Computing Amoebas

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CONTENTS

We study computational aspects of amoebas associated with varieties in $(\mathbb{C}^*)^n$, both from an exact and from an experimental point of view. In particular, we give explicit characterizations for the amoebas of classes of linear and nonlinear varieties and present homotopy-based techniques to compute the boundary of two-dimensional amoebas.

1. INTRODUCTION

The notion of *amoebas*, introduced by Gel′fand, Kapranov, and Zelevinsky in 1994 [Gel′fand et al. 94], serves to study the solution set $X \subset \mathbb{C}^n$ of a system of polynomial equations. Namely, it addresses this question from the following viewpoint. Given $w \in [0, \infty)^n$, does there exist a vector $z \in X$ with $|z_1| = w_1, \ldots, |z_n| = w_n$? How can the subset of all vectors $w = (w_1, \ldots, w_n) \in [0, \infty)^n$ be characterized for which the answer is "yes"? For reasons explained below, it is convenient to work in the algebraic torus $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and look at $\log|z_i|$ rather than $|z_i|$ itself.

Formally, the *amoeba* of a subset $X \subset (\mathbb{C}^*)^n$ is the image of X under the map

$$\begin{aligned} \mathrm{Log} : (\mathbb{C}^*)^n &\to \mathbb{R}^n\,, \\ z &\mapsto (\log|z_1|, \ldots, \log|z_n|)\,, \end{aligned}$$

where log denotes the natural logarithm. The restriction $\mathrm{Log}_{|X}$ is called the *amoeba map* of X. As we will see later in detail, if X is an algebraic curve in the plane ($n = 2$), then its amoeba looks like one of those microscopic animals, embracing convex regions and growing tentacles towards infinity in various directions (Figure 1).

Amoebas have recently been used in several fields of mathematics. We mention two examples. In *topology*, amoebas were used to provide significant contributions to Hilbert's 16th Problem (which is still widely open). Hilbert's problem asks for a classification of the topological types of real algebraic manifolds and has initiated the corresponding branch of mathematics. Recently, Mikhalkin used amoebas to prove topological uniqueness

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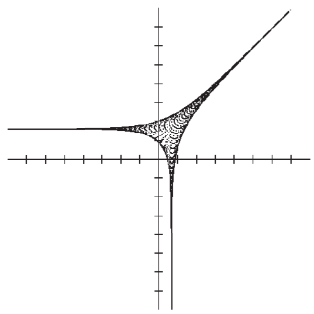


FIGURE 1. Amoeba Log $\mathcal{V}(f)$ for $f(z_1, z_2) = \frac{1}{2}z_1 + \frac{1}{5}z_2 - 1$.

of real plane algebraic curves maximally arranged with respect to three lines [Mikhalkin 00].

In the field of *dynamical systems*, actions of $\mathbb{Z}^n$ on compact metric spaces can be characterized in terms of expansive behavior along the half-spaces of $\mathbb{R}^n$. In [Einsiedler et al 01], amoebas have been applied to characterize this expansive behavior for *algebraic* $\mathbb{Z}^n$-actions, i.e., actions of $\mathbb{Z}^n$ by automorphisms of a compact abelian group.

Other mathematical habitats of amoebas include complex analysis [Forsberg et al. 00, Rullgård 01], mirror symmetry [Ruan 00], and measure theory [Mikhalkin and Rullgård 01, Passare and Rullgård 00]. The computational handling of amoebas still involves many difficulties and unsolved problems.

In the present paper, we study some concrete computational questions both from an exact and from an experimental point of view. In particular, we will be concerned with the case where X is a subvariety of the torus $(\mathbb{C}^*)^n$ with $X = \mathcal{V}(I)$ for some ideal $I \subset \mathbb{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$.

From the exact point of view, we provide explicit characterizations for certain classes of linear varieties, thus extending the results of [Forsberg et al. 00] on hyperplane amoebas. We also give an exact characterization for a class of nonlinear varieties which includes the Grassmannian of lines in 3-space. These characterizations can be used to answer algorithmic questions, such as membership of a given point in the amoeba.

For amoebas of plane algebraic curves which do not fit into these specific classes, we show how the topological results of [Mikhalkin 00] can be used to establish homotopy-based numerical techniques to compute the boundary of the amoeba. Experimentally, we have used these techniques and present some results (in terms of visualizations) illustrating this approach.

The paper is structured as follows. In Section 2, we review some basic properties and theorems concerning amoebas, accompanied by experiments visualizing the shape of amoebas. Then we introduce the relevant algorithmic questions. In Sections 3 and 4, we give new explicit characterizations for some classes of linear and nonlinear varieties, respectively. We complement these characterizations by some computer-algebraic experiments investigating some cases not covered by the theorems. Finally, in Section 5, we study homotopy-based techniques to draw two-dimensional amoebas.

2. PRELIMINARIES

Let $\mathbb{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ denote the ring of complex *Laurent polynomials* in n variables, i.e., sums of the form $\sum_{\alpha \in J} c_\alpha z^\alpha$ with finite index sets $J \subset \mathbb{Z}^n$ (see, e.g., [Cox et al. 98]). For Laurent polynomials $f_1, \dots, f_m$, let $\mathcal{V}(f_1, \dots, f_m)$ denote the set of common zeroes of $f_1, \dots, f_m$ in $(\mathbb{C}^*)^n$.

2.1 Hypersurface Amoebas

If X is an algebraic hypersurface in $(\mathbb{C}^*)^n$, then we call the amoeba of X a *hypersurface amoeba* [Forsberg et al. 00]. We assume that X is the zero set of a single Laurent polynomial $f(z) = \sum_{\alpha \in J} c_\alpha z^\alpha$.

Example 2.1.

(a) The shaded area in Figure 1 shows the amoeba Log $\mathcal{V}(f)$ for the linear function

$$f(z_1, z_2) = \frac{1}{2}z_1 + \frac{1}{5}z_2 - 1\,.$$

Note that this amoeba is a two-dimensional set. When denoting the coordinates in the amoeba plane by w_1 and w_2, the three tentacles have the asympotics $w_1 = \log 2$, $w_2 = \log 5$, and $w_2 = w_1 + \log(5/2)$. Note that the amoeba of a two-dimensional variety $\mathcal{V}(f) \in (\mathbb{C}^*)^2$ is not always a two-dimensional set. Consider for example, $f(z_1, z_2) := z_1 + z_2$, where Log $\mathcal{V}(f) = \{(w_1, w_2) \in \mathbb{R}^2 : w_1 = w_2\}$.

(b) If $f \in \mathbb{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ is a binomial in n variables,

$$f(z) = z^\alpha - z^\beta$$

with $\alpha \neq \beta \in \mathbb{Z}^n$, then the amoeba Log $\mathcal{V}(f)$ is a hyperplane in $\mathbb{R}^n$ which passes through the origin. To see this, first note that for any complex solution z of $z^\alpha = z^\beta$, the real vector $|z| = (|z_1|, \dots, |z_n|)$ is a solution as well. So it suffices to consider vectors $z \in (0, \infty)^n$. We can rewrite $|z|^\alpha = |z|^\beta$ as $|z|^{\alpha - \beta} = 1$, and by using the dot product of vectors, we obtain

$$(\alpha - \beta) \cdot \mathrm{Log}\ z = 0\,.$$

Since $\alpha \neq \beta$, this equation defines a hyperplane in the coordinates $\log|z_1|, \ldots, \log|z_n|$ which passes through the origin.

The following basic properties of amoebas have been stated in [Gel′fand et al. 94, Forsberg et al. 00]. They are the reason why it is often convenient to look at $\log|z_i|$ rather than $|z_i|$ itself.

Theorem 2.2. *The complement of a hypersurface amoeba* $\operatorname{Log} \mathcal{V}(f)$ *consists of finitely many convex regions, and these regions are in bijective correspondence with the different Laurent expansions of the rational function* $1/f$.

