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Positive solutions of a Kirchhoff–Schrödinger–Newton system with critical nonlocal term

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Abstract. This paper deals with the following Kirchhoff–Schrödinger–Newton system with critical growth

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = \phi|u|^{2^*-3}u + \lambda|u|^{p-2}u, & \text{in } \Omega,\\ -\Delta\phi = |u|^{2^*-1}, & \text{in } \Omega,\\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a smooth bounded domain, $M(t) = 1 + bt^{\theta-1}$ with $t > 0$, $1 < \theta < \frac{N+2}{N-2}$, $b > 0$, $1 < p < 2$, $\lambda > 0$ is a parameter, $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent. By using the variational method and the Brézis–Lieb lemma, the existence and multiplicity of positive solutions are established.

Keywords: Kirchhoff–Schrödinger–Newton, positive solutions, critical growth.

2020 Mathematics Subject Classification: 35J20, 35J60, 35B09.

1 Introduction and main result

Consider the following Kirchhoff–Schrödinger–Newton system involving critical growth

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = \phi|u|^{2^*-3}u + \lambda|u|^{p-2}u, & \text{in } \Omega,\\ -\Delta\phi = |u|^{2^*-1}, & \text{in } \Omega,\\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a smooth bounded domain, $M(t) = 1 + bt^{\theta-1}$ with $t > 0$, $1 < \theta < \frac{N+2}{N-2}$, $b > 0$, $1 < p < 2$, $\lambda > 0$ is a parameter, $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent.

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This system is derived from the Schrödinger–Poisson system

$$\begin{cases} -\Delta u + V(x)u + \eta\phi f(u) = h(x,u), & \text{in } \mathbb{R}^3, \\ -\Delta\phi = 2F(u), & \text{in } \mathbb{R}^3. \end{cases} \tag{1.2}$$

System as (1.2) has been studied extensively by many researchers because (1.2) has a strong physical meaning, which describes quantum particles interacting with the electromagnetic field generated by the motion. The Schrödinger–Poisson system (also called Schrödinger–Maxwell system) was first introduced by Benci and Fortunato in [6] as a physical model describing a charged wave interacting with its own electrostatic field in quantum mechanic. For more information on the physical aspects about (1.2), we refer the reader to [6,7].

Many recent studies of (1.2) have focused on existence of multiple solutions, ground states, positive and non-radial solutions. When $h(x,u) = |u|^{p-2}u$, Alves et al. in [4] considered the existence of ground state solutions for (1.2) with $4 < p < 6$. In [10], Cerami and Vaira proved the existence of positive solutions of (1.2) when $h(x,u) = a(x)|u|^{p-2}u$ with $4 < p < 6$ and $a(x)$ is a nonnegative function. The same result was established in [11,18,22,23] for $2 < p < 6$. In [20,25,26,28], by using variational methods, the authors proved the existence of ground state solutions of (1.2) with subcritical and critical growths. In addition, the existence of solutions for Schrödinger–Poisson system involving critical nonlocal term has been paid much attention by many authors, we can see [2,13,16,19,24,27] and so on.

In [5], Arora et al. considered a nonlocal Kirchhoff type equation with a critical Sobolev nonlinearity, using suitable variational techniques, the authors showed how to overcome the lack of compactness at critical levels. In [15], by using the variational method and the concentration compactness principle, Lei and Suo established the existence and multiplicity of nontrivial solutions. Luyen and Cuong [21] obtained the existence of multiple solutions for a given boundary value problem, using the minimax method and Rabinowitz's perturbation method. In [29], Zhou, Guo and Zhang combined the variational method and the mountain pass theorem, to get the existence of weak solutions, this time on the Heisenberg group.

Specially, Azzollini, D'Avenia and Vaira [3] studied the following Schrödinger–Newton type system with critical growth

$$\begin{cases} -\Delta u = \lambda u + |u|^{2^*-3}u\phi, & \text{in } \Omega, \\ -\Delta\phi = |u|^{2^*-1}, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a smooth bounded domain. By the variational method, they obtained the existence and nonexistence results of positive solutions when $N = 3$ and the existence of solutions in both the resonance and the non-resonance case for higher dimensions.

Lei and Gao [14] considered the Schrödinger–Newton system with sign-changing potential

$$\begin{cases} -\Delta u = f_\lambda(x)|u|^{p-2}u + |u|^3u\phi, & \text{in } \Omega, \\ -\Delta\phi = |u|^5, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain, $1 < p < 2$, $f_\lambda = \lambda f^+ + f^-$, $\lambda > 0$, $f\pm = \max\{\pm f, 0\}$. By using the variational method and analytic techniques, the authors proved the existence and multiplicity of positive solutions.

