde{n}$ such that $t_n \geq \frac{\bar{t}}{\gamma}$ for all $n \geq n^*$. Then, for all $n \geq n^*$, taking also (2.5) into account, we obtain

$$\begin{aligned}
0 &< \int_\Omega (g(u_n)u_n - 2G(u_n))dx \\
&= \int_{\Omega\setminus E} (g(u_n)u_n - 2G(u_n))dx + \int_E (g(u_n)u_n - 2G(u_n))dx \\
&\leq \delta|\Omega| - (\delta+1)|\Omega| < 0,
\end{aligned}$$

a contradiction. Therefore, the sequence $\{t_n\}_{n\geq\bar{n}}$ is bounded in $\mathbb{R}$ and (2.8) yields the boundedness of $\{w_n\}_{n\geq\bar{n}}$ in $H_0^1(\Omega)$, as well. As a consequence, we get the boundedness of $\{u_n\}_{n\geq\bar{n}}$ in $H_0^1(\Omega)$, as desired. □

Collecting the results of the previous lemmas, it is now easy to derive our first existence result.

Theorem 2.3. *For all $\lambda > 0$, problem (P_λ) has at least one non-zero solution.*

Proof. Let $\{u_n\}$ be the sequence of solutions to (P_n) in Lemma 2.1. By Lemma 2.2 there exists $u^* \in H_0^1(\Omega)$ such that, up to a subsequence,

$$u_n \rightharpoonup u^* \text{ in } H_0^1(\Omega), \quad u_n \to u^* \text{ in } L^p(\Omega), \text{ for all } p \in [1, 2^*).$$

Fixing $v \in H_0^1(\Omega)$ and taking the limit as $n \to +\infty$ in the identity $I_n'(u_n)(v) = 0$, we get $I_\lambda'(u^*)(v) = 0$, i.e. u^* is a weak solution to (P_λ). To justify that $u^* \neq 0$, observe that, by (2.5) one has

$$\begin{aligned}
0 &< \left(\frac{1}{2} - \frac{1}{k}\right) R^2 \\
&\leq \lambda \int_\Omega (g(u_n)u_n dx - 2G(u_n))\, dx \\
&\leq \lambda k_1 \left(\|u_n\|_1 + \|u_n\|_q^q\right) + 2\lambda k_1 \left(\|u_n\|_1 + \frac{1}{q}\|u_n\|_q^q\right),
\end{aligned}$$

and so, letting $n \to +\infty$, the conclusion is achieved. □

We now show that, when λ approaches zero, every non-zero solution to (P_λ) is actually positive. To this aim, for all $\lambda > 0$, set

$$S_\lambda := \{u \in H_0^1(\Omega) \setminus \{0\} : u \text{ is a solution to } (P_\lambda)\},$$

and denote by $\mathcal{P}$ the interior of the positive cone of $C_0^1(\overline{\Omega})$, i.e.

$$\mathcal{P} := \left\{u \in C_0^1(\overline{\Omega}) : u > 0 \text{ in } \Omega,\ \frac{\partial u}{\partial \nu} < 0 \text{ on } \partial\Omega\right\},$$

ν being the unit outer normal to $\partial\Omega$. Our second result reads as follows:

Theorem 2.4. *There exists $\Lambda^* > 0$ such that for each $\lambda \in (0, \Lambda^*)$, $S_\lambda \subset \mathcal{P}$.*

Proof. We first observe that, by the regularity theory of elliptic equations, for all $\lambda > 0$ and $u_\lambda \in S_\lambda$, one has $u_\lambda \in C^{1,\alpha}(\overline{\Omega})$, for some $\alpha \in (0,1)$.