The shape of the amoeba is also related to the support

$$\operatorname{supp}(f) = \{\alpha \in \mathbb{Z}^n \,:\, c_\alpha \neq 0\}$$

of the function f and to the Newton polytope

$$\operatorname{New}(f) = \operatorname{conv}(\operatorname{supp}(f)) \,.$$

Example 2.3. Figure 2 shows the Newton polygon of a dense quartic polynomial f in two variables. Since we are not aware of any visualizations of "real-life" amoebas of interesting degree in the literature (in the sense that the pictures do not only focus on topological correctness), we present some experiments which illustrate both the topological and the geometric structure of an amoeba. Figure 3 depicts a series of amoebas $\operatorname{Log} \mathcal{V}(f)$ for dense quartic polynomials $f \in \mathbb{R}[z_1, z_2]$. In the first picture in this series, f is the product of four linear functions f_1, f_2, f_3, f_4. The amoeba of $\mathcal{V}(f)$ is the union of the amoebas of $\mathcal{V}(f_1)$, $\mathcal{V}(f_2)$, $\mathcal{V}(f_3)$, and $\mathcal{V}(f_4)$. This polynomial f is perturbed by adding or subtracting to every coefficient c_α of f (with the exception of the coefficient corresponding to the constant term) independently a random value in the interval $[0, \frac{1}{5}|c_\alpha|)$; see the right picture in the top row. This perturbation process is then iterated another four times.

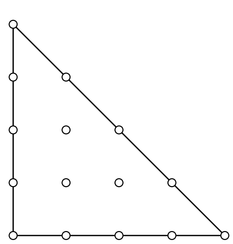


FIGURE 2. Newton polygon of a dense quartic in two variables.

The series of pictures has been produced with a MAPLE program which imposes an appropriate grid on the complex plane for one of the variables, say z_1, then solving the resulting quartic polynomials for z_2.

By Theorem 2.2, the complement ${}^c\operatorname{Log} \mathcal{V}(f)$ of an amoeba $\operatorname{Log} \mathcal{V}(f)$ consists of finitely many components. This gives rise to the following computational definition of an order in terms of multidimensional complex analysis, originating from the computation of multidimensional residues [Forsberg et al. 00].

Definition 2.4. The *order* of a point $w \in {}^c\operatorname{Log} \mathcal{V}(f)$ is defined by the vector $\nu \in \mathbb{Z}^n$ whose components are

$$\nu_j = \frac{1}{(2\pi i)^n} \int\limits_{\operatorname{Log}^{-1}(w)} \frac{z_j \partial_j f(z)}{f(z)} \frac{dz_1 \wedge \cdots \wedge dz_n}{z_1 \cdots z_n}$$

$$1 \leq j \leq n \,.$$

It can be shown that two different points $w, w' \in {}^c\operatorname{Log} \mathcal{V}(f)$ have the same order if and only if they are contained in the same connected component E of ${}^c\operatorname{Log} \mathcal{V}(f)$. Hence, ν can also be called the order of the component E. Moreover, it can be shown that the order ν of any component of ${}^c\operatorname{Log} \mathcal{V}(f)$ is contained in the Newton polytope $\operatorname{New}(f)$. To compute an order, the following description is useful.

Lemma 2.5. [Forsberg et al. 00] *For any vector* $s \in \mathbb{Z}^n \setminus \{0\}$ *and* $w \in {}^c\operatorname{Log} \mathcal{V}(f)$, *the directional order* $\langle s, \nu(f, w) \rangle$ *is equal to the number of zeroes (minus the order of the pole at the origin) of the one-variable Laurent polynomial*

$$u \mapsto f(c_1 u^{s_1}, \ldots, c_n u^{s_n})$$

inside the unit circle $|u| = 1$. *Here,* $c \in (\mathbb{C}^*)^n$ *is any vector with* $\operatorname{Log}(c) = w$.

All these results refer to the case where X is an algebraic hypersurface. A main difficulty in the treatment of amoebas of arbitrary varieties comes from the following simple observation. If X, Y, and Z are subvarieties of $(\mathbb{C}^*)^n$ with $X \cap Y = Z$, then $\operatorname{Log} Z \subset \operatorname{Log} X \cap \operatorname{Log} Y$, but in general the inclusion is proper.

2.2 Basic Computational Questions

Probably the most natural computational problem on amoebas is the one of membership which has been raised by Douglas Lind in connection with [Einsiedler et al 01].

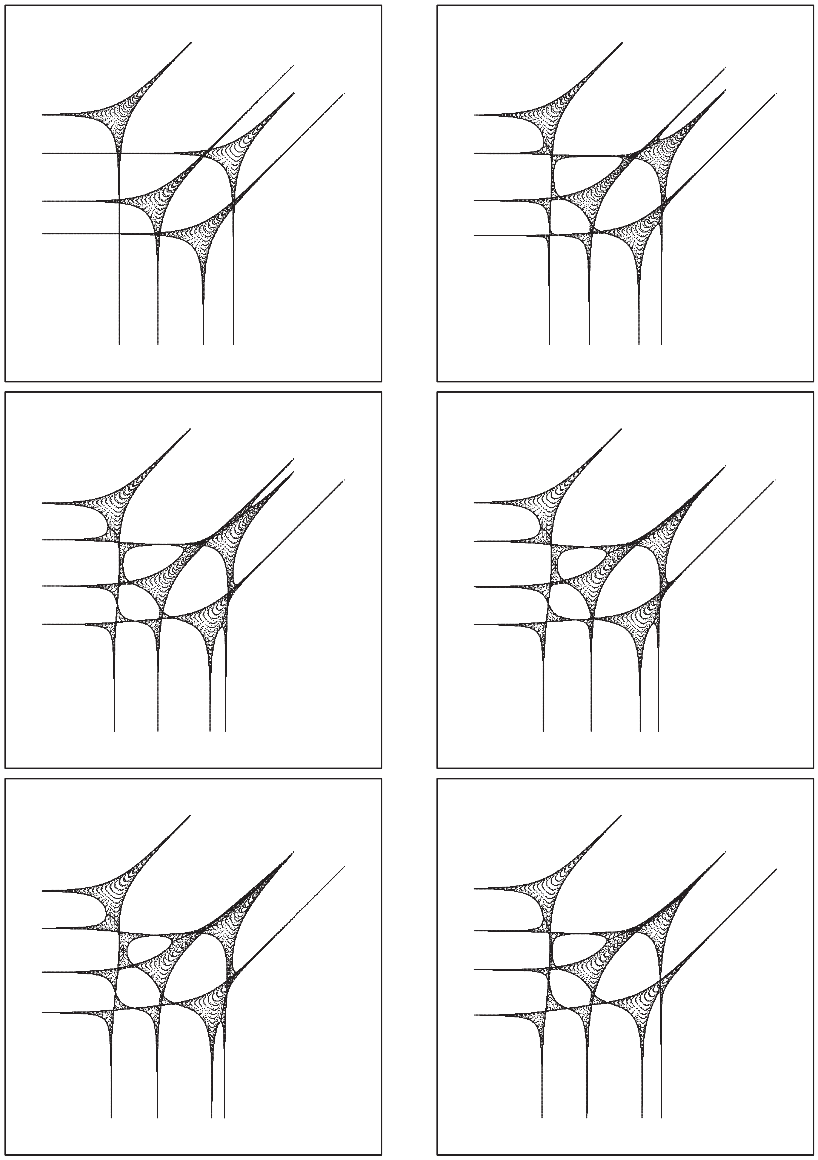


FIGURE 3. A series of quartic amoebas in two variables. The first picture shows the amoeba of $\mathcal{V}(f_1 \cdot f_2 \cdot f_3 \cdot f_4)$, where $f_1(z_1, z_2) = \left(\frac{1}{30}z_1 + \frac{1}{30}z_2 - 1\right)$, $f_2(z_1, z_2) = \left(\frac{1}{5}z_1 + 4z_2 - 1\right)$, $f_3(z_1, z_2) = \left(3z_1 + \frac{4}{7}z_2 - 1\right)$, $f_4(z_1, z_2) = \left(30z_1 + \frac{1}{300}z_2 - 1\right)$.

Membership:

Instance: Given $n, m \in \mathbb{N}$, $f_1, \ldots, f_m \in \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$, $x \in (0, \infty)^n$.

Question: Does there exist $z \in \mathcal{V}(f_1, \ldots, f_m)$ with $|z_k| = x_k$ for $1 \le k \le n$? (i.e., is $(\log x_1, \ldots, \log x_n) \in \operatorname{Log} \mathcal{V}(f_1, \ldots, f_m)$?)

Expressing every complex number z_k in the form $z_k = u_k + iv_k$ with $u_k, v_k \in \mathbb{R}$, the membership problem is a decision problem over the real numbers. It is known from Tarski's results that those problems are decidable [Tarski 51]. From the complexity-theoretical point of view, let us recall that in the binary Turing machine model, the size of the input is defined as the length of the binary encoding of the input data [Garey and Johnson 79], so these statements refer to rational input vectors and rational input polynomials (i.e., polynomials with rational coefficients).