In [17], Li et al. proved the existence, nonexistence and multiplicity of positive radially symmetric solutions for the following Schrödinger–Poisson system

$$\begin{cases} -\Delta u + u + \lambda\phi|u|^3u = \mu|u|^{p-2}u, & \text{in } \mathbb{R}^3, \\ -\Delta\phi = |u|^5, & \text{in } \mathbb{R}^3, \end{cases}$$

where $p \in (2, 6)$, $\lambda \in \mathbb{R}$ and $\mu \geq 0$ are parameters.

With the help of the Lax–Milgram theorem, for every $u \in H_0^1(\Omega)$, the second equation of system (1.1) has a unique solution $\phi_u \in H_0^1(\Omega)$, we substitute ϕ_u to the first equation of system (1.1), then system (1.1) transforms into the following equation

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = \phi_u|u|^{2^*-3}u + \lambda|u|^{p-2}u, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

The variational functional associated with (1.3) is defined by

$$I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{b}{2\theta}\left(\int_\Omega |\nabla u|^2 dx\right)^\theta - \frac{1}{2(2^*-1)}\int_\Omega \phi_u|u|^{2^*-1}dx - \frac{\lambda}{p}\int_\Omega |u|^p dx.$$

We say that $u \in H_0^1(\Omega)$ is a weak solution of (1.3), for all $\psi \in H_0^1(\Omega)$, then u satisfies

$$\left[1 + b\left(\int_\Omega |\nabla u|^2 dx\right)^{\theta-1}\right]\int_\Omega \nabla u \nabla\psi dx = \int_\Omega \phi_u|u|^{2^*-3}u\psi dx + \lambda\int_\Omega |u|^{p-2}u\psi dx.$$

Our technique based on the Ekeland variational principle and the mountain pass theorem. Since system (1.1) contains a nonlocal critical growth term, which leads to the cause of the lack of compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ and the Palais–Smale condition for the corresponding energy functional could not be checked directly. Then we overcome the compactness by using the Brézis–Lieb lemma.

Now we state our main result.

Theorem 1.1. *Assume that* $1 < \theta < \frac{N+2}{N-2}$, $\frac{N}{N-2} < p < 2$ *and* $N > 4$, $b > 0$ *is small enough. Then there exists* $\Lambda_* > 0$ *such that for all* $\lambda \in (0, \Lambda_*)$, *system* (1.1) *has at least two positive solutions.*

Throughout this paper, we make use of the following notations:

- The space $H_0^1(\Omega)$ is equipped with the norm $\|u\|^2_{H_0^1(\Omega)} = \int_\Omega |\nabla u|^2 dx$, the norm in $L^p(\Omega)$ is denoted by $\|\cdot\|_p$.
- Let $D^{1,2}(\mathbb{R}^N)$ be the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|u\|^2_{D^{1,2}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\nabla u|^2 dx$.
- $C, C_1, C_2, \ldots$ denote various positive constants, which may vary from line to line.
- We denote by S_ρ (respectively, B_ρ) the sphere (respectively, the closed ball) of center zero and radius ρ, i.e. $S_\rho = \{u \in H_0^1(\Omega) : \|u\| = \rho\}$, $B_\rho = \{u \in H_0^1(\Omega) : \|u\| \leq \rho\}$.
- Let S be the best constant for Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$, namely

$$S = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega |u|^{2^*} dx\right)^{2/2^*}}.$$

2 Proof of the theorem

Firstly, we have the following important lemma in [3].

Lemma 2.1. *For all $u \in H_0^1(\Omega)$, there exists a unique solution $\phi_u \in H_0^1(\Omega)$ of*

$$\begin{cases} -\Delta\phi = |u|^{2^*-1}, & \text{in } \Omega, \\ \phi = 0, & \text{on } \partial\Omega. \end{cases}$$

Moreover,

(1) $\phi_u \geq 0$ *for* $x \in \Omega$ *and for each* $t > 0$, $\phi_{tu} = t^{2^*-1}\phi_u$.

