A Method to Estimate the Degree of C^0-Sufficiency of Analytic Functions

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CONTENTS

In this paper we develop a method for an estimation of the degree of C^0-sufficiency of an analytic function germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. The difference between the estimation we give and the degree of C^0-sufficiency of f will be ≤ 1, by virtue of the results of Chang-Lu and Bochnak-Kucharz.

1. INTRODUCTION

The problem of determining the degree of C^0-sufficiency $s(f)$ of a given analytic function germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is well known in singularity theory. There is some previous work which gives an upper estimate for this number (see Section 2 for its definition). For instance, in the paper [Lichtin 81], the case $n = 2$ is considered. In the paper [Fukui 91], an estimate for the degree of C^0-sufficiency of Newton non-degenerate function germs (in the sense of [Kouchnirenko 76]) is obtained for any n, thus generalizing the cited work of Lichtin.

In this paper, we use some facts from commutative algebra, particularly from multiplicity theory, to give a method providing an estimation for $s(f)$, where $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is any analytic function germ with an isolated singularity at the origin. The number thus obtained differs from $s(f)$ at most by one unit.

As we shall see, the characterization of $s(f)$ using Łojasiewicz type inequalities given by the results of [Chang and Lu 73] and [Bochnak and Kucharz 79] will play a fundamental role in our approach. The link between the algebraic tools we use and the language of Łojasiewicz type inequalities comes from [Lejeune and Teissier 74] characterizing the integral closure of an ideal in the ring $\mathcal{O}_n$ of analytic function germs $(\mathbb{C}^n, 0) \to \mathbb{C}$ (see Theorem 2.4).

2000 AMS Subject Classification: Primary 58A20; Secondary 13H15

Keywords: Integral closure of ideals, Lojasiewicz inequalities, C^0-sufficiency of jets

1058-6458/2001 $0.50 per page

2. C^0-SUFFICIENCY OF JETS AND THE INTEGRAL CLOSURE OF AN IDEAL

Let $x_1, \dots, x_n$ be a coordinate system in $\mathbb{C}^n$ that shall be fixed throughout the text. If $k = (k_1, \dots, k_n) \in \mathbb{N}^n$, where $\mathbb{N}$ is the set of nonnegative integers, then we denote by x^k the monomial $x_1^{k_1} \cdots x_n^{k_n}$. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an analytic function germ and let $f(x) = \sum_k a_k x^k$ be the Taylor expansion of f near the origin. If r is a positive integer then the *r-jet of f* is defined as the polynomial $j^r f(x) = \sum_{|k| \le r} a_k x^k$, where $|k| = k_1 + \cdots + k_n$.

Definition 2.1. We say that the r-jet $j^r f$ is *C^0-sufficient* if, for each analytic map germ $g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ such that $j^r g(x) = j^r f(x)$, there exists a germ of homeomorphism $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $f = g \circ \varphi$. We call *degree of C^0-sufficiency of f*, denoted by $s(f)$, the least $r \in \mathbb{N}$ such that $j^r f$ is C^0-sufficient.

If $f \in \mathcal{O}_n$, let us denote by $J(f)$ the ideal of $\mathcal{O}_n$ generated by the partial derivatives of f and by grad f the map germ $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ given by

$$\text{grad } f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right).$$

The next theorem is a nice characterization of the notion of C^0-sufficiency.

Theorem 2.2. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an analytic map germ, then the r-jet of f is C^0-sufficient if and only if there exist some $C, \delta > 0$ such that*

$$|x|^{r-\delta} \le C \big| (\text{grad } f)(x) \big|,$$

for all x in some open neighbourhood of 0 in $\mathbb{C}^n$.

The *if part* of the above theorem is proved in [Chang and Lu 73] and the *only if part* is proved in [Bochnak and Kucharz 79].

In the hypotheses of Theorem 2.2, we denote by $\alpha_0(f)$ the greatest upper bound of those $\alpha > 0$ such that

$$|x|^{\alpha} \le C \big| (\text{grad } f)(x) \big|,$$

for all x in some open neighbourhood of 0 in $\mathbb{C}^n$ and some constant $C > 0$. This number exists by [Łojasiewicz 76, p. 136] and it is a rational number by [Risler 74]. Therefore, $s(f) = [\alpha_0(f)] + 1$, where $[a]$ denotes the greatest integer $\le a$.