The time complexity is measured in terms of the overall input encoding. If the dimension n is fixed, then the theory of real closed fields can be decided in polynomial time [Collins 75, Ben-Or et al. 86]. More precisely, the following holds:

Theorem 2.6. *For fixed dimension n, the following decision problem can be decided in polynomial time: Given rational polynomials $p_1(x_1, \ldots, x_n), \ldots, p_s(x_1, \ldots, x_n)$, a Boolean formula $\varphi(x_1, \ldots, x_n)$ which is a Boolean combination of polynomial equations and inequalities, i.e., $p_i(x_1, \ldots, x_n) = 0$ or $p_i(x_1, \ldots, x_n) \le 0$, and quantifiers $Q_1, \ldots, Q_n$, decide the truth of the statement*

$$Q_1(x_1 \in \mathbb{R}) \ldots Q_n(x_n \in \mathbb{R}) \quad \varphi(x_1, \ldots, x_n) .$$

We can conclude:

Corollary 2.7. *For fixed dimension n, membership of a point in an amoeba can be solved in polynomial time.*

However, despite this (theoretical) fact that for fixed dimension these problems can be decided in polynomial time, current implementations are only capable to deal with very small dimensions, up to three real variables. Generally, there are two approaches towards practical solutions of decision problems over the reals. One is based on Collins' cylindrical algebraic decomposition (CAD) [Collins 75], and the other one is the critical point method ([Grigor′ev and Vorobjov, Jr.]; for the state of the art, see [Aubry et al. 02]).

Another natural computational task is to compute (at least in a numerical sense) the (relative) boundary for the amoeba of a given ideal, e.g., for visualization purposes. This will be done in Section 5.

2.3 Known Results on the Membership Problem

The best way to answer questions like the membership problem is to know an explicit representation of the amoeba, say, in terms of equalities and inequalities. Example 2.1(b) contains a representation of this kind for the class of binomials. In [Forsberg et al. 00], those representations have been derived for the case of hypersurface amoebas $\operatorname{Log} \mathcal{V}(f)$, where f is a product of *linear* functions $f_1, \ldots, f_m$. Since $\operatorname{Log} \mathcal{V}(g \cdot h) = \operatorname{Log} \mathcal{V}(g) \cup \operatorname{Log} \mathcal{V}(h)$ for any Laurent polynomials g, h, all factors of f can be considered separately; hence, we can assume $m = 1$.

Let $\mathbb{P}^n_{\mathbb{R}}$ and $\mathbb{P}^n_{\mathbb{C}}$ denote the n-dimensional real projective space and n-dimensional complex projective space, respectively. In order to derive an explicit representation of a hyperplane amoeba, it is helpful to decompose the logarithmic map into two mappings. Firstly, the moment map

$$\begin{array}{rcl} \mathbb{P}^n_{\mathbb{C}} & \to & \Delta_n \\ (z_0, \ldots, z_n) & \mapsto & \dfrac{(|z_0|, |z_1|, \ldots, |z_n|)}{\sum_{i=0}^n |z_i|} , \end{array}$$

where Δ_n is the regular simplex,

$$\Delta_n = \{(t_0, \ldots, t_n) \in \mathbb{R}^n \, : \, t_0, \ldots, t_n \ge 0, \sum_{i=0}^n t_i = 1\} .$$

This moment map can be considered on the whole variety $\mathcal{V}(f)$ in $\mathbb{C}^n$ or $\mathbb{P}^n_{\mathbb{C}}$ rather than only on the subvariety of $(\mathbb{C}^*)^n$. The second mapping

$$\begin{array}{rcl} \operatorname{int}(\Delta_n) & \to & \mathbb{R}^n \\ (t_0, \ldots, t_n) & \mapsto & \left(\log \frac{t_1}{t_0}, \ldots, \log \frac{t_n}{t_0}\right) , \end{array}$$

is a homeomorphism from the interior of Δ_n to $\mathbb{R}^n$. Following the notation in [Gel′fand et al. 94], the image of a set X under the first mapping is called the *compactified amoeba* of X. In particular, the following theorem from [Forsberg et al. 00] shows that it maps hyperplanes to polytopes.

Theorem 2.8. [Forsberg et al. 00] *The compactified amoeba of a hyperplane*

$$X = \{z \in \mathbb{P}^n_{\mathbb{C}} \, : \, \sum_{i=0}^n a_i z_i = 0\} ,$$

$a_i \in \mathbb{C}$, is the polytope in Δ_n defined by the inequalities

$$|a_j| t_j \le \sum_{k \ne j} |a_k| t_k , \qquad 0 \le j \le n .$$

If no two of the coefficients a_i are zero then the polytope has $\binom{n+1}{2}$ vertices given by

$$\frac{1}{|a_i| + |a_j|} (|a_j| e_i + |a_i| e_j) , \qquad 0 \le i < j \le n ,$$

where e_k denotes the k-th unit vector. In particular, for $n = 2$, the compactified amoeba is the triangle in Δ_2 with vertices

$$\frac{1}{|a_0| + |a_1|} (|a_1|, |a_0|, 0) ,$$
$$\frac{1}{|a_0| + |a_2|} (|a_2|, 0, |a_0|) ,$$
$$\frac{1}{|a_1| + |a_2|} (0, |a_2|, |a_1|) .$$

Figure 4 depicts the compactified amoeba of the (projective closure of the) linear variety $\mathcal{V}(f)$ with $f(z_1, z_2) = z_1/2 + z_2/5 - 1$ from Example 2.1.

Hence, in order to check whether a given point $w \in \mathbb{R}^n$ is contained in the amoeba $\operatorname{Log} \mathcal{V}(f)$ of a hyperplane $\mathcal{V}(f)$ we compute the corresponding point t in the compactified variant by $t_i = e^{w_i} / (\sum_{i=0}^n e^{w_i})$, $0 \le i \le n$. By

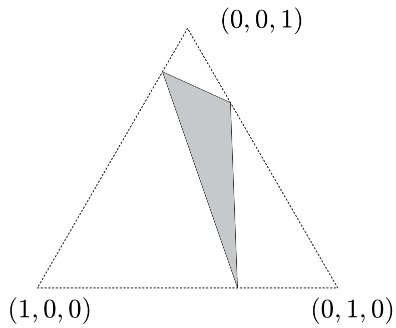
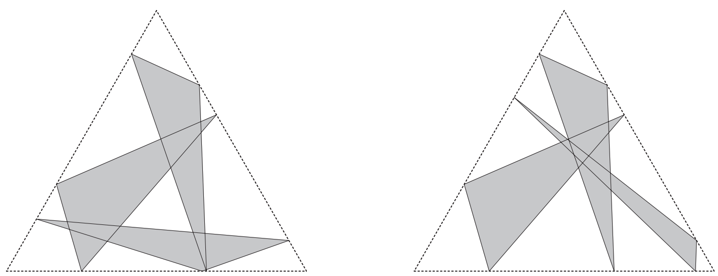


FIGURE 4. Compactified amoeba of $f(z_1, z_2) = \frac{1}{2}z_1 + \frac{1}{5}z_2 - 1$.



(a) With hole (b) Without hole

FIGURE 5. Compactified amoeba of plane cubic curves which factor into three linear terms.

Theorem 2.8, we just have to check containment of t in a polytope that is given as an intersection of finitely many halfspaces.

Figure 5 shows what can happen when considering the amoeba of a plane cubic curve that factors into three lines. The amoeba of that curve is the union of the amoebas of each line. For some of these curves, the amoeba contains a "hole," i.e., an additional bounded component in the complement (as in Figure 5(a)), and for some of these curves, the amoeba does not contain such a hole (as in Figure 5(b)).

3. AMOEBAS OF LINEAR VARIETIES

In this section, we consider linear varieties in $\mathbb{P}^n_{\mathbb{C}}$ of dimension less than $n-1$. In general, the compactified amoeba of a variety of this kind is *not* a polytope, even if the variety is defined by linear equations with real coefficients. A line $\ell \subset \mathbb{P}^n_{\mathbb{C}}$ which is defined by linear equations with real coefficients is called a *real line* in $\mathbb{P}^n_{\mathbb{C}}$. Figure 6(a) shows the compactified amoeba of a real line in $\mathbb{P}^3_{\mathbb{C}}$.