(2) $$\int_\Omega |\nabla\phi_u|^2 dx = \int_\Omega \phi_u |u|^{2^*-1} dx \leq S^{-2^*}\|u\|^{2(2^*-1)}.$$

(3) *If $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, then*

$$\int_\Omega \phi_{u_n}|u_n|^{2^*-1}dx - \int_\Omega \phi_{u_n-u}|u_n-u|^{2^*-1}dx = \int_\Omega \phi_u|u|^{2^*-1}dx + o_n(1).$$

Lemma 2.2. *There exist constants $\delta, \rho, \Lambda_0 > 0$, for all $\lambda \in (0, \Lambda_0)$ such that the functional I_λ satisfies the following conditions:*

(i) $I_\lambda|_{u\in S_\rho} \geq \delta > 0$; $\inf_{u\in B_\rho} I_\lambda(u) < 0$.

(ii) *There exists $e \in H_0^1(\Omega)$ with $\|e\| > \rho$ such that $I_\lambda(e) < 0$.*

Proof. (i) Using the Hölder inequality and the Sobolev inequality, we get

$$\int_\Omega |u|^p dx \leq \left(\int_\Omega |u|^{2^*}dx\right)^{\frac{p}{2^*}} \left(\int_\Omega 1^{\frac{2^*}{2^*-p}}dx\right)^{\frac{2^*-p}{2^*}} \leq |\Omega|^{\frac{2^*-p}{2^*}} S^{-\frac{p}{2}}\|u\|^p. \tag{2.1}$$

Therefore, it follows from (2.1) and the Sobolev inequality that

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{b}{2\theta}\left(\int_\Omega |\nabla u|^2 dx\right)^\theta - \frac{1}{2(2^*-1)}\int_\Omega \phi_u|u|^{2^*-1}dx - \frac{\lambda}{p}\int_\Omega |u|^p dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{2(2^*-1)}S^{-2^*}\|u\|^{2(2^*-1)} - \frac{\lambda}{p}|\Omega|^{\frac{2^*-p}{2^*}}S^{-\frac{p}{2}}\|u\|^p \\ &= \|u\|^p\left(\frac{1}{2}\|u\|^{2-p} - \frac{1}{2(2^*-1)}S^{-2^*}\|u\|^{2(2^*-1)-p} - \frac{\lambda}{p}|\Omega|^{\frac{2^*-p}{2^*}}S^{-\frac{p}{2}}\right). \end{aligned}$$

Let $H(t) = \frac{1}{2}t^{2-p} - \frac{1}{2(2^*-1)}S^{-2^*}t^{2(2^*-1)-p}$ for $t > 0$, thus, there exists a constant

$$\rho = \left[\frac{(2^*-1)(2-p)S^{2^*}}{(2(2^*-1)-p)}\right]^{\frac{1}{2(2^*-2)}} > 0$$

such that $\max_{t>0} h(t) = h(\rho) > 0$. Setting $\Lambda_0 = \frac{pS^{\frac{p}{2}}}{|\Omega|^{\frac{2^*-p}{2^*}}}h(\rho)$, there exists a constant $\delta > 0$ such that $I_\lambda|_{u\in S_\rho} \geq \delta$ for each $\lambda \in (0, \Lambda_0)$. Moreover, for every $u \in H_0^1(\Omega)\backslash\{0\}$, we get

$$\lim_{t\to 0^+}\frac{I_\lambda(tu)}{t^p} = -\frac{\lambda}{p}\int_\Omega |u|^p dx < 0.$$

So we obtain $I_\lambda(tu) < 0$ for all $u \neq 0$ and tu small enough. Hence, for $\|u\|$ small enough, we have

$$m \triangleq \inf_{u \in B_\rho} I_\lambda(u) < 0.$$

(ii) Set $u \in H_0^1(\Omega)$, for all $t > 0$, we get

$$I_\lambda(tu) = \frac{t^2}{2}\|u\|^2 + \frac{bt^{2\theta}}{2\theta}\|u\|^{2\theta} - \frac{t^{2(2^*-1)}}{2(2^*-1)}\int_\Omega \phi_u |u|^{2^*-1}dx - \frac{\lambda t^p}{p}\int_\Omega |u|^p dx \to -\infty$$

as $t \to \infty$, which implies that $I_\lambda(tu) < 0$ for $t > 0$ large enough. Consequently, we can find $e \in H_0^1(\Omega)$ with $\|e\| > \rho$ such that $I_\lambda(e) < 0$. The proof is complete. □

Definition 2.3. A sequence $\{u_n\} \subset H_0^1(\Omega)$ is called $(PS)_c$ sequence of I_λ if $I_\lambda(u_n) \to c$ and $I'_\lambda(u_n) \to 0$ as $n \to \infty$. We say that I_λ satisfies $(PS)_c$ condition if every $(PS)_c$ sequence of I_λ has a convergent subsequence in $H_0^1(\Omega)$.