Now, we are going to give the definition of integral closure of an ideal. As we shall see, this concept is related to the type of inequalities shown above.

Definition 2.3. Given a noetherian ring R and an ideal $I \subseteq R$, we say that $h \in R$ is *integral over I* when h satisfies a relation of the form $h^m + a_1 h^{m-1} + \cdots + a_{m-1} h + a_m = 0$, where $m \ge 1$ and $a_i \in I^i$, for all $i = 1, \dots, m$. The set of those elements which are integral over I forms an ideal $\overline{I} \subseteq R$ called *the integral closure of I.*

Obviously, we have $I \subseteq \overline{I}$. When the equality $I = \overline{I}$ holds, the ideal I is said to be *integrally closed.* The integral closure of an ideal can be characterized in analytical terms, as the following theorem shows.

Theorem 2.4. *Let $I \subseteq \mathcal{O}_n$ be an ideal and $h \in \mathcal{O}_n$. Let $g_1, \dots, g_s$ be a system of generators of I. Then $h \in \overline{I}$ if and only if there exists a constant $C > 0$ and an open neighbourhood U of 0 in $\mathbb{C}^n$ such that*

$$|h(x)| \le C \sup\{|g_i(x)| : i = 1, \dots, s\},$$

for all $x \in U$ [Lejeune and Teissier 74, p. 602].

The above result can also be found in [Teissier 81, p. 338]. We recall that an ideal I in a local ring (R, m) is said to be *m-primary*, where m is the maximal ideal of R, when there exists some $\ell \ge 1$ such that $m^\ell \subseteq I$. If m_n denotes the maximal ideal of $\mathcal{O}_n$, then an ideal $I \subseteq \mathcal{O}_n$ is m_n-primary if and only if $V(I) = \{0\}$, where $V(I)$ is the zero set of I. In turn, this is equivalent to saying that $\dim_{\mathbb{C}} \mathcal{O}_n / I < \infty$ ([Eisenbud 94, p. 74]). We shall refer to the number $\dim_{\mathbb{C}} \mathcal{O}_n / I$ as the *codimension of I*. Given an m_n-primary ideal $I \subseteq \mathcal{O}_n$, we set $\alpha(I) = \min\{\ell \ge 1 : m^\ell \subseteq \overline{I}\}$.

If $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is an analytic function germ with an isolated singularity at the origin (which means that $J(f)$ is an m_n-primary ideal), then $\alpha(J(f))$ is the least integer greater than or equal to $\alpha_0(f)$, by Theorem 2.4. Hence, if $\alpha_0(f) \notin \mathbb{N}$, then $s(f) = \alpha(J(f))$ and, if $\alpha_0(f) \in \mathbb{N}$, then $s(f) = \alpha(J(f)) + 1$. Therefore the number $\alpha(J(f)) + 1$ gives an estimate of $s(f)$ differing from $s(f)$ at most by 1. The next section is devoted to computing $\alpha(I)$, for an arbitrary m_n-primary ideal in $\mathcal{O}_n$. This computation can be done, for instance, using the program "Singular" [Greuel et al. 98].

3. THE INTEGRAL CLOSURE AND THE MULTIPLICITY OF AN IDEAL

The integral closure of an ideal is a notion closely related to the concepts of multiplicity and reduction of an ideal. Given two ideals $J \subseteq I$ in a local ring (R, m), we say that J is a *reduction* of I when there exists an integer

$r \geq 1$ such that $I^{r+1} = JI^r$. If I is an m-primary ideal of R, then we denote by $e(I)$ the *multiplicity of* I in the Hilbert-Samuel sense, this number can be seen as follows ([Matsumura 86, p. 107]):

$$e(I) = \lim_{n\to\infty} \frac{d!}{n^d}\ell(R/I^n),$$

where $d = \dim R$ (the Krull dimension of R) and $\ell(R/I^n)$ denotes the length of R/I^n as an R-module, for all $n \geq 0$. Suppose that $J \subseteq I$ is another m-primary ideal, then we observe that $e(J) \geq e(I)$.