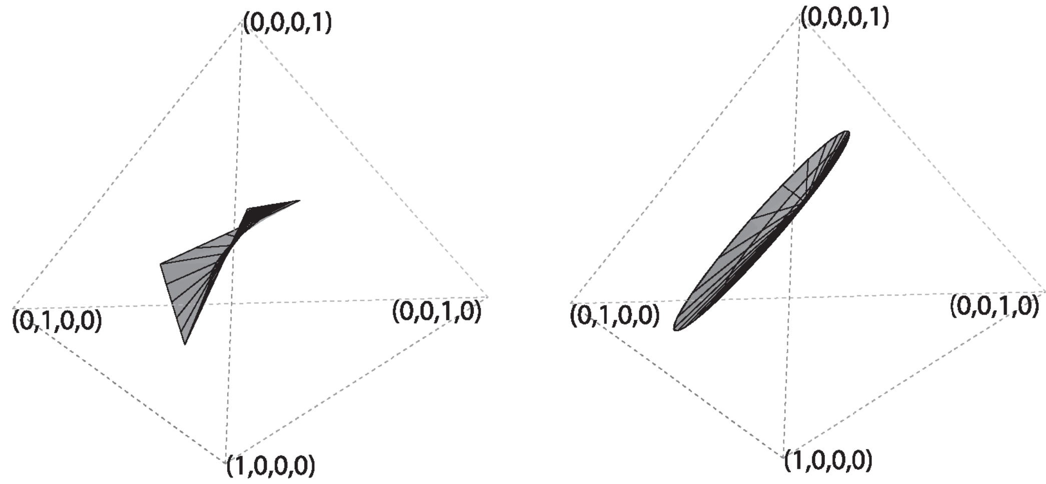
In order to answer membership questions for real lines in $\mathbb{P}^n_{\mathbb{C}}$, we consider the following *quadratic amoeba* ([Ruan 00]) defined by the map

$$\begin{array}{rcl} \mathbb{P}^n_{\mathbb{C}} & \to & \Delta_n \\ (z_0, z_1, \ldots, z_n) & \mapsto & \dfrac{(|z_0|^2, \ldots, |z_n|^2)}{|z_0|^2 + \ldots + |z_n|^2} . \end{array} \tag{3–1}$$

Analogous to Section 2, if we know an explicit representation of a quadratic amoeba, then we can easily solve the membership problem.

A line $\ell \subset \mathbb{P}^n_{\mathbb{C}}$ can be represented by its n-dimensional Plücker coordinate $(p_{ij})_{0 \le i < j \le n} \in \mathbb{P}_{\mathbb{C}}^{\binom{n+1}{2}}$ as follows (see, e.g., [Hodge and Pedoe 47, Cox et al. 96]). If $a, b \in \mathbb{P}^n_{\mathbb{C}}$ are two different points on ℓ, then let $p_{ij} = a_i b_j - a_j b_i$, $0 \le i < j \le n$. It is well-known that the p_{ij} satisfy certain quadratic relations, the *Plücker relations*. For example, for $n = 3$, we have $p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0$. The following theorem shows that the quadratic amoeba of a real line in complex n-space is the convex hull of an ellipse. See Figure 6(b) for an example.

Remark 3.1. Figures 6(a) and (b) have been produced with a three-dimensional surface plot in MAPLE, where the line $\ell \subset \mathbb{P}^3_{\mathbb{C}}$ is considered as a two-dimensional affine subspace over the reals.



(a) Compactified amoeba (b) Quadratic amoeba

FIGURE 6. Amoebas of the line $\{(0,1,2)+\lambda(1,-1,-1) : \lambda \in \mathbb{C}\} \subset \mathbb{C}^3$.

Theorem 3.2. *Let $n \geq 3$, and let ℓ be a real line in $\mathbb{P}^n_{\mathbb{C}}$ with Plücker coordinate $(p_{ij})_{0 \leq i < j \leq n} \in \mathbb{P}^n_{\mathbb{R}}$. Furthermore, let none of the coefficients p_{ij} be zero. A point $w \in \Delta_n$ is contained in the quadratic amoeba of ℓ if and only if the following equations and inequality are satisfied:*

$$\begin{aligned} &p_{12}p_{1j}p_{2j}w_0 - p_{02}p_{0j}p_{2j}w_1 + p_{01}p_{0j}p_{1j}w_2 \\ &\quad - p_{01}p_{02}p_{12}w_j = 0\,, \qquad 3 \leq j \leq n \end{aligned} \tag{3–2}$$

and

$$\begin{aligned} &2p_{13}^2p_{23}^2w_1w_2 + 2p_{12}^2p_{23}^2w_1w_3 + 2p_{12}^2p_{13}^2w_2w_3 \\ &\quad - p_{23}^4w_1^2 - p_{13}^4w_2^2 - p_{12}^4w_3^2 \geq 0\,. \end{aligned} \tag{3–3}$$

Since the theorem assumes that none of the coefficients p_{ij} is zero, the $n-2$ equations in (3–2) define a two-dimensional subspace. Further note that for a line whose Plücker coefficients are not all nonzero, equations (3–2) and inequality (3–3) might vanish identically (e.g., for $\ell = \{(0,0,0)+\lambda(1,2,3) : \lambda \in \mathbb{C}\}$. However, all these special cases can be treated separately.

Proof: Consider the points $A = (p_{01}, 0, -p_{12}, -p_{13}, \ldots, -p_{1n})$ and $B = (-p_{02}, -p_{12}, 0, p_{23}, \ldots, p_{2n})$ on ℓ. Then ℓ can be written in the parameterized form $\lambda A + \mu B$ with $\lambda, \mu \in \mathbb{C}$, $(\lambda, \mu) \neq (0,0)$. Without loss of generality, we can assume $\lambda \in \mathbb{R}$.

In order to prove that the image of every point $z \in \ell$ under the quadratic amoeba mapping satisfies (3–2) and (3–3), let z have the form $\lambda A + \mu B$. To simplify notation, let w denote only the numerator of the image defined in (3–1). Then we have

$$w_0 = |\lambda p_{01} - \mu p_{02}|^2\,, \tag{3–4}$$
$$w_1 = |\mu|^2 p_{12}^2\,, \tag{3–5}$$
$$w_2 = \lambda^2 p_{12}^2\,, \tag{3–6}$$
$$w_j = |-\lambda p_{1j} + \mu p_{2j}|^2\,, \qquad 3 \leq j \leq n\,. \tag{3–7}$$

We expand the sum on the left-hand side of (3–2) via (3–4)–(3–7) and $|a|^2 = a\overline{a}$, and separately consider the coefficients of λ^2, $|\mu|^2$, and $\lambda(\mu+\overline{\mu})$ in this expansion. The coefficient of λ^2 is

$$-p_{01}p_{12}p_{1j}(p_{01}p_{2j} - p_{02}p_{1j} + p_{0j}p_{12})\,.$$

The expression in the brackets evaluates to zero by the Plücker relations. Since the coefficients of $|\mu|^2$ and of $\lambda(\mu+\overline{\mu})$ vanish as well, Equation (3–2) is satisfied for $3 \leq j \leq n$.

Expanding the sum on the left-hand side of (3–3), the coefficients of λ^4, $\lambda^3(\mu+\overline{\mu})$, $\lambda|\mu|^2(\mu+\overline{\mu})$, and $|\mu|^4$ vanish. With regard to terms of degree 2 in both variables, there are terms both containing $\lambda^2|\mu|^2$ and terms containing $\lambda^2(\mu+\overline{\mu})^2$. Namely, we obtain the expression

$$4p_{12}^2p_{13}^2p_{23}^4\lambda^2|\mu|^2 - p_{12}^2p_{13}^2p_{23}^4\lambda^2(\mu+\overline{\mu})^2\,.$$

Since $p_{ij} \in \mathbb{R}$, $\lambda \in \mathbb{R}$ and $(\mu+\overline{\mu})^2 = 4(\mathrm{Re}\,\mu)^2 \leq 4|\mu|^2$, inequality (3–3) is fulfilled.

Conversely, assume that a point $w \in \Delta_n$ satisfies (3–2) and (3–3). We will explicitly compute the parameters $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{C}$ of a point $z \in \ell$ with $\sum_{i=0}^n |z_i|^2 = 1$ such that w is the image of z under the quadratic amoeba mapping.

