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THE SMALLEST HYPERBOLIC 6-MANIFOLDS

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ABSTRACT. By gluing together copies of an all right-angled Coxeter polytope a number of open hyperbolic 6-manifolds with Euler characteristic -1 are constructed. They are the first known examples of hyperbolic 6-manifolds having the smallest possible volume.

1. INTRODUCTION

The last few decades have seen a surge of activity in the study of finite volume hyperbolic manifolds—that is, complete Riemannian n-manifolds of constant sectional curvature -1. Not surprisingly for geometrical objects, volume has been, and continues to be, the most important invariant for understanding their sociology. The possible volumes in a fixed dimension form a well-ordered subset of $\mathbb{R}$, indeed a discrete subset except in 3 dimensions (where the orientable manifolds at least have ordinal type ω^ω). Thus it is a natural problem with a long history to construct examples of manifolds with minimum volume in a given dimension.

In 2 dimensions the solution is classical, with the minimum volume in the compact orientable case achieved by a genus 2 surface, and in the noncompact orientable case by a once-punctured torus or thrice-punctured sphere (the identities of the manifolds are of course also known in the nonorientable case). In 3 dimensions the compact orientable case remains an open problem with the Matveev-Fomenko-Weeks manifold [16, 30] obtained via $(5,-2)$-Dehn surgery on the sister of the figure-eight knot complement conjecturally the smallest. Amongst the noncompact orientable 3-manifolds the figure-eight knot complement realizes the minimum volume [17], and the Gieseking manifold (obtained by identifying the sides of a regular hyperbolic tetrahedron as in [14, 20]) does so for the nonorientable ones [1]. One could also add "arithmetic" to our list of adjectives and so have eight optimization problems to play with (so that the Matveev-Fomenko-Weeks manifold is known to be the minimum volume orientable, arithmetic compact 3-manifold; see [5]).

When $n \geq 4$ the picture is murkier, although in even dimensions we have recourse to the Gauss-Bonnet Theorem, so that in particular the minimum volume a $2m$-dimensional hyperbolic manifold could possibly have, is when the Euler characteristic χ satisfies $|\chi| = 1$. The first examples of noncompact 4-manifolds with

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$\chi = 1$ were constructed in [23] (see also [13]). The compact case remains a difficult unsolved problem, although if we restrict to arithmetic manifolds, then it is known [2, 8] that a minimum volume arithmetic compact orientable 4-manifold M has $\chi \leq 16$ and M is isometric to the orbit space of a torsion-free subgroup of the hyperbolic Coxeter group $[5, 3, 3, 3]$. The smallest compact hyperbolic 4-manifold currently known to exist has $\chi = 8$ and is constructed in [8]. Manifolds of very small volume have been constructed in 5 dimensions [13, 24], but the smallest volume 6-dimensional example hitherto known has $\chi = -16$ [13].

In this paper we announce the discovery of a number of noncompact nonorientable hyperbolic 6-manifolds with Euler characteristic $\chi = -1$. The method of construction is classical in that the manifolds are obtained by identifying the sides of a 6-dimensional hyperbolic Coxeter polytope.

2. Coxeter polytopes

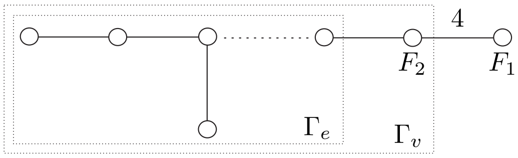
Let C be a convex (not necessarily bounded) polytope of finite volume in a simply connected space X^n of constant curvature. Call C a Coxeter polytope if the dihedral angle subtended by two intersecting $(n-1)$-dimensional sides is π/m for some integer $m \geq 2$. When $X^n = S^n$ or the Euclidean space E^n, such polyhedra have been completely classified [11], but in the hyperbolic space H^n, a complete classification remains a difficult problem (see for example [27] and the references there).

If Γ is the group generated by reflections in the $(n-1)$-dimensional sides of C, then Γ is a discrete cofinite subgroup of the Lie group $\operatorname{Isom} X^n$, and every discrete cofinite reflection group in $\operatorname{Isom} X^n$ arises in this way from some Coxeter polytope, which is uniquely defined up to isometry. The Coxeter symbol for C (or Γ) has nodes indexed by the $(n-1)$-dimensional sides, and an edge labeled m joining the nodes corresponding to sides that intersect with angle π/m (label the edge joining the nodes of nonintersecting sides by ∞). In practice the labels 2 and 3 occur often, so that edges so labeled are respectively removed or left unlabeled.