Lemma 2.4. *Assume that $1 < \theta < \frac{N+2}{N-2}$ and $1 < p < 2$, the functional I_λ satisfies the $(PS)_c$ condition for each $c < c_* = \frac{2}{N+2}S^{\frac{N}{2}} - D\lambda^{\frac{2}{2-p}}$, where $D = \frac{[2(2^*-1)-p]^{\frac{2}{2-p}}}{2(2^*-1)(2^*-2)^{\frac{p}{2-p}}p^{\frac{2}{2-p}}}(S^{-\frac{p}{2}}|\Omega|^{\frac{2^*-p}{2^*}})^{\frac{2}{2-p}}$.*

Proof. Let $\{u_n\} \subset H_0^1(\Omega)$ be a (PS) sequence for I_λ at the level c, that is

$$I_\lambda(u_n) \to c \quad \text{and} \quad I'_\lambda(u_n) \to 0 \quad \text{as } n \to \infty. \tag{2.2}$$

Combining with (2.1) and (2.2), we have

$$\begin{aligned}
c + 1 + o(\|u_n\|) &\geq I_\lambda(u_n) - \frac{1}{2(2^*-1)}\langle I'_\lambda(u_n), u_n\rangle \\
&= \left(\frac{1}{2} - \frac{1}{2(2^*-1)}\right)\|u_n\|^2 + b\left(\frac{1}{2\theta} - \frac{1}{2(2^*-1)}\right)\|u_n\|^{2\theta} \\
&\quad - \lambda\left(\frac{1}{p} - \frac{1}{2(2^*-1)}\right)|\Omega|^{\frac{2^*-p}{2^*}}S^{-\frac{p}{2}}\|u_n\|^p \\
&\geq \left(\frac{1}{2} - \frac{1}{2(2^*-1)}\right)\|u_n\|^2 - \lambda\left(\frac{1}{p} - \frac{1}{2(2^*-1)}\right)|\Omega|^{\frac{2^*-p}{2^*}}S^{-\frac{p}{2}}\|u_n\|^p.
\end{aligned}$$

Therefore $\{u_n\}$ is bounded in $H_0^1(\Omega)$ for all $1 < p < 2$. Thus, we may assume up to a subsequence, still denoted by $\{u_n\}$, that there exists $u \in H_0^1(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u, & \text{weakly in } H_0^1(\Omega), \\ u_n \to u, & \text{strongly in } L^q(\Omega)\ (1 \leq q < 2^*), \\ u_n(x) \to u(x), & \text{a.e. in } \Omega, \end{cases} \tag{2.3}$$

as $n \to \infty$. By (2.1) and the Young inequality, one has

$$\lambda\int_\Omega |u|^p dx \leq \lambda S^{-\frac{p}{2}}|\Omega|^{\frac{2^*-p}{2^*}}\|u\|^p \leq \eta\|u\|^2 + C(\eta)\lambda^{\frac{2}{2-p}}, \tag{2.4}$$

where $C(\eta) = \eta^{-\frac{p}{2-p}}(S^{-\frac{p}{2}}|\Omega|^{\frac{2^*-p}{2^*}})^{\frac{2}{2-p}}$, it follows from (2.2) and (2.4) that

$$\begin{aligned}
I_\lambda(u) &= I_\lambda(u) - \frac{1}{2(2^*-1)}\langle I'_\lambda(u), u\rangle \\
&\geq \left(\frac{1}{2} - \frac{1}{2(2^*-1)}\right)\|u\|^2 - \left(\frac{1}{p} - \frac{1}{2(2^*-1)}\right)\lambda\int_\Omega |u|^p dx \\
&\geq \left(\frac{2^*-2}{2(2^*-1)} - \frac{2(2^*-1)-p}{2(2^*-1)p}\eta\right)\|u\|^2 - \frac{2(2^*-1)-p}{2(2^*-1)p}C(\eta)\lambda^{\frac{2}{2-p}}.
\end{aligned}$$