Since none of the Plücker coefficients p_{ij} is zero, the representations (3–5) and (3–6) of w in terms of λ, μ imply $|\mu|^2 = w_1/p_{12}^2$ and $\lambda^2 = w_2/p_{12}^2$. Furthermore, since the case $w_1 = w_2 = 0$ would lead to a contradiction, we have $|\mu|^2 > 0$ or $\lambda^2 > 0$. Equation (3–7) for $j = 3$ implies

$$-\lambda(\mu + \overline{\mu}) = \frac{w_3 - \lambda^2 p_{13}^2 - |\mu| p_{23}^2}{p_{23}p_{13}} . \qquad (3\text{–}8)$$

In case $\lambda \neq 0$, squaring this equation and substituting the expressions for $|\mu|^2$ and λ^2 yields

$$(\text{Re } \mu)^2 = \frac{(p_{12}^2 w_3 - p_{13}^2 w_2 - p_{23}^2 w_1)^2}{4p_{12}^2 p_{13}^2 p_{23}^2 w_2} .$$

This equation together with the equation for $|\mu|^2$ give a solution for μ if and only if the right-hand side is less than or equal to $|\mu|^2$, which yields the condition

$$(p_{12}^2 w_3 - p_{13}^2 w_2 - p_{23}^2 w_1)^2 \leq 4p_{13}^2 p_{23}^2 w_1 w_2 .$$

However, the latter condition is equivalent to inequality (3–3). Hence, there exists a solution for λ and μ satisfying (3–5), (3–6), and (3–7) for $j = 3$. It remains to show that this solution also satisfies (3–4) and (3–7) for $4 \leq j \leq n$. With regard to (3–4), substituting $\lambda(\mu + \overline{\mu})$ in (3–4) by (3–8) and substituting λ^2, $|\mu|^2$ in the resulting equation gives the linear equation in w,

$$\begin{aligned}(p_{02}^2 p_{13}p_{23} - p_{01}p_{02}p_{23}^2)w_1 + (p_{01}^2 p_{13}p_{23} - p_{13}^2 p_{01}p_{02})w_2 \\ + p_{01}p_{02}p_{12}^2 w_3 = p_{12}^2 p_{13}p_{23} w_0 .\end{aligned}$$

By applying the Plücker relations on the terms in the brackets, this equation is equivalent to (3–2). Analogously, it can be checked that (3–7) is satisfied for $4 \leq j \leq n$. Finally, the case $\lambda = 0$ implies $w_2 = 0$ and can be checked directly. □

The following corollaries express the quadratic amoeba directly in terms of the defining inequalities of a real line ℓ in 3- or 2-space.

Corollary 3.3. *Let ℓ be a line in $\mathbb{P}^3_{\mathbb{C}}$ given as the solution of the system of linear equations*

$$\begin{aligned} a_0 z_0 + a_1 z_1 + a_2 z_2 + a_3 z_3 &= 0 , \\ b_0 z_0 + b_1 z_1 + b_2 z_2 + b_3 z_3 &= 0 \end{aligned}$$

with real *coefficients a_i, b_i. Further, let $q = (q_{01}, \ldots, q_{23}) \in \mathbb{P}^5_{\mathbb{R}}$, $q_{ij} = a_i b_j - a_j b_i$, denote the dual Plücker coordinate of ℓ, and let none of the dual Plücker coefficients q_{ij} be zero. Then the quadratic amoeba of ℓ is given by the set of points $w \in \Delta_3$ satisfying*

$$\begin{aligned} q_{01}q_{02}q_{03}w_0 - q_{01}q_{12}q_{13}w_1 + q_{02}q_{12}q_{23}w_2 \\ - q_{03}q_{13}q_{23}w_3 = 0 \end{aligned} \qquad (3\text{–}9)$$

and

$$\begin{aligned} 2q_{01}^2 q_{02}^2 w_1 w_2 + 2q_{01}^2 q_{03}^2 w_1 w_3 + 2q_{02}^2 q_{03}^2 w_2 w_3 - q_{01}^4 w_1^2 \\ - q_{02}^4 w_2^2 - q_{03}^4 w_3^2 \geq 0 . \end{aligned} \qquad (3\text{–}10)$$

Proof: The statement follows immediately from Theorem 3.2 and the well-known relation that the vectors $(p_{01}, \ldots, p_{23})$ and $(q_{23}, -q_{13}, q_{12}, q_{03}, -q_{02}, q_{01})$ coincide in $\mathbb{P}^5$ (see, e.g., [Hodge and Pedoe 47]). □

Similar to Theorem 3.2 it can be shown:

Corollary 3.4. *Let ℓ be a line in $\mathbb{P}^2_{\mathbb{C}}$ given as the solution of the linear equation*

$$a_0 z_0 + a_1 z_1 + a_2 z_2 = 0$$

with real coefficients a_i. Then the quadratic amoeba of ℓ is given by the inequality

$$2a_0^2 a_1^2 w_0 w_1 + 2a_0^2 a_2^2 w_0 w_2 + 2a_1^2 a_2^2 w_1 w_2 - \sum_{i=0}^{2} a_i^4 w_i^2 \geq 0 .$$

The following statement gives a partial answer to the question of how the quadratic amoebas of hyperplanes look.

Theorem 3.5. *The quadratic amoeba of a hyperplane*

$$X = \{z \in \mathbb{P}^n_{\mathbb{C}} : \sum_{i=0}^{n} a_i z_i = 0\} ,$$

$a_i \in \mathbb{C}$, has a boundary which is contained in a hypersurface of degree 2^{n-1}. For $n = 3$ this surface is given by

$$\begin{aligned} & W_2 W_3 (8W_1 + 4(W_0 - W_1 - W_2 - W_3))^2 \\ & \quad - (-4W_1(W_2 + W_3) + (W_0 - W_1 - W_2 - W_3)^2 \\ & \qquad + 4W_2 W_3)^2 = 0 , \end{aligned}$$

where $W_i := |a_i| w_i$.

Proof: According to Theorem 2.8, the facets of the polytope of the compactified amoeba are given by equations of the form

$$|a_0| t_0 = \sum_{i=1}^{n} |a_i| t_i \tag{3–11}$$

in the variables $t_0, \ldots, t_n$.

By passing over to the quadratic amoeba, described in the variables $w_0, \ldots, w_n$, we obtain instead

$$\sqrt{|a_0| w_0} = \sum_{i=1}^{n} \sqrt{|a_i| w_i}. \tag{3–12}$$

Without loss of generality, we assume $n \geq 2$. By $n-1$ squaring steps, we can eliminate the square roots of $w_0, \ldots, w_{n-2}$. Since the original equation is homogeneous, this gives an equation in which the only square root is $\sqrt{w_{n-1} w_n}$. This square root can be removed by another squaring operation. In particular, for $n = 3$, the squaring operations are applied on Equation (3–12),

$$\sqrt{W_1} \cdot 2(\sqrt{W_2} + \sqrt{W_3}) = W_0 - W_1 - W_2 - W_3 - 2\sqrt{W_2 W_3}\,,$$

and on

$$\sqrt{W_2 W_3}(8W_1 + 4(W_0 - W_1 - W_2 - W_3)) \\ = -4W_1(W_2 + W_3) + (W_0 - W_1 - W_2 - W_3)^2 + 4W_2 W_3.$$

We obtain the equation stated in the theorem. Since the equations of the other facets in Theorem 2.8 differ from (3–11) just by various signs (which become irrelevant within the squaring process), they lead to the same equation.

The same method for computing the hypersurface equation can be used for any $n \geq 2$. □

For all the classes of varieties treated in this section, we observe: If the quadratic amoeba is defined by equations with real coefficients, then the relative boundary of the amoeba is given by the images of real points in the variety V. In particular, for a point w in the amoeba with a real preimage in V, the inequalities (3–3) and (3–10) become equalities. If we neglect the common denominator of all components, then for the real points in V, the quadratic amoeba mapping is a Veronese mapping $\mathbb{P}^n_{\mathbb{C}} \to \mathbb{P}^n_{\mathbb{C}}$, $z \mapsto (z_0^2, \ldots, z_n^2)$. So the problem to characterize the quadratic amoeba images for the real points of a d-dimensional linear subspace in $\mathbb{P}^n_{\mathbb{C}}$ corresponds to finding the algebraic relations of the squares of $n+1$ homogeneous linear forms on a d-dimensional projective space. From this point of view, Corollary 3.3 implies that the squares of four homogeneous linear forms (in general position) on a one-dimensional projective space satisfy a linear and a quadratic relation.