Let Λ be an $(n+1)$-dimensional Lorentzian lattice, that is, an $(n+1)$-dimensional free $\mathbb{Z}$-module equipped with a $\mathbb{Z}$-valued bilinear form of signature $(n, 1)$. For each n, there is a unique such Λ, denoted $I_{n,1}$, that is odd and self-dual (see [25, Theorem V.6], or [18, 19]). By [4], the group $\mathrm{O}_{n,1}\mathbb{Z}$ of automorphisms of $I_{n,1}$ acts discretely, cofinitely by isometries on the hyperbolic space H^n obtained by projectivising the negative norm vectors in the Minkowski space-time $I_{n,1} \otimes \mathbb{R}$ (to get a faithful action one normally passes to the centerless version $\mathrm{PO}_{n,1}\mathbb{Z}$).

Vinberg and Kaplinskaya showed [28, 29] that the subgroup Reflec_n of $\mathrm{PO}_{n,1}\mathbb{Z}$ generated by reflections in positive norm vectors has finite index if and only if $n \leq 19$, thus yielding a family of cofinite reflection groups and corresponding finite volume Coxeter polytopes in the hyperbolic spaces H^n for $2 \leq n \leq 19$. Indeed, Conway and Sloane showed ([9, Chapter 28] or [10]) that for $n \leq 19$ the quotient of $\mathrm{PO}_{n,1}\mathbb{Z}$ by Reflec_n is a subgroup of the automorphism group of the Leech lattice. Borcherds [3] showed that the (non-self-dual) even sublattice of $I_{21,1}$ also acts cofinitely, yielding the highest-dimensional example known of a Coxeter group acting cofinitely on hyperbolic space.

When $4 \leq n \leq 9$ the group $\Gamma = \mathrm{Reflec}_n$ has Coxeter symbol,



with $n+1$ nodes and C a noncompact, finite volume n-simplex Δ^n (when $n > 9$, the polytope C has a more complicated structure).

Let v be the vertex of Δ^n opposite the side F_1 marked on the symbol, and let Γ_v be the stabilizer in Γ of this vertex. This stabilizer is also a reflection group with symbol as shown, and is finite for $4 \leq n \leq 8$ (being the Weyl group of type A_4, D_5, E_6, E_7 and E_8 respectively) and infinite for $n = 9$ (when it is the affine Weyl group of type $\widetilde{E}_8$). Let

$$P_n = \bigcup_{\gamma \in \Gamma_v} \gamma(\Delta^n),$$

a convex polytope obtained by gluing $|\Gamma_v|$ copies of the simplex Δ^n together. Thus, P_n has finite volume precisely when $4 \leq n \leq 8$, although it is noncompact, with a mixture of finite vertices in H^n and cusped ones on ∂H^n. In any case, P_n is an all right-angled Coxeter polytope: its sides meet with dihedral angle $\pi/2$ or are disjoint. This follows immediately from the observation that the sides of P_n arise from the Γ_v-images of the side of Δ^n opposite v, and this side intersects the other sides of Δ^n in dihedral angles $\pi/2$ or $\pi/4$. Vinberg has conjectured that $n = 8$ is the highest dimension in which finite volume all right-angled polytopes exist in hyperbolic space.

The volume of the polytope P_n is given by

$$\mathrm{vol}(P_n) = |\Gamma_v|\mathrm{vol}(\Delta^n) = |\Gamma_v|[\mathrm{PO}_{n,1}\mathbb{Z} : \Gamma]\mathrm{covol}(\mathrm{PO}_{n,1}\mathbb{Z}),$$

where $\mathrm{covol}(\mathrm{PO}_{n,1}\mathbb{Z})$ is the volume of a fundamental region for the action of $\mathrm{PO}_{n,1}\mathbb{Z}$ on H^n (and for $4 \leq n \leq 9$ the index $[\mathrm{PO}_{n,1}\mathbb{Z} : \Gamma] = 1$). When n is even, we have by [26] and [22],

$$\mathrm{covol}(\mathrm{PO}_{n,1}\mathbb{Z}) = \frac{(2^{\frac{n}{2}} \pm 1)\pi^{\frac{n}{2}}}{n!} \prod_{k=1}^{\frac{n}{2}} |B_{2k}|,$$

with B_{2k} the $2k$-th Bernoulli number and with the plus sign if $n \equiv 0, 2 \mod 8$ and the minus sign otherwise.