A ring R is said to be *equidimensional* if $\dim R = \dim R/P$, for every minimal prime ideal P of R (see [Eisenbud 94, p. 458]). The following result is known as Rees' theorem.

Theorem 3.1. *Let R be an equidimensional local ring and let $J \subseteq I \subseteq R$ be a pair of m-primary ideals of R, then the following statements are equivalent:* [Rees 61]

(1) J is a reduction of I;

(2) $\overline{I} = \overline{J}$;

(3) $e(I) = e(J)$.

We have the following immediate consequence of the above theorem.

Corollary 3.2. *Let I be an m-primary ideal of R and $h \in R$. Then $h \in \overline{I}$ if and only if $e(I) = e(I + hR)$.*

The following theorem, the proof of which can be found in [Matsumura 86, p. 112] and [Northcott and Rees 54, p. 153], is essential in order to apply computational methods to obtain the number $\alpha(I)$.

Theorem 3.3. *Let $I = \langle g_1, \dots, g_s\rangle \subseteq \mathcal{O}_n$ be an m_n-primary ideal of $\mathcal{O}_n$. Then there exists a Zariski open set $W \subseteq \mathbb{C}^s \times \mathbb{C}^n$ such that whenever $(a_{11}, \dots, a_{1s}, \dots, a_{n1}, \dots, a_{ns})$ is an element of W, then the ideal of $\mathcal{O}_n$ generated by $h_i = \sum_j a_{ij}g_j$, $i = 1, \dots, n$, is a reduction of I.*

If $I = \langle g_1, \dots, g_s\rangle \subseteq \mathcal{O}_n$ is an ideal and $a = (a_{11}, \dots, a_{1s}, \dots, a_{n1}, \dots, a_{ns}) \in \mathbb{C}^s \times \mathbb{C}^n$, then we denote by $I(a)$ the ideal of $\mathcal{O}_n$ generated by $h_i = \sum_j a_{ij}g_j$, $i = 1, \dots, n$.

Corollary 3.4. *Let $I = \langle g_1, \dots, g_s\rangle \subseteq \mathcal{O}_n$ be a m_n-primary ideal of $\mathcal{O}_n$. Then there exists a Zariski open set $W \subseteq \mathbb{C}^s \times \mathbb{C}^n$ such that whenever $a \in W$, then*

$$e(I) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{I(a)}.$$

Proof: If W is the Zariski open set given in Theorem 3.3, then $I(a)$ is a reduction of I, for all $a \in W$, and by Theorem 3.1 both ideals have the same multiplicity. But the multiplicity of $I(a)$ is equal to its codimension $\dim_{\mathbb{C}} \mathcal{O}_n/I(a)$, since it is generated by n elements in $\mathcal{O}_n$ (see Theorem 17.11 of [Matsumura 86]). □

By the above corollary, if $I = \langle g_1, \dots, g_s\rangle$ is an m_n-primary ideal in $\mathcal{O}_n$, the multiplicity of I is the minimum value of the codimensions $\dim_{\mathbb{C}} \mathcal{O}_n/J$ of those ideals J generated by n general linear combinations of $g_1, \dots, g_s$. It is worth to remark that $e(I) \geq \dim_{\mathbb{C}} \mathcal{O}_n/I$ and that the equality holds if and only if I is generated by n elements.

Definition 3.5. If $g(x) = \sum_k a_k x^k \in \mathcal{O}_n$, the *support of* g, denoted by supp(g), is the set of those $k \in \mathbb{N}^n$ such that $a_k \neq 0$. Given any set $S \subseteq \mathcal{O}_n$, we define the *support of* S as the union of the supports of the elements belonging to S and we denote this set by supp(S). The *Newton polyhedron of* S is defined as the convex hull in $\mathbb{R}^n_+$ of $\{k + v : k \in \text{supp}(S), v \in \mathbb{R}^n_+\}$, where $\mathbb{R}_+ = [0, +\infty[$. If I is any ideal of $\mathcal{O}_n$, it is easy to check that $\Gamma_+(I)$ is equal to the convex hull of $\Gamma_+(g_1) \cup \dots \cup \Gamma_+(g_s)$, where $g_1, \dots, g_s$ is any finite generating system of I.