If $u_\lambda \in S_\lambda$, it is straightforward to check that $v_\lambda := \lambda^{-1} u_\lambda$ is a solution to the problem

$$\begin{cases} -\Delta u = \lambda_1 u + g(\lambda u) & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\tilde{P}_\lambda$}$$

clearly equivalent to (P_λ). Note that (g_2) ensures the existence of some $a > 0$ such that $g(t) < 0$ for all $t \in (0, a)$, and moreover it must hold

$$\|v_\lambda\|_\infty \geq \frac{a}{\lambda}, \tag{2.9}$$

otherwise we would get $g(u_\lambda) < 0$ in $\Omega \setminus u_\lambda^{-1}(0)$, and so

$$\|u_\lambda\|^2 - \lambda_1 \|u_\lambda\|_2^2 = \lambda \int_\Omega g(u_\lambda) u_\lambda dx < 0,$$

against the definition of λ_1. From now on, we will then focus on ($\tilde{P}_\lambda$). We split the proof in several steps.

Step 1. We show that there exist two constants $C^*, \Lambda_0 > 0$ such that, for any $\lambda \in (0, \Lambda_0]$ and for any $v_\lambda \in S_\lambda$,

$$\|v_\lambda\| \geq \frac{C^*}{\lambda}. \tag{2.10}$$

Fix $\beta > \max\{\frac{N}{2}, \frac{1}{q-1}\}$. By [1, Theorem 8.2] and the embedding $W^{2,\beta}(\Omega) \hookrightarrow C^1(\overline{\Omega})$, one has $v_\lambda \in W^{2,\beta}(\Omega)$ and there exists a constant $C_0 > 0$, independent of λ, such that

$$\|v_\lambda\|_{C^1(\overline{\Omega})} \leq C_0 \left((\lambda_1 + 1)\|v_\lambda\|_\beta + \|g(\lambda v_\lambda)\|_\beta\right). \tag{2.11}$$

So, by (g_1) and Hölder's inequality, we get

$$\begin{aligned} \int_\Omega |g(\lambda v_\lambda)|^\beta dx &\leq k_1^\beta \int_\Omega \left(1 + (\lambda v_\lambda)^{q-1}\right)^\beta dx \\ &\leq 2^{\beta-1} k_1^\beta \left(|\Omega| + \lambda^{\beta(q-1)} \|v_\lambda\|_\infty^{\beta(q-1)-1} \|v_\lambda\|_1\right), \end{aligned}$$

and therefore

$$\|v_\lambda\|_\infty \leq C_0 \left((\lambda_1 + 1) \|v_\lambda\|_\infty^{\frac{\beta-1}{\beta}} \|v_\lambda\|_1^{\frac{1}{\beta}} \right.$$
$$\left. + 2^{\frac{\beta-1}{\beta}} k_1 \left(|\Omega|^{\frac{1}{\beta}} + \lambda^{q-1} \|v_\lambda\|_\infty^{q-1-\frac{1}{\beta}} \|v_\lambda\|_1^{\frac{1}{\beta}} \right) \right).$$

Now, dividing by $\|v_\lambda\|_\infty^{\frac{\beta-1}{\beta}}$ both sides of the previous inequality and taking (2.9) into account, we obtain,

$$\begin{aligned} \left(\frac{a}{\lambda}\right)^{\frac{1}{\beta}} \leq \|v_\lambda\|_\infty^{\frac{1}{\beta}} &\leq C_1 \left(\|v_\lambda\|_1^{\frac{1}{\beta}} + \|v_\lambda\|_\infty^{\frac{1-\beta}{\beta}} + \lambda^{q-1} \|v_\lambda\|_\infty^{q-2} \|v_\lambda\|_1^{\frac{1}{\beta}} \right) \\ &\leq C_1 \left(\|v_\lambda\|_1^{\frac{1}{\beta}} + a^{\frac{1-\beta}{\beta}} \lambda^{\frac{\beta-1}{\beta}} + a^{q-2}\lambda \|v_\lambda\|_1^{\frac{1}{\beta}} \right) \\ &\leq C_2 \left((1+\lambda) \|v_\lambda\|^{\frac{1}{\beta}} + \lambda^{\frac{\beta-1}{\beta}} \right). \end{aligned} \tag{2.12}$$