Alternatively (when n is even), we have recourse to the Gauss-Bonnet Theorem, so that $\mathrm{vol}(P_n) = \kappa_n|\Gamma_v|\chi(\Gamma)$, where $\chi(\Gamma)$ is the Euler characteristic of the Coxeter group Γ and $\kappa_n = 2^n(n!)^{-1}(-\pi)^{n/2}(n/2)!$. The Euler characteristic of Coxeter groups can be easily computed from their symbol (see [6, 7] or [13, Theorem 9]). Indeed, when $n = 6$, $\chi(\Gamma) = -1/\mathcal{L}$ where $\mathcal{L} = 2^{10}\,3^4\,5$ and so $\mathrm{vol}(P_6) = 8\pi^3|E_6|/15\mathcal{L} = \pi^3/15$.

The Coxeter symbol for P_n has a nice description in terms of finite reflection groups. If v' is the vertex of Δ^n opposite the side F_2, let Γ_e be the pointwise stabilizer of $\{v, v'\}$: the elements thus stabilize the edge e of Δ^n joining v and v'.

Now consider the Cayley graph $\mathcal{C}_v$ for Γ_v with respect to the generating reflections in the sides of the symbol for Γ_v. Thus, $\mathcal{C}_v$ has vertices in one-to-one correspondence with the elements of Γ_v and for each generating reflection s_α, an undirected edge labeled s_α connecting vertices γ_1 and γ_2 if and only if $\gamma_2 = \gamma_1 s_\alpha$ in Γ_v. In particular, $\mathcal{C}_v$ has s_2 labeled edges corresponding to the reflection in F_2. Removing these s_2-edges decomposes $\mathcal{C}_v$ into components, each of which is a copy of the Cayley graph $\mathcal{C}_e$ for Γ_e, with respect to the generating reflections.

Take as the nodes of the symbol for P_n these connected components. If two components have an s_2-labeled edge running between any two of their vertices in $\mathcal{C}_v$, then leave the corresponding nodes unconnected; otherwise, connect them by an edge labeled ∞. The resulting symbol (respectively the polytope P_n) thus has $|\Gamma_v|/|\Gamma_e|$ nodes (resp. sides). The number of sides of P_n for $n = 4, 5, 6, 7, 8$ is $10, 16, 27, 56$ and 240 respectively.

3. Constructing the manifolds

We now restrict our attention to the case $n = 6$. We work in the hyperboloid model of hyperbolic 6-space

$$H^6 = \{x \in \mathbb{R}^7 : x_1^2 + x_2^2 + \cdots + x_6^2 - x_7^2 = -1 \text{ and } x_7 > 0\}$$

and represent the isometries of H^6 by Lorentzian 7×7 matrices that preserve H^6. The right-angled polytope P_6 has 27 sides each congruent to P_5. We position P_6 in H^6 so that 6 of its sides are bounded by the 6 coordinate hyperplanes $x_i = 0$ for $i = 1, \ldots, 6$ and these 6 sides intersect at the center e_7 of H^6. Let K_6 be the group of 64 diagonal Lorentzian 7×7 matrices $\mathrm{diag}(\pm 1, \ldots, \pm 1, 1)$. The set $Q_6 = K_6 P_6$, which is the union of 64 copies of P_6, is a right-angled convex polytope with 252 sides. We construct hyperbolic 6-manifolds, with $\chi = -8$, by gluing together the sides of Q_6 by a proper side-pairing with side-pairing maps of the form rk with k in K_6 and r a reflection in a side S of Q_6. The side-pairing map rk pairs the side $S' = kS$ to S (see §11.1 and §11.2 of [21] for a discussion of proper side-pairings). We call such a side-pairing of Q_6 simple. We searched for simple side-pairings of Q_6 that yield a hyperbolic 6-manifold M with a freely acting $\mathbb{Z}/8$ symmetry group that permutes the 64 copies of P_6 making up M in such a way that the resulting quotient manifold is obtained by gluing together 8 copies of P_6. Such a quotient manifold has $\chi = -8/8 = -1$. This is easier said than done, since the search space of all possible side-pairings of Q_6 is very large. We succeeded in finding desired side-pairings of Q_6 by employing a strategy that greatly reduces the search space. The strategy is to extend a side-pairing in dimension 5 with the desired properties to a side-pairing in dimension 6 with the desired properties.

Let $Q_5 = \{x \in Q_6 : x_1 = 0\}$. Then Q_5 is a right-angled convex 5-dimensional polytope with 72 sides. Note that Q_5 is the union $K_5 P_5$ of 32 copies of P_5 where $P_5 = \{x \in P_6 : x_1 = 0\}$ and K_5 is the group of 32 diagonal Lorentzian 7×7 matrices $\mathrm{diag}(1, \pm 1, \ldots, \pm 1, 1)$. A simple side-pairing of Q_6 that yields a hyperbolic 6-manifold M restricts to a simple side-pairing of Q_5 that yields a hyperbolic 5-manifold which is a totally geodesic hypersurface of M. All the orientable hyperbolic 5-manifolds that are obtained by gluing together the sides of Q_5 by a simple side-pairing are classified in [24].