In order to investigate these algebraic relations for higher dimensions, we can apply computer algebra systems, such as Macaulay 2 [Grayson and Stillman 01] (see, e.g., [Eisenbud 01, page 19] for a related treatment of the twisted cubic curve). In this computer experiment, we work over the finite field $\mathbb{F} := \mathbb{Z}_{32749}$, taking into account the experience that for these kind of computations, we obtain the same qualititative results we would get in characteristic 0.

The Macaulay 2 program shown below chooses $n+1$ random homogeneous linear forms $L_0(z_0, \ldots, z_d), \ldots, L_n(z_0, \ldots, z_d)$ in $d+1$ homogeneous variables,

$$\begin{array}{ccc} \mathbb{P}^d_{\mathbb{F}} & \to & \mathbb{P}^n_{\mathbb{F}}\,, \\ (z_0, \ldots, z_d) & \mapsto & (L_0(z_0, \ldots, z_d), \ldots, L_n(z_0, \ldots, z_d))\,. \end{array}$$

Assuming that the linear forms are generic, the image of this map defines a d-dimensional subspace of an n-dimensional projective space. The kernel of the map defines an ideal $I \subset \mathbb{Z}_{32749}[y_0, \ldots, y_n]$ which consists of the algebraic relations among the elements in the image (for the algorithmic techniques underlying the computation of this ideal see [Burundu and Stillman 93]).

```
d = 1
n = 3
R = ZZ/32749[z_0..z_d]
S = ZZ/32749[y_0..y_n]
I = kernel( map( R, S, apply(toList(0..n),
                     i -> (random(1,R))^2 )));
d, n, dim I, degree I,
    apply(first entries mingens I, f -> degree f)
```

Besides the values of d, n, the dimension of I, and the degree of I, the last line prints the degrees of a minimal set of generators of I. The output is

```
(1, 3, 2, 2, {{1}, {2}})
```

The degrees $\{1, 2\}$ of a minimal set of generators correspond to the linear and the quadratic relation of Corollary 3.3. For amoebas of planes in 4-space we have to consider $d = 2$ and $n = 4$. The corresponding Macaulay 2 computation shows that the homogeneous ideal of algebraic relations for the squares of the five linear forms is generated by seven cubics:

```
(2, 4, 3, 4, {{3}, {3}, {3}, {3}, {3}, {3}, {3}})
```

So these computations give some indication how the quadratic amoeba images of the *real* points in the linear variety can be characterized. However, we do not know in how far these techniques can be exploited to find good characterizations also of the images of the complex points.

4. AMOEBAS OF NONLINEAR VARIETIES

In this section, we explain the computation of an amoeba when the defining equations of the variety have a simpler expression in terms of algebraically independent monomials. Let $\phi_1, \dots, \phi_d$ be d Laurent monomials in n variables, say, $\phi_i = z^{a_i} = z_1^{a_{i1}} z_2^{a_{i2}} \cdots z_n^{a_{in}}$, where $a_i = (a_{i1}, \dots, a_{in}) \in \mathbb{Z}^n$. They define a homomorphism ϕ of algebraic groups from $(\mathbb{C}^*)^n$ to $(\mathbb{C}^*)^d$. Let V be any subvariety of $(\mathbb{C}^*)^d$. Then its inverse image $\phi^{-1}(V)$ is a subvariety of $(\mathbb{C}^*)^n$. Our objective is to compute the amoeba of $\phi^{-1}(V)$ in terms of the amoeba of V.

Lemma 4.1. *The following three conditions are equivalent:*

(i) The map ϕ is onto.

(ii) The monomials $\phi_1, \dots, \phi_d$ are algebraically independent.

(iii) The vectors $a_1, \dots, a_d$ are linearly independent.

Proof: Equivalence of (ii) and (iii) is stated, e.g., in the proof of [Sturmfels 96, Lemma 4.2]: Every $\mathbb{Z}$-linear relation among $a_1, \dots, a_n$ translates into an algebraic relation of the form $\phi_{i_1}^{d_1} \cdots \phi_{i_r}^{d_r} - \phi_{j_1}^{e_1} \cdots \phi_{j_s}^{e_s} = 0$ with $d_1, \dots, d_r, e_1, \dots, e_s \in \mathbb{N}$. The ideal of all algebraic relations among our monomials is generated by such binomials.

In order to show that (iii) implies (i), for a given $y \in (\mathbb{C}^*)^d$, choose $x \in \mathbb{C}^d$ with $e^{x_i} = y_i$, $1 \le i \le d$. If $a_1, \dots, a_d$ are linearly independent, then there exists $z \in \mathbb{C}^n$ with $a_{i1}z_1 + \ldots + a_{in}z_n = x_i$ for $1 \le i \le d$; hence $\phi(e^{z_1}, \dots, e^{z_n}) = y$.

Finally, in order to show that (i) implies (iii), it suffices to show that the integer vectors $a_1, \dots, a_d$ are linearly independent over $\mathbb{R}$. For a given $x \in \mathbb{R}^d$, let z be the preimage of $(e^{x_1}, \dots, e^{x_d})$ under ϕ. We can assume $z \in (0, \infty)^n$ because otherwise we can pass over to $(|z_1|, \dots, |z_n|)$. Since $a_{i1}z_1 + \ldots + a_{in}z_n = x_i$, $1 \le i \le d$, we can conclude the linear independence. □

Let ϕ' denote the restriction of ϕ to the multiplicative subgroup $(0, \infty)^n$. Consider the following commutative diagram of multiplicative abelian groups:

$$\begin{array}{ccc} (\mathbb{C}^*)^n & \xrightarrow{\phi} & (\mathbb{C}^*)^d \\ \downarrow & & \downarrow \\ (0,\infty)^n & \xrightarrow{\phi'} & (0,\infty)^d \end{array}$$

The vertical maps are taking coordinate-wise absolute value. For vectors $p = (p_1, \dots, p_n)$ in $(\mathbb{C}^*)^n$, we write $|p| = (|p_1|, \dots, |p_n|) \in (0, \infty)^n$, and similarly for vectors of length d. Further, for $V \subset (\mathbb{C}^*)^n$ let $|V| := \{|p| : p \in V\}$.

Lemma 4.2. *Suppose that the three equivalent conditions in Lemma 4.1 hold. Then $|\phi^{-1}(V)| = \phi'^{-1}(|V|)$.*

Proof: It is straightforward to check, without any assumptions on ϕ, that ϕ' maps $|\phi^{-1}(V)|$ into $|V|$. In other words, $|\phi^{-1}(V)|$ is always a subset of $\phi'^{-1}(|V|)$. What we must prove is $\phi'^{-1}(|V|) \subset |\phi^{-1}(V)|$. Let $u \in \phi'^{-1}(|V|)$. Then $\phi'(u) \in |V|$. Fix any point ξ in the subvariety V of $(\mathbb{C}^*)^d$ such that $|\xi| = \phi'(u)$. Now use the assumption that ϕ is surjective: We choose any preimage η of ξ under ϕ. Thus η is a point in the subvariety $\phi^{-1}(V)$ of $(\mathbb{C}^*)^n$. Consider now the point $\eta \cdot u \cdot (|\eta|)^{-1}$ in the algebraic group $(\mathbb{C}^*)^n$. We have

$$\begin{aligned} \phi\big(\eta \cdot u \cdot (|\eta|)^{-1}\big) &= \phi(\eta) \cdot \phi(u) \cdot |\phi(\eta)|^{-1} \\ &= \xi \cdot \phi'(u) \cdot |\xi|^{-1} = \xi \in V. \end{aligned}$$

Thus $\eta \cdot u \cdot (|\eta|)^{-1}$ lies in $\phi^{-1}(V)$. Its image under the absolute value map equals

$$\big|\eta \cdot u \cdot (|\eta|)^{-1}\big| = |\eta| \cdot |u| \cdot (|\eta|)^{-1} = |u|,$$

and we conclude that u lies in $|\phi^{-1}(V)|$, as desired. □

Lemma 4.2 applies to the logarithmic amoeba, the compactified amoeba, and the quadratic amoeba of $\phi^{-1}(V)$, since all of these amoebas are images of $|\phi^{-1}(V)|$.