Now, if $0 < \lambda \leq \min\{1, a(2C_2)^{-\beta}\} := \Lambda_0$, one has

$$\|v_\lambda\|^{\frac{1}{\beta}} \geq \frac{1}{2C_2} \left(\frac{a}{\lambda}\right)^{\frac{1}{\beta}} - \frac{1}{2} \geq \frac{1}{4C_2} \left(\frac{a}{\lambda}\right)^{\frac{1}{\beta}}$$

and hence (2.10) is fulfilled with $C^* = a(4C_2)^{-\beta}$. Since of course $\|v_\lambda\| \to +\infty$ as $\lambda \to 0^+$, by (2.12) we can determine $C_3 > 0$ and $\Lambda_1 \in (0, \Lambda_0]$ such that $\|v_\lambda\| \geq 1$ and

$$\|v_\lambda\|_\infty \leq C_3 \|v_\lambda\| \tag{2.13}$$

for any $\lambda \in (0, \Lambda_1]$. For the rest of the proof, we assume $\lambda \in (0, \Lambda_1]$.

Step 2. We now show that, writing v_λ as

$$v_\lambda = t_\lambda \phi_1 + w_\lambda,$$

with $t_\lambda \in \mathbb{R}$ and $w_\lambda \in \text{span}\{\phi_1\}^\perp$, then it holds

$$\|w_\lambda\|_{C^1(\overline{\Omega})} \leq \tilde{C} \|v_\lambda\|^{\frac{q}{2}}, \tag{2.14}$$

for some $\tilde{C} > 0$. By the same arguments as [3], it is easily seen that $t_\lambda > 0$ and that w_λ is a weak solution to

$$\begin{cases} -\Delta u = \lambda_1 u + g(\lambda v_\lambda) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.15}$$

The relation $I'_\lambda(v_\lambda)(\phi_1) = 0$ and the definition of ϕ_1 imply that

$$\int_\Omega \nabla v_\lambda \nabla \phi_1 dx - \lambda_1 \int_\Omega v_\lambda \phi_1 dx - \int_\Omega g(\lambda v_\lambda)\phi_1 dx = -\int_\Omega g(\lambda v_\lambda)\phi_1 dx = 0,$$

and therefore

$$\int_\Omega g(\lambda v_\lambda) w_\lambda dx = \int_\Omega g(\lambda v_\lambda)(v_\lambda - t_\lambda \phi_1) dx = \int_\Omega g(\lambda v_\lambda) v_\lambda dx.$$

So, we get

$$\begin{aligned}\|w_\lambda\|^2 &= \lambda_1 \|w_\lambda\|_2^2 + \int_\Omega g(\lambda v_\lambda) w_\lambda dx \\ &\le \frac{\lambda_1}{\lambda_2}\|w_\lambda\|^2 + \int_\Omega g(\lambda v_\lambda) v_\lambda dx \\ &\le \frac{\lambda_1}{\lambda_2}\|w_\lambda\|^2 + k_1\left(\|v_\lambda\|_1 + \lambda^{q-1}\|v_\lambda\|_q^q\right) \\ &\le \frac{\lambda_1}{\lambda_2}\|w_\lambda\|^2 + C_4\|v_\lambda\|^q,\end{aligned}$$

from which we deduce the estimate

$$\|w_\lambda\|^2 \le C_5 \|v_\lambda\|^q, \tag{2.16}$$

being $C_5 = \frac{\lambda_2 C_4}{\lambda_2 - \lambda_1}$. By applying the same arguments as before to the function w_λ and bearing in mind also (2.13) and (2.16), we obtain