We started with the hyperbolic 5-manifold N, numbered 27 in [24], obtained by gluing together the sides of Q_5 by the simple side-pairing with side-pairing code `2B7JB47JG81`. The manifold N has a freely acting $\mathbb{Z}/8$ symmetry group that permutes the 32 copies of P_5 making up N in such a way that the resulting quotient

TABLE 1. Side-pairing codes and homology groups of the seven examples.

N	SP	H_1	H_2	H_3	H_4	H_5
		0248	0248	0248	0248	0248
1	`GW8dNEEdN4ZJ01k2l1PIY`	0401	1910	4821	1500	0000
2	`HX9dNFEcM5aKU6f3f6UKa`	0401	1810	8710	5500	0000
3	`HX9dNFEcM5YIO1l3l1OIY`	0401	2900	7810	4400	1000
4	`HX9dNFEcM5YIO6l3l6OIY`	0401	2800	7910	4400	1000
5	`HX9dNFEcM5YIOxl3lyOIY`	0211	2800	4821	1400	1000
6	`HX9dNFEcM5YIOyl3lxOIY`	0211	2800	4930	1400	1000
7	`HX9dNFEcM5aKUxf3fyUKa`	0301	1900	5630	2500	0000

manifold is obtained by gluing together 4 copies of P_5. A generator of the $\mathbb{Z}/8$ symmetry group of N is represented by the Lorentzian 6×6 matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \\ -1 & 0 & -1 & -1 & 0 & 2 \end{pmatrix}.$$

The strategy is to search for simple side-pairings of Q_6 that yield a hyperbolic 6-manifold with a freely acting $\mathbb{Z}/8$ symmetry group with generator represented by the following Lorentzian 7×7 matrix that extends the above Lorentzian 6×6 matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & -1 & 0 & -1 & -1 & 0 & 2 \end{pmatrix}.$$

For such a side-pairing the resulting quotient manifold can be obtained by gluing together 8 copies of P_6 by a proper side-pairing. By a computer search we found 14 proper side-pairings of 8 copies of P_6 in this way, and hence we found 14 hyperbolic 6-manifolds with $\chi = -1$. Each of these 14 manifolds is noncompact with volume $8\mathrm{vol}(P_6) = 8\pi^3/15$ and five cusps. These 14 hyperbolic 6-manifolds represent at least 7 different isometry types, since they represent 7 different homology types. Table 1 lists side-pairing codes for 7 simple side-pairings of Q_6 whose $\mathbb{Z}/8$ quotient manifold has homology groups isomorphic to $\mathbb{Z}^a \oplus (\mathbb{Z}/2)^b \oplus (\mathbb{Z}/4)^c \oplus (\mathbb{Z}/8)^d$ for nonnegative integers a, b, c, d encoded by $abcd$ in the table. In particular, all 7 manifolds in Table 1 have a finite first homology group.

All of our examples, with $\chi = -1$, can be realized as the orbit space H^6/Γ of a torsion-free subgroup Γ of $\mathrm{PO}_{6,1}\mathbb{Z}$ of minimum index. These manifolds are the first examples of hyperbolic 6-manifolds having the smallest possible volume. All these manifolds are nonorientable. In the near future, we hope to construct orientable examples of noncompact hyperbolic 6-manifolds having $\chi = -1$.

References

[1] C. Adams, *The noncompact hyperbolic 3-manifold of minimum volume*, Proc. AMS **100** (1987), 601–606. MR0894423 (88m:57018)

[2] M. Belolipetsky, *On volumes of arithmetic quotients of $SO(1,n)$*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (to appear).

[3] R. Borcherds, *Automorphism groups of Lorentzian lattices*, J. Algebra **111** (1987), no. 1, 133–153. MR0913200 (89b:20018)

[4] A. Borel and Harish-Chandra, *Arithmetic subgroups of algebraic groups*, Ann. of Math. **75** (1962), no. 3, 485–535. MR0147566 (26:5081)

[5] T. Chinburg, E. Friedman, K. N. Jones, and A. W. Reid, *The arithmetic hyperbolic 3-manifold of smallest volume*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **30** (2001), no. 4, 1–40. MR1882023 (2003a:57027)

[6] I. M. Chiswell, *The Euler characteristic of graph products and Coxeter groups*, in Discrete Groups and Geometry, W. J. Harvey and Colin Maclachlan (Editors), London Math. Soc. Lect. Notes, vol. 173, 1992, pp. 36–46. MR1196914 (94a:05090)

[7] I. M. Chiswell, *Euler characteristics of groups*, Math. Z. **147** (1976), 1–11. MR0396785 (53:645)

[8] M. D. E. Conder and C. Maclachlan, *Small volume compact hyperbolic 4-manifolds*, to appear, Proc. Amer. Math. Soc.