Letting $\eta = \frac{p(2^*-2)}{2(2^*-1)-p}$ and $D = \frac{[2(2^*-1)-p]^{\frac{2}{2-p}}}{2(2^*-1)(2^*-2)^{\frac{p}{2-p}}p^{\frac{2}{2-p}}}(S^{-\frac{p}{2}}|\Omega|^{\frac{2^*-p}{2^*}})^{\frac{2}{2-p}}$, we have $I_\lambda(u) \geq -D\lambda^{\frac{2}{2-p}}$. Next, we prove that $u_n \to u$ strongly in $H_0^1(\Omega)$. Set $w_n = u_n - u$ and $\lim_{n\to\infty}\|w_n\| = l$, by using the Brézis–Lieb lemma [9], we have

$$\begin{aligned}\|u_n\|^2 &= \|w_n\|^2 + \|u\|^2 + o(1),\\ \|u_n\|^{2\theta} &= \left(\|w_n\|^2 + \|u\|^2 + o(1)\right)^\theta,\\ \int_\Omega \phi_{u_n}|u_n|^{2^*-1}dx &= \int_\Omega \phi_{w_n}|w_n|^{2^*-1}dx + \int_\Omega \phi_u|u|^{2^*-1}dx + o(1).\end{aligned}$$

From (2.2), (2.3) and Lemma 2.1, one has

$$\begin{aligned}&\|w_n\|^2 + \|u\|^2 + b\left(\|w_n\|^2 + \|u\|^2 + o(1)\right)^\theta\\ &\qquad - \int_\Omega \phi_{w_n}|w_n|^{2^*-1}dx - \int_\Omega \phi_u|u|^{2^*-1}dx - \lambda\int_\Omega |u|^p dx = o(1), \end{aligned} \tag{2.5}$$

and

$$\|u\|^2 + b\|u\|^{2\theta} - \int_\Omega \phi_u|u|^{2^*-1}dx - \lambda\int_\Omega |u|^p dx = 0. \tag{2.6}$$

It follows from (2.5) and (2.6) that

$$\|w_n\|^2 + b\left[\left(\|w_n\|^2 + \|u\|^2 + o(1)\right)^\theta - \|u\|^{2\theta}\right] - \int_\Omega \phi_{w_n}|w_n|^{2^*-1}dx = o(1). \tag{2.7}$$

Since $\|w_n\| \to l$, we have

$$\left(\|w_n\|^2 + \|u\|^2 + o(1)\right)^\theta - \|u\|^{2\theta} \to \left(l^2 + \|u\|^2 + o(1)\right)^\theta - \|u\|^{2\theta} = l_1 \geq 0, \quad \text{as } n \to \infty.$$

If follows from (2.7) that

$$\int_\Omega \phi_{w_n}|w_n|^{2^*-1}dx \to l^2 + bl_1.$$

Applying the Sobolev inequality, we get

$$\|w_n\|^{2(2^*-1)} \geq S^{2^*}\int_\Omega \phi_{w_n}|w_n|^{2^*-1}dx + o(1). \tag{2.8}$$

Thus, by (2.8), we can deduce that

$$l^{2(2^*-1)} \geq S^{2^*}(l^2 + bl_1) \geq S^{2^*}l^2 \quad \text{as } n \to \infty,$$

which implies that $l \geq S^{\frac{N}{4}}$ as $n \to \infty$. Since $I(u_n) = c + o(1)$, we obtain

$$\frac{1}{2}\|w_n\|^2 + \frac{b}{2\theta}[(\|w_n\|^2 + \|u\|^2 + o(1))^\theta - \|u\|^{2\theta}] - \frac{1}{2(2^*-1)}\int_\Omega \phi_{w_n}|w_n|^{2^*-1}dx = c - I_\lambda(u) + o(1).$$

Hence, there holds

$$\begin{aligned}c &= \left(\frac{1}{2} - \frac{1}{2(2^*-1)}\right)l^2 + \left(\frac{1}{2\theta} - \frac{1}{2(2^*-1)}\right)bl_1 + I_\lambda(u)\\ &\geq \frac{2}{N+2}S^{\frac{N}{2}} - D\lambda^{\frac{2}{2-p}} \geq c_*,\end{aligned}$$

as $n \to \infty$. This is a contradiction. Hence, we can conclude that $u_n \to u$ in $H_0^1(\Omega)$. The proof is complete. □

Choose the extremal function

$$U_\varepsilon(x) = \frac{[N(N-2)\varepsilon^2]^{\frac{N-2}{4}}}{(\varepsilon^2+|x|^2)^{\frac{N-2}{2}}}, \qquad x \in \mathbb{R}^N,\ \varepsilon > 0.$$

It is a positive solution of the following problem

$$-\Delta U_\varepsilon = U_\varepsilon^{2^*-1} \quad \text{in } \mathbb{R}^N,$$

and satisfies

$$\int_{\mathbb{R}^N} |\nabla U_\varepsilon|^2 dx = \int_{\mathbb{R}^N} |U_\varepsilon|^{2^*} dx = S^{\frac{N}{2}}.$$