$$\begin{aligned}\|w_\lambda\|_{C^1(\overline{\Omega})} &\le C_6\left((\lambda_1+1)\|w_\lambda\|_\beta + \|g(\lambda v_\lambda)\|_\beta\right) \\ &\le C_6\left((\lambda_1+1)\|w_\lambda\|_\infty^{\frac{\beta-1}{\beta}}\|w_\lambda\|_1^{\frac{1}{\beta}} + 2^{\frac{\beta-1}{\beta}}k_1\left(|\Omega|^{\frac{1}{\beta}} + \lambda^{q-1}\|v_\lambda\|_\infty^{q-1-\frac{1}{\beta}}\|v_\lambda\|_1^{\frac{1}{\beta}}\right)\right) \\ &\le C_7\left(\|w_\lambda\|_{C^1(\overline{\Omega})}^{\frac{\beta-1}{\beta}}\|v_\lambda\|^{\frac{q}{2\beta}} + 1 + \lambda^{q-1}\|v_\lambda\|^{q-1}\right) \\ &\le C_7\left(\|w_\lambda\|_{C^1(\overline{\Omega})}^{\frac{\beta-1}{\beta}}\|v_\lambda\|^{\frac{q}{2\beta}} + 2\|v_\lambda\|^{q-1}\right).\end{aligned}$$

So, either

$$\|w_\lambda\|_{C^1(\overline{\Omega})} \le 2C_7\|w_\lambda\|_{C^1(\overline{\Omega})}^{\frac{\beta-1}{\beta}}\|v_\lambda\|^{\frac{q}{2\beta}}$$

or

$$\|w_\lambda\|_{C^1(\overline{\Omega})} \le 4C_7\|v_\lambda\|^{q-1}.$$

In any case, we get

$$\|w_\lambda\|_{C^1(\overline{\Omega})} \le \tilde{C}\|v_\lambda\|^{\frac{q}{2}}, \tag{2.17}$$

where $\tilde{C} = 4C_7$, as desired.

Step 3 (conclusion). Taking (2.10) and (2.16) into account, for $0 < \lambda \le \min\{1, \Lambda_0, \Lambda_1, \Lambda_2\}$, where $\Lambda_2 := \left(\frac{1}{2C_5}\right)^{\frac{1}{2-q}} C^*$, we obtain

$$t_\lambda^2 \ge \frac{\|v_\lambda\|^2 - C_5\|v_\lambda\|^q}{\|\phi_1\|^2} \ge \frac{\|v_\lambda\|^2}{\|\phi_1\|^2}\left(1 - \frac{C_5 C^{*q-2}}{\lambda^{q-2}}\right) \ge \frac{\|v_\lambda\|^2}{2\|\phi_1\|^2} = C_8\|v_\lambda\|^2, \tag{2.18}$$

where $C_8 = \frac{1}{2\|\phi_1\|^2}$. For this range of λ, in view of (2.17), we then obtain

$$\left\|t_\lambda^{-1}v_\lambda - \phi_1\right\|_{C^1(\overline{\Omega})} = t_\lambda^{-1}\|w_\lambda\|_{C^1(\overline{\Omega})} \le \tilde{C}C_8^{-\frac{1}{2}}\|v_\lambda\|^{\frac{q}{2}-1} \le C_9\lambda^{1-\frac{q}{2}}$$

with $C_9 = \tilde{C}C_8^{-\frac{1}{2}}C^{*\frac{q}{2}-1}$. Since $\phi_1 \in \mathcal{P}$ and $\mathcal{P}$ is an open subset of $C^1(\overline{\Omega})$, there exists $\delta > 0$ such that

$$\{u \in C^1(\overline{\Omega}) : \|u - \phi_1\|_{C^1(\overline{\Omega})} < \delta\} \subset \mathcal{P}.$$

So, setting $\Lambda_3 := \left(\frac{\delta}{C_9}\right)^{\frac{2}{2-q}}$, for all $0 < \lambda \le \min\{1, \Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3\} := \Lambda^*$, one has $t_\lambda^{-1}v_\lambda \in \mathcal{P}$ and hence $v_\lambda \in \mathcal{P}$. This concludes the proof. □

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