[9] J. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, Second Edition, Springer, 1993. MR1194619 (93h:11069)

[10] J. Conway and N. J. A. Sloane, *Leech roots and Vinberg groups*, Proc. Roy. Soc. London Ser. A **384** (1982), no. 1787, 233–258. MR0684311 (84b:10048)

[11] H. S. M. Coxeter, *Discrete groups generated by reflections*, Ann. of Math. **35** (1934), no. 2, 588–621. MR1503182

[12] M. W. Davis, *A hyperbolic 4-manifold*, Proc. Amer. Math. Soc. **93** (1985), 325–328. MR0770546 (86h:57016)

[13] B. Everitt, *Coxeter groups and hyperbolic manifolds*, Math. Ann. **330** (2004), no. 1, 127–150. MR2091682

[14] B. Everitt, *3-Manifolds from Platonic solids*, Top. App. **138** (2004), 253–263. 2035484 (2004m:57031)

[15] B. Everitt, J. Ratcliffe and S. Tschantz, *Arithmetic hyperbolic 6-manifolds of smallest volume*, (in preparation).

[16] V. Matveev and A. Fomenko, *Constant energy surfaces of Hamilton systems, enumeration of three-dimensional manifolds in increasing order of complexity, and computation of volumes of closed hyperbolic manifolds*, Russian Math. Surveys **43** (1988), 3–24. MR0937017 (90a:58052)

[17] C. Cao and G. R. Meyerhoff, *The orientable cusped hyperbolic 3-manifolds of minimum volume*, Invent. Math. **146** (2001), no. 3, 451–478. MR1869847 (2002i:57016)

[18] J. Milnor and D. Husemoller, *Symmetric Bilinear Forms*, Springer, 1973. MR0506372 (58:22129)

[19] A. Neumaier and J. J. Seidel, *Discrete Hyperbolic Geometry*, Combinatorica **3** (1983), 219–237. MR0726460 (85h:51025)

[20] I. Prok, *Classification of dodecahedral space forms*, Beiträge Algebra Geom. **39** (1998), no. 2, 497–515. MR1642716 (99i:52029)

[21] J. Ratcliffe, *Foundations of hyperbolic manifolds*, Graduate Texts in Mathematics 149, Springer 1994. MR1299730 (95j:57011)

[22] J. Ratcliffe and S. Tschantz, *Volumes of integral congruence hyperbolic manifolds*, J. Reine Angew. Math. **488** (1997), 55–78. MR1465367 (99b:11076)

[23] J. Ratcliffe and S. Tschantz, *The volume spectrum of hyperbolic 4-manifolds*, Experiment. Math. **9** (2000), no. 1, 101–125. MR1758804 (2001b:57048)

[24] J. Ratcliffe and S. Tschantz, *Integral congruence two hyperbolic 5-manifolds*, Geom. Dedicata **107** (2004), 187–209. MR2110762

[25] J.-P. Serre, *A Course in Arithmetic*, Graduate Texts in Mathematics **7**, Springer, 1973. MR0344216 (49:8956)

[26] C. L. Siegel, *Über die analytische Theorie der quadratischen Formen II*, Ann. Math. **37** (1936), 230–263. MR1503276

[27] E. B. Vinberg, *Hyperbolic reflection groups*, Uspekhi Mat. Nauk **40** (1985), no. 1, 29–66; English transl., Russian Math. Surveys **40** (1985), no. 1, 31–75. MR0783604 (86m:53059)
[28] E. B. Vinberg and I. M. Kaplinskaya, *On the groups* $O_{18,1}(\mathbb{Z})$ *and* $O_{19,1}(\mathbb{Z})$, Dokl. Akad. Nauk. SSSR **238** (1978), no. 6, 1273–1275; English transl., Soviet Math. Dokl. **19** (1978), no. 1, 194–197. MR0476640 (57:16199)
[29] E. B. Vinberg, *On the groups of units of certain quadratic forms*, Math. Sb. **87** (1972), no. 1, 18–36; English transl., Math. USSR, Sb. **16** (1972), no. 1, 17–35. MR0295193 (45:4261)
[30] J. Weeks, *Hyperbolic structures on 3-manifolds*, Ph.D. thesis, Princeton University, 1985.

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