Pick a cut-off function $\varphi \in C_0^\infty(\Omega)$ such that $\varphi(x) = 1$ on $B(0, \frac{r}{2})$, $\varphi(x) = 0$ on $\mathbb{R}^N - B(0,r)$ and $0 \le \varphi(x) \le 1$ on $\mathbb{R}^N$. Set $u_\varepsilon(x) = \varphi(x)U_\varepsilon(x)$, from [8], we have

$$\begin{cases} \displaystyle\int_\Omega |\nabla u_\varepsilon|^2 dx = S^{\frac{N}{2}} + O(\varepsilon^{N-2}), \\ \displaystyle\int_\Omega |u_\varepsilon|^{2^*} dx = S^{\frac{N}{2}} + O(\varepsilon^N). \end{cases} \tag{2.9}$$

To estimate the value c observe that, multiplying the second equation of system (1.1) by $|u|$ and integrating, we get

$$\int_\Omega |u|^{2^*} dx = \int_\Omega \nabla\phi_u \nabla|u| dx \le \frac{1}{2}\|\phi_u\|^2 + \frac{1}{2}\|u\|^2. \tag{2.10}$$

Then, we define a new functional $H_\lambda : H_0^1(\Omega) \to \mathbb{R}$ by

$$\begin{aligned} H_\lambda(u) &\triangleq \frac{2^*}{2(2^*-1)}\|u\|^2 + \frac{b}{2\theta}\|u\|^{2\theta} - \frac{1}{2^*-1}\int_\Omega |u|^{2^*}dx - \frac{\lambda}{p}\int_\Omega |u|^p dx \\ &= \frac{2^*}{2^*-1}\left[\frac{1}{2}\|u\|^2 + \frac{(2^*-1)b}{2\theta 2^*}\|u\|^{2\theta} - \frac{1}{2^*}\int_\Omega |u|^{2^*}dx - \lambda\frac{2^*-1}{2^*p}\int_\Omega |u|^p dx\right] \\ &\triangleq \frac{2^*}{2^*-1}J_\lambda(u), \end{aligned}$$

where

$$J_\lambda(u) = \frac{1}{2}\|u\|^2 + \frac{(2^*-1)b}{2\theta 2^*}\|u\|^{2\theta} - \frac{1}{2^*}\int_\Omega |u|^{2^*}dx - \lambda\frac{2^*-1}{2^*p}\int_\Omega |u|^p dx.$$

By (2.10), which implies that

$$I_\lambda(u) \le H_\lambda(u) = \frac{2^*}{2^*-1}J_\lambda(u), \tag{2.11}$$

for every $u \in H_0^1(\Omega)$, and $c \le \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \sup_{t\ge 0} J_\lambda(tu)$. If we consider the following problem

$$\begin{cases} -\left[1 + \dfrac{(2^*-1)b}{2^*}\left(\displaystyle\int_\Omega |\nabla u|^2 dx\right)^{\theta-1}\right]\Delta u = |u|^{2^*-2}u + \lambda\dfrac{2^*-1}{2^*}|u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.12}$$

Then we find that the weak solution of problem (2.12) correspond to the critical points of the functional J_λ. Next, we compute $\sup_{t\ge 0} J_\lambda(tu_\varepsilon) = J_\lambda(t_\varepsilon u_\varepsilon)$.

Lemma 2.5. *Assume that $1 < \theta < \frac{N+2}{N-2}$, $\frac{N}{N-2} < p < 2$ and $N > 4$, then there exist $\Lambda_3, b_0 > 0$ such that for all $\lambda \in (0, \Lambda_3)$ and $b \in (0, b_0)$, it holds*

$$\sup_{t\geq 0} J_\lambda(tu_\varepsilon) < \frac{1}{N}S^{\frac{N}{2}} - \frac{N+2}{2N}D\lambda^{\frac{2}{2-p}}.$$

In particular,

$$\sup_{t\geq 0} I_\lambda(tu_\varepsilon) < \frac{2}{N+2}S^{\frac{N}{2}} - D\lambda^{\frac{2}{2-p}}.$$

Proof. For convenience, we consider the functional $J_b^* : H_0^1(\Omega) \to \mathbb{R}$ as follows

$$J_b^*(u) = \frac{1}{2}\|u\|^2 + \frac{(2^*-1)b}{2\theta 2^*}\|u\|^{2\theta} - \frac{1}{2^*}\int_\Omega |u|^{2^*}dx.$$