Corollary 4.3. *Let $f = \sum_{i=1}^d c_i \cdot z_1^{a_{i1}} \cdots z_n^{a_{in}}$ be a Laurent polynomial with algebraically independent terms. Then the compactified (respectively, quadratic) amoeba of $\mathcal{V}(f)$ is the inverse image under ϕ' of the compactified (respectively, quadratic) amoeba of the hyperplane $\sum_{i=1}^d c_i y_i = 0$. The logarithmic amoeba* Log $\mathcal{V}(f)$ *is the inverse image of the logarithmic hyperplane amoeba under the linear map defined by the matrix (a_{ij}).*

Example 4.4. The Grassmann variety $\mathbb{G}_{1,3}$ of lines in 3-space is the variety in $\mathbb{P}^5_{\mathbb{C}}$ defined by

$$p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0\,.$$

Here, we consider $\mathbb{G}_{1,3}$ as a subvariety of $(\mathbb{C}^*)^6$. The three terms in this quadratic equation involve distinct variables and are hence algebraically independent. Note that $\mathbb{G}_{1,3}$ equals $\phi^{-1}(V)$ where

$$\phi : (\mathbb{C}^*)^6 \to (\mathbb{C}^*)^3\,,$$
$$(p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23}) \mapsto (p_{01}p_{23},\, p_{02}p_{13},\, p_{03}p_{12})$$

and V denotes the plane in 3-space defined by the linear equation

$$x - y + z = 0\,.$$

As we saw earlier in Corollary 3.4, the quadratic amoeba of V is defined by the inequality

$$X^2 + Y^2 + Z^2 \leq 2XY + 2XZ + 2YZ\,.$$

Corollary 4.3 implies that the quadratic amoeba of $\mathbb{G}_{1,3}$ is defined by

$$\begin{aligned}&P_{01}^2 P_{23}^2 + P_{02}^2 P_{13}^2 + P_{03}^2 P_{12}^2 \leq \\ &2P_{01}P_{02}P_{13}P_{23} + 2P_{01}P_{03}P_{12}P_{23} + 2P_{02}P_{03}P_{12}P_{13}\,.\end{aligned}$$

5. DRAWING TWO-DIMENSIONAL AMOEBAS

After having investigated specific classes of varieties, we now want to "compute" the geometry of an arbitrary two-dimensional amoeba in the sense of drawing it. As already seen in Section 2, the main task is to understand the boundary structure and topology of the amoeba. In [Mikhalkin 00], the logarithmic Gauss map was used to investigate the border of two-dimensional amoebas from a topological point of view. Here, we will use these ideas to establish a homotopy-based numerical algorithm for drawing an amoeba. For general references on homotopy-based numerical techniques in solving systems of polynomial equations we refer to [Cox et al. 98, Verschelde 99].

Let $f \in \mathbb{C}[z_1, z_2]$ and assume $z \in (\mathbb{C}^*)^2$ is a nonsingular point in $\mathcal{V}(f)$. We fix a small neighborhood U around z and one branch of the *holomorphic* logarithm function for this neighborhood. The image of this local logarithm function log applied to $U \cap \mathcal{V}(f)$ defines a one-dimensional complex manifold in $\mathbb{C}^2$. In particular, the normal direction of this manifold at $w = \log z$ is given by the logarithmic Gauss map $\gamma : U \cap \mathcal{V}(f) \to \mathbb{P}^1_{\mathbb{C}}$,

$$\begin{aligned}\gamma(z) &= \frac{d(f \circ e^w)}{dw}\Big|_{w=\log z} \\ &= \left(\frac{\partial f}{\partial z_1}(e^w), \frac{\partial f}{\partial z_2}(e^w)\right) \cdot \operatorname{diag}(e^{w_1}, e^{w_2})\Big|_{w=\log z} \\ &= \left(z_1 \frac{\partial f}{\partial z_1}(z_1, z_2)\,,\, z_2 \frac{\partial f}{\partial z_2}(z_1, z_2)\right)\,.\end{aligned}$$

Let $\mathrm{crit}_{\mathrm{Log}}(f)$ denote the critical points of the amoeba mapping, i.e., the points z where the differential mapping of the amoeba mapping is not surjective. In order to exhibit the geometric relationships, let us review the following theorem from [Mikhalkin 00].

Theorem 5.1. *Let $f \in \mathbb{C}[z_1, z_2]$ be a polynomial with real coefficients, and $\mathcal{V}(f)$ be nowhere singular. Further, let $\gamma : \mathcal{V}(f) \to \mathbb{P}^1_{\mathbb{C}}$ be its logarithmic Gauss map. Then the set of critical points of the amoeba mapping is given by $\mathrm{crit}_{\mathrm{Log}}(f) = \gamma^{-1}(\mathbb{P}^1_{\mathbb{R}})$.*

Proof: A point z is a critical point of the amoeba mapping if and only if the hypersurface defined by f contains a tangent direction $(t_1, t_2) \in \mathbb{C}^2 \setminus \{0\}$ such that $t_k = i c_k z_k$ for some real constants c_k, $k \in \{1, 2\}$. Combining this with the tangent condition,

$$t_1 \frac{\partial f}{\partial z_1}(z) + t_2 \frac{\partial f}{\partial z_2}(z) = 0\,,$$

we obtain the condition

$$c_1 \gamma_1(z) + c_2 \gamma_2(z) = 0\,.$$

This equation has a nonzero real solution for (c_1, c_2) if and only if $\gamma(z) \in \mathbb{P}^1_{\mathbb{R}} \subset \mathbb{P}^1_{\mathbb{C}}$. $\square$

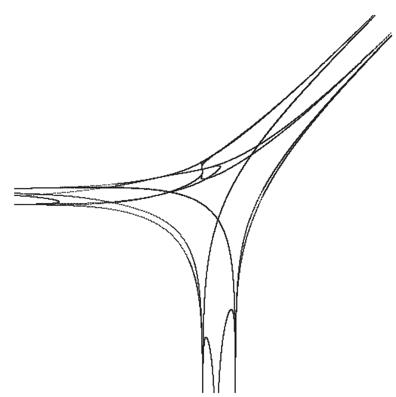


FIGURE 7. Critical points of the amoeba of a cubic function.

Every boundary point of the amoeba is a critical point of the amoeba mapping. Quite interestingly, we can also have a look at what happens in the situations when there are fewer holes than the maximum possible number given by the number of lattice points in the Newton polygon. Figure 7 shows an amoeba and its critical points for a cubic polynomial whose amoeba does not have a hole. We observe that the critical points bound a nonconvex region.

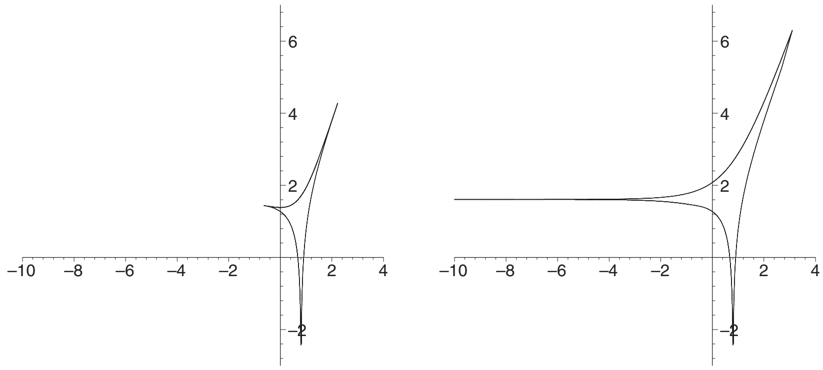


FIGURE 9. The two traces of the set of critical points

However, Figure 7 also shows that besides the boundary points and the critical points bounding a nonconvex region, there are even more critical points. In order to extract useful boundary information from the critical points, we propose to use a homotopy-based method to trace the different branches within the set of all critical points separately. To illustrate this idea, consider the parabola $\mathcal{V}(f)$ in $\mathbb{C}^2$ defined by $f(z_1, z_2) = z_2 - z_1^2 + 2z_1 - 5$.

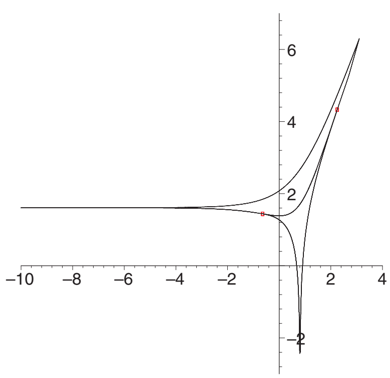


FIGURE 8. The critical points of the amoeba map for the function $z_2 - z_1^2 + 2z_1 - 5$.