If $J \subseteq I$ are two ideals in $\mathcal{O}_n$, then we observe that $\Gamma_+(J) \subseteq \Gamma_+(I)$. Hence, if I is an m_n-primary ideal of $\mathcal{O}_n$ then $\Gamma_+(I)$ intersects all the coordinate axis in $\mathbb{R}^n$, since there is some power of m_n contained in I.

We say that an ideal I of $\mathcal{O}_n$ is *monomial* when it is generated by monomials $x^k = x_1^{k_1} \cdots x_n^{k_n}$, $k_i \in \mathbb{N}$.

Lemma 3.6. *Let $I \subseteq \mathcal{O}_n$ be a monomial ideal, then $\overline{I}$ is equal to the ideal generated by those monomials x^k such that $k \in \Gamma_+(I)$.* [Eisenbud 94, p. 141]

Corollary 3.7. *Let $I \subseteq \mathcal{O}_n$ be any ideal, then $\Gamma_+(I) = \Gamma_+(\overline{I})$.*

Proof: It is obvious that $I \subseteq \overline{I}$. Let I_0 be the ideal generated by those monomials x^k such that $k \in \Gamma_+(I)$. Then, we have that $\Gamma_+(I_0) = \Gamma_+(\overline{I_0})$, by Lemma 3.6. In particular, it follows that

$$\Gamma_+(I) \subseteq \Gamma_+(\overline{I}) \subseteq \Gamma_+(\overline{I_0}) = \Gamma_+(I_0) = \Gamma_+(I).$$

□

We will denote by K_I the ideal of $\mathcal{O}_n$ generated by those monomials x^k belonging to $\overline{I}$. Observe that this is an integrally closed ideal, by Lemma 3.6. We also denote by $e_1, \ldots, e_n$ the canonical basis in $\mathbb{R}^n$.

Lemma 3.8. *Let $I \subseteq \mathcal{O}_n$ be an m_n-primary ideal of $\mathcal{O}_n$ and let $\alpha_i = \min\{\alpha > 0 : \alpha e_i \in \Gamma_+(K_I)\}$, for all $i = 1, \ldots, n$. Then $\alpha(I) = \max\{\alpha_1, \ldots, \alpha_n\}$.*

Proof: Let $r = \max\{\alpha_1, \ldots, \alpha_n\}$, if $m_n^\ell \subseteq \overline{I}$, for some $\ell \geq 1$, then $\ell \geq \alpha_i$, for all $i = 1, \ldots, n$. In particular, we have that $\ell \geq r$. On the other hand, the set of all monomials x^k whose exponents are in the Newton polyhedron determined by $x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}$, belong to K_I, by Lemma 3.6. In particular, this means that $m_n^r \subseteq \overline{I}$ and that $r \geq \alpha(I)$. □

Given an m_n-primary ideal $I = \langle g_1, \ldots, g_s \rangle \subseteq \mathcal{O}_n$ and an element $h \in \mathcal{O}_n$, we shall denote by $e(I, h)$ the multiplicity of the ideal generated by $g_1, \ldots, g_s, h$. We also define the vector $\beta(I) = (\beta_1, \ldots, \beta_n)$, where $\beta_i = \min\{\beta > 0 : \beta e_i \in \Gamma_+(I)\}$, for all $i = 1, \ldots, n$.

We now describe an algorithm to determine the number $\alpha(I)$:

(1) First, we compute $e(I)$ as follows. If $s = n$, then $e(I) = \dim_{\mathbb{C}} \mathcal{O}_n/I$. If $s > n$, then we compute the codimension of the ideal generated by n generic linear combinations of $g_1, \ldots, g_s$ using the library `LIB"random.lib"` of Singular, thus obtaining $e(I)$, by Corollary 3.4.