Define

$$h_b(t) = J_b^*(tu_\varepsilon) = \frac{t^2}{2}\|u_\varepsilon\|^2 + \frac{(2^*-1)bt^{2\theta}}{2\theta 2^*}\|u_\varepsilon\|^{2\theta} - \frac{t^{2^*}}{2^*}\int_\Omega |u_\varepsilon|^{2^*}dx, \quad \text{for all } t \geq 0.$$

It is clear that $\lim_{t\to 0} h_b(t) = 0$ and $\lim_{t\to\infty} h_b(t) = -\infty$. Therefore there exists $t_{b,\varepsilon} > 0$ such that $h(t_{b,\varepsilon}) = \max_{t\geq 0} h_b(t)$, that is

$$0 = h_0'(t_{0,\varepsilon}) = t_{0,\varepsilon}\left(\|u_\varepsilon\|^2 - t_{0,\varepsilon}^{2^*-2}\int_\Omega |u_\varepsilon|^{2^*}dx\right),$$

one has

$$t_{0,\varepsilon} = \left(\frac{\|u_\varepsilon\|^2}{\int_\Omega |u_\varepsilon|^{2^*}dx}\right)^{\frac{1}{2^*-2}}.$$

Hence, we deduce from (2.9) that

$$\begin{aligned}\sup_{t\geq 0} J_b^*(tu_\varepsilon) &= h_b(t_{b,\varepsilon}u_\varepsilon) \leq h_0(t_{b,\varepsilon}u_\varepsilon) \leq h_0(t_{0,\varepsilon}u_\varepsilon)\\ &= \frac{1}{N}S^{\frac{N}{2}} - \frac{N+2}{2N}D\lambda^{\frac{2}{2-p}} + \frac{N+2}{2N}D\lambda^{\frac{2}{2-p}} + O(\varepsilon^{N-2}).\end{aligned} \tag{2.13}$$

By using the definitions of J and u_ε, we have

$$J_\lambda(tu_\varepsilon) \leq \frac{t^2}{2}\|u_\varepsilon\|^2 + \frac{b(2^*-1)t^{2\theta}}{2\theta 2^*}\|u_\varepsilon\|^{2\theta},$$

for all $t \geq 0$ and $\lambda > 0$. It follows from (2.9) that there exist $T \in (0,1)$, Λ_1, $b_0 > 0$ and $\varepsilon_1 > 0$ such that

$$\sup_{0\leq t\leq T} J_\lambda(tu_\varepsilon) < \frac{1}{N}S^{\frac{N}{2}} - \frac{N+2}{2N}D\lambda^{\frac{2}{2-p}},$$

for every $0 < \lambda < \Lambda_1$, $0 < b < b_0$ and $0 < \varepsilon < \varepsilon_1$. According to the definition of u_ε, there exists $C_1 > 0$, such that we have

$$\begin{aligned}\int_\Omega |u_\varepsilon|^p dx &\geq C \int_{B_{r/2}(0)} \frac{\varepsilon^{\frac{p(N-2)}{2}}}{(\varepsilon^2+|x|^2)^{\frac{p(N-2)}{2}}} dx \\ &= C\varepsilon^{\frac{p(N-2)}{2}} \int_0^{r/2} \frac{t^{N-1}}{(\varepsilon^2+t^2)^{\frac{p(N-2)}{2}}} dt \\ &= C\varepsilon^{N-\frac{p(N-2)}{2}} \int_0^{r/2\sqrt{\varepsilon}} \frac{y^{N-1}}{(1+y^2)^{\frac{p(N-2)}{2}}} dy \\ &\geq C\varepsilon^{N-\frac{p(N-2)}{2}} \int_0^{1} \frac{y^{N-1}}{(1+y^2)^{\frac{p(N-2)}{2}}} dy \\ &\geq C_1\varepsilon^{N-\frac{p(N-2)}{2}}.\end{aligned} \tag{2.14}$$

Thus, it follows from (2.13) and (2.14) that

$$\begin{aligned}\sup_{t\geq T} J(tu_\varepsilon) = \sup_{t\geq T} &\left(J_b(tu_\varepsilon) - \lambda \frac{2^*-1}{2^*p} t^p \int_\Omega |u_\varepsilon|^p dx \right) \\ &\leq \frac{1}{N} S^{\frac{N}{2}} - \frac{N+2}{2N} D\lambda^{\frac{2}{2-p}} \\ &\quad + \frac{N+2}{2N} D\lambda^{\frac{2}{2-p}} + C_2\varepsilon^{N-2} - C_1\lambda\varepsilon^{N-\frac{p(N-2)}{2}} \\ &< \frac{1}{N} S^{\frac{N}{2}} - \frac{N+2}{2N} D\lambda^{\frac{2}{2-p}},\end{aligned} \tag{2.15}$$