Figure 8 shows the critical points of this function. By Theorem 5.1, they can be computed as follows. For all real $s \in \mathbb{R}$, we want to solve

$$f(z_1, z_2) = 0\,, \tag{5–1}$$

$$g(z_1, z_2, s) := z_1 \frac{\partial f}{\partial z_1}(z_1, z_2) - s z_2 \frac{\partial f}{\partial z_2}(z_1, z_2) = 0 \tag{5–2}$$

for z_1 and z_2. In order to avoid solving many systems of polynomial equations from scratch, we can apply the following numerical homotopy technique. If we know a solution z to the system of equations for a given starting parameter s, then we can trace the corresponding one-dimensional branch of solutions by successively perturbing s and numerically computing the new preimage z_{new}.

For the parabola, we obtain the two traces depicted in Figure 9. Note that these two traces coincide in the lower right part. The two points in which the two traces split are singular points for these curves; these points are also depicted in Figure 8. Since there does not exist a unique tangent direction in these two points, they satisfy (5–1), (5–2), as well as the equation

$$\det \begin{pmatrix} \frac{\partial f}{\partial z_1}(z_1, z_2) & \frac{\partial f}{\partial z_2}(z_1, z_2) \\ \frac{\partial g}{\partial z_1}(z_1, z_2, s) & \frac{\partial g}{\partial z_2}(z_1, z_2, s) \end{pmatrix} = 0\,.$$

Namely, in case of a nonzero determinant, the Implicit Function Theorem would guarantee a unique tangent direction. Altogether, this gives a system of three polynomial equations in the variables x, y, s for computing the candidates of the splitting points.

Since the set of critical points is a superset of the amoeba boundary, they decompose the amoeba into smaller regions. The next task is to decide algorithmically which of the regions in the whole plane belong to the amoeba and which of them are the complement components. Numerically, we can proceed as follows. For every critical point z which we compute during the homotopy method, we sample the neighborhood of z on the complex variety $\mathcal{V}(f)$ by numerically computing several points $z^{(1)}, \ldots, z^{(r)} \in \mathcal{V}(f)$ close to z. For any of these points $z^{(i)}$, we compute and draw the image $\operatorname{Log} z^{(i)}$. Figure 10 shows the images of the sampling points in grey color. By definition, these additional points lie inside the amoeba. Hence, every region which contains at least one image of a sampling point belongs to the amoeba.

Note that in Figure 10, sampling the neighborhood of those critical points whose images are contained in the interior of the amoeba only gives image points towards

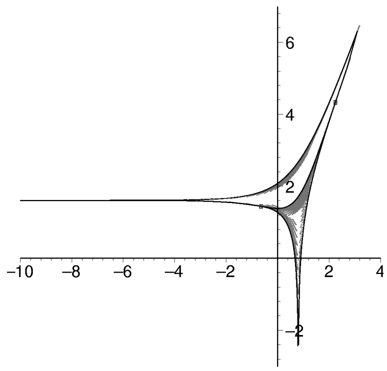


FIGURE 10. Numerically drawing the boundary

the lower-right side. Hence, they do not give a certificate that the upper-left region is part of the amoeba. However, this certificate is established by the critical points on the upper-left boundary. For related topological investigations compare [Mikhalkin 00]. (For example, the nonsingular critical points which are contained in both curves of Figure 9 stem from nonreal preimages. The nonsingular critical points which appear in only one curve stem from a real preimage.)

Now, assuming an underlying grid on the whole plane $\mathbb{R}^2$, techniques from computer graphics like filling algorithms can be applied to fill all the regions in which a noncritical point exists.

We remark that for the distinction of amoeba regions from the complement regions, it would also be helpful to have good algorithmic characterizations of the tentacle directions. Those characterizations in terms of universal Gröbner bases are currently investigated by Bernd Sturmfels [Sturmfels 02].

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REFERENCES

[Aubry et al. 02] Ph. Aubry, F. Rouillier, and M. Safey El Din. "Real Solving for Positive Dimensional Systems." *J. Symb. Comp.* 34 (2002), 543–560.

[Ben-Or et al. 86] M. Ben-Or, D. Kozen, and J. Reif. "The Complexity of Elementary Algebra and Geometry." *J. Computer and System Sciences* 32 (1986), 251–264.

[Burundu and Stillman 93] M. Burundu and M. Stillman. "Computing the Equations of a Variety." *Trans. Am. Math. Soc.* 337 (1993), 677–690.

[Collins 75] G.E. Collins. "Quantifier Elimination for Real Closed Fields by Cylindrical Algebraic Decomposition." In *Proc. Automated Theory and Formal Languages*, pp. 134–183, Lecture Notes in Computer Science vol. 33. Berlin: Springer-Verlag, 1975.

[Cox et al. 96] D. Cox, J. Little, and D. O'Shea. *Ideals, Varieties, and Algorithms,* 2nd edition. New York: Springer-Verlag, 1996.

[Cox et al. 98] D. Cox, J. Little, and D. O'Shea. *Using Algebraic Geometry*, Graduate Texts in Mathematics, Volume 185. New York: Springer-Verlag, 1998.

[Einsiedler et al 01] M. Einsiedler, D. Lind, R. Miles, and T. Ward. "Expansive Subdynamics for Algebraic $\mathbb{Z}^d$-Actions. *Ergodic Theory and Dynamical Systems* 21 (2001), 1695–1729.

[Eisenbud 01] D. Eisenbud. "Projective Geometry and Homological Algebra." In *Computations in Algebraic Geometry with Macaulay 2*, edited by D. Eisenbud, D. Grayson, M. Stillman, and B. Sturmfels. Berlin: Springer-Verlag, 2001.

[Forsberg et al. 00] M. Forsberg, M. Passare, and A. Tsikh. "Laurent Determinants and Arrangements of Hyperplane Amoebas." *Adv. Math.* 151 (2000), 45–70.

[Garey and Johnson 79] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP–Completeness.* New York: Freeman, 1979.

[Gel′fand et al. 94] I. M. Gel′fand, M. M. Kapranov, and A. V. Zelevinsky. *Discriminants, Resultants, and Multidimensional Determinants.* Noston, MA: Birkhäuser Boston Inc., 1994.

[Grayson and Stillman 01] D. R. Grayson and M. E. Stillman. MACAULAY 2. *A Software System for Research in Algebraic Geometry and Commutative Algebra.* Available from World Wide Web: (http://www.math.uiuc.edu/Macaulay2/), 2001.

[Grigor′ev and Vorobjov, Jr.] D. Y. Grigor′ev and N. N. Vorobjov, Jr. "Solving Systems of Polynomial Inequalities in Subexponential Time." *J. Symb. Comp.* 5 (1988), 37–64.

[Hodge and Pedoe 47] W. Hodge and D. Pedoe. *Methods of Algebraic Geometry*, Volume 1. Cambridge: Cambridge University Press, 1947.

[Mikhalkin 00] G. Mikhalkin. "Real Algebraic Curves, the Moment Map and Amoebas." *Ann. of Math.* 151 (2000), 309–326.

[Mikhalkin and Rullgård 01] G. Mikhalkin and H. Rullgård. “Amoebas of Maximal Area.” *Internat. Math. Res. Notices* (2001), 441–451.

[Passare and Rullgård 00] M. Passare and H. Rullgård. *Amoebas, Monge-Ampère Measures and Triangulations of the Newton Polytope.* Research Report No. 10, Stockholm University, 2000.

[Ruan 00] W.-D. Ruan. *Newton Polygon and String Diagram.* Available from World Wide Web: (www.arXiv.org/abs/math.DG/0011012), 2000.

[Rullgård 01] H. Rullgård, “Polynomial Amoebas and Convexity.” Ph.D. diss. Stockholm University, 2001, available as Research Report No. 8, 2001.

[Sturmfels 96] B. Sturmfels. *Gröbner Bases and Convex Polytopes*, American Mathematical Society, Providence, RI, 1996.

[Sturmfels 02] B. Sturmfels. *Solving Systems of Polynomial Equations*, to appear in the CBMS series of the AMS, 2002.

[Tarski 51] A. Tarski. *A Decision Method for Elementary Algebra and Geometry*, 2nd edition. Berkeley, CA: University of California Press, 1951.

[Verschelde 99] J. Verschelde. “PHCpack: A General-Purpose Solver for Polynomial Systems by Homotopy Continuation.” *ACM Trans. Math. Software* (1999), 251–276.

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