(2) Consider the vector $\beta(I) = (\beta_1, \ldots, \beta_n)$. Let us fix an index $i = 1, \ldots, n$.

(3) We compute $e(I, x_i^{\beta_i})$ as in item (1).

(4) We know that $e(I) \geq e(I, x_i^{\beta_i})$. If $e(I) = e(I, x_i^{\beta_i})$, we set $\alpha_i = \beta_i$. Otherwise, we compute $e(I, x_i^{\beta_i+1})$.

(5) If $e(I) = e(I, x_i^{\beta_i+1})$ then we define $\alpha_i = \beta_i + 1$. Otherwise, we apply the same process to $\beta_i + 2$.

(6) Since the ideal K_I is also an m_n-primary ideal, the Newton polyhedron of K_I intersects all the coordinate axis. Then, this process stops and we obtain the number $\alpha_i = \min\{\alpha > 0 : \alpha e_i \in \Gamma_+(K_I)\} = \min\{\alpha > 0 : e(I) = e(I, x_i^\alpha)\}$.

(7) We compute the number α_i, for all $i = 1, \ldots, n$, following items (3)-(6).

(8) Finally, $\alpha(I) = \max\{\alpha_1, \ldots, \alpha_n\}$, by Lemma 3.8.

In the next example, we apply the above ideas to our initial objective, that is, the one of giving a sharp upper estimate to the degree of C^0-sufficiency of an arbitrary function germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with an isolated singularity at the origin.

Example 3.9. Consider the map germ $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ given by $f(x, y, z) = x^8 + y^7 + xz^5 + yz^3 + (xy^2 - x^2y)^2$. It is easy to check that f has an isolated singularity at the origin and that $\beta(J(f)) = (7, 6, 3)$. On the other hand, using "Singular," the Milnor number of f is $\dim_{\mathbb{C}} \mathcal{O}_n/J(f) = 79$. If we apply the process described above, we find that $x^7, y^6 \in \overline{J(f)}$, $z^3 \notin \overline{J(f)}$ and $z^4 \in \overline{J(f)}$. Then $\alpha(J(f)) = 7$ and this means that $j^8 f$ is C^0-sufficient (see the comment before Section 3).

For the sake of completeness, we also give the explicit computations that have lead us to state that $z^3 \notin \overline{J(f)}$:

```
>ring R= 0,(x,y,z),ds;
>poly f=x8+y7+xz5+yz3+(xy2-x2y)∧2;
>ideal I=jacob(f);
>LIB"random.lib";
>ideal A=I, z3;
>ideal B=randomid(A,3);
>ideal C=std(B);
>vdim (C);
>77
```

We see that $e(J(f), z^3) = 77 < 79 = e(J(f))$, then $z^3 \notin \overline{J(f)}$ by Corollary 3.2. Analogously we also conclude that $e(J(f), z^3) = 79$, so $z^4 \in \overline{J(f)}$.

Remark 3.10. The map given in Example 3.9 is not Newton non-degenerate in the sense of [Kouchnirenko 76]; therefore the result in [Fukui 91] on the estimation of $\alpha_0(f)$ can not be applied in this case. Moreover, the described method to compute $\alpha(J(f)) + 1$ can be used as a tool to test the sharpness of other results estimating the number $\alpha_0(f)$.

By the papers [Kuo 69] and [Kuiper 72], the *if part* of Theorem 2.2 also holds for real analytic function germs (see [Bochnak and Łojasiewicz 71] for the version of Theorem 2.2 for real variables). If $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ is analytic, we can complexify the coordinates in $\mathbb{R}^n$ in order to obtain a complex analytic function germ $f_{\mathbb{C}} : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$.

Suppose that $f_{\mathbb{C}}$ defines an isolated singularity at the origin. Then the number $\alpha(J(f_{\mathbb{C}})) + 1$ is also an upper bound for the degree of C^0-sufficiency of f.

ACKNOWLEDGEMENTS

The author wishes to express his gratitude to Prof. G. M. Greuel for his helpful comments. This work was supported by DGCYT, Grant BFM 2000-1110.

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Received January 18, 2001; accepted July 31, 2001.