where the constant $C_2 > 0$. Here we have used the fact that $\frac{N}{N-2} < p < 2$ and $\frac{(N-2)(2-2p)+2N}{(N-2)(2-p)} < \frac{2}{2-p}$, let $\varepsilon = \lambda^{\frac{2}{(N-2)(2-p)}}$, $0 < \lambda < \Lambda_2 = \min\left\{1, (\frac{C_1}{C_3})^{\frac{(N-2)(2-p)}{2p(N-2)-2N}}\right\}$, then

$$\begin{aligned}\frac{N+2}{2N} D\lambda^{\frac{2}{2-p}} + C_2\varepsilon^{N-2} - C_1\lambda\varepsilon^{N-\frac{p(N-2)}{2}} &\leq C_3\lambda^{\frac{2}{2-p}} - C_1\lambda\varepsilon^{N-\frac{p(N-2)}{2}} \\ &= C_3\lambda^{\frac{2}{2-p}} - C_1\lambda^{\frac{(N-2)(2-2p)+2N}{(N-2)(2-p)}} \\ &< 0,\end{aligned} \tag{2.16}$$

where $C_3 > 0$. Therefore, we have

$$\sup_{t\geq 0} J_\lambda(tu_\varepsilon) < \frac{1}{N} S^{\frac{N}{2}} - \frac{N+2}{2N} D\lambda^{\frac{2}{2-p}},$$

for all $0 < \lambda < \Lambda_3 = \min\{\Lambda_1, \Lambda_2, \varepsilon_1\}$ and $0 < b < b_0$. The proof is complete. □

Theorem 2.6. *Assume that $0 < \lambda < \Lambda_0$ (Λ_0 is as in Lemma 2.2). Then system (1.1) has a positive solution u_λ satisfying $I_\lambda(u_\lambda) < 0$.*

Proof. Applying Lemma 2.2, we have

$$m \triangleq \inf_{u\in \overline{B_\rho(0)}} I_\lambda(u) < 0.$$

By the Ekeland variational principle [12], there exists a minimizing sequence $\{u_n\} \subset \overline{B_\rho(0)}$ such that

$$I_\lambda(u_n) \leq \inf_{u\in \overline{B_\rho(0)}} I_\lambda(u) + \frac{1}{n}, \qquad I_\lambda(v) \geq I_\lambda(u_n) - \frac{1}{n}\|v-u_n\|, \qquad v \in \overline{B_\rho(0)}.$$

Thus, we obtain that $I_\lambda(u_n) \to m$ and $I'_\lambda(u_n) \to 0$. By Lemma 2.4, we have $u_n \to u_\lambda$ in $H_0^1(\Omega)$ with $I_\lambda(u_n) \to m < 0$, which implies that $u_\lambda \not\equiv 0$. Note that $I_\lambda(u_n) = I_\lambda(|u_n|)$, we have $u_\lambda \geq 0$. Then, by using the strong maximum principle, we obtain that u_λ is a positive solution of system (1.1) such that $I_\lambda(u_\lambda) < 0$. □

Theorem 2.7. *Assume that* $0 < \lambda < \Lambda_*(\Lambda_* = \min\{\Lambda_0, \Lambda_3\})$. *Then the system* (1.1) *has a positive solution* $u_* \in H_0^1(\Omega)$ *with* $I_\lambda(u_*) > 0$.

Proof. According to the mountain pass theorem [1] and Lemma 2.2, there exists a sequence $\{u_n\} \subset H_0^1(\Omega)$ such that

$$I_\lambda(u_n) \to c > 0 \quad \text{and} \quad I'_\lambda(u_n) \to 0 \quad \text{as } n \to \infty,$$

where

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda(\gamma(t)),$$

and

$$\Gamma = \left\{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0, \gamma(1) = e\right\}.$$

From Lemma 2.4, we know that $\{u_n\} \subset H_0^1(\Omega)$ has a convergent subsequence, still denoted by $\{u_n\}$, such that $u_n \to u_*$ in $H_0^1(\Omega)$ as $n \to \infty$,

$$I_\lambda(u_*) = \lim_{n\to\infty} I_\lambda(u_n) = c > 0,$$

which implies that $u_* \not\equiv 0$. It is similar to Theorem 2.6 that $u_* > 0$, we obtain that u_* is a positive solution of system (1.1) such that $I_\lambda(u_*) > 0$. Combining the above facts with Theorem 2.6 the proof of Theorem 1.1 is complete. □

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