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INVERSE SOURCE PROBLEM IN A 3D BALL FROM BEST MEROMORPHIC APPROXIMATION ON 2D SLICES*

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Dedicated to Ed Saff on the occasion of his 60th birthday

Abstract. We show that the inverse monopolar or dipolar source problem in a 3D ball from overdetermined Dirichlet-Neumann data on the boundary sphere reduces to a family of 2D inverse branchpoint problems in cross sections of the sphere, at least when there are finitely many sources. We adapt from [7] an approach to these 2D inverse problem which is based on meromorphic approximation, and we present numerical results.

Key words. inverse source problems, potential theory, meromorphic approximation

AMS subject classifications. 31A25, 30E10, 30E25, 35J05

1. Introduction. In this paper we consider a classical inverse problem for the Laplace operator, that consists in recovering finitely many pointwise sources located in a 3D domain from measurements of their potential and its normal derivative on the boundary. This type of inverse problem dates back to Newton, and it has received renewed interest from the scientific community in recent years. For instance in medical engineering, the inverse EEG (ElectroEncephaloGraphy) problem consists in detecting pointwise dipolar current sources (modeling epileptic foci) located in the brain, from measures of the electric potential on the scalp [14, 17, 18, 19, 22, 33]. We shall study more generally the mixed case of monopolar and dipolar sources, which is reasonably comprehensive with respect to such applications.

We will show that if the 3D domain is a ball, then the above issue is equivalent to a sequence of 2D inverse problems each of which consists in recovering the branchpoints that a holomorphic function f has in a disk from the knowledge of f on the boundary circle. Subsequently, we apply to these 2D problems the technique of [7] that rests with approximating f on the boundary circle by a meromorphic function with n poles, and locating the branchpoints from the cluster of the poles when n goes large; initially, this technique was proposed for crack detection [9, 10]. Finally, we provide numerical experiments to illustrate our results.

The meromorphic approximants that we used for the experiments are local best approximants with n poles for the L^2-norm on the circle [4]. Several other kinds of approximants would do as well: a noteworthy alternative is best *uniform* approximation on the circle by meromorphic functions with n-poles, which is made constructive by the Adamjan-Arov-Krein theory [1].

The approach can be extended to smooth 3D domains whose sections along some axis are so called R-domains [16], but we shall not carry out such generalization here.

The outline is as follows. In section 2 we state the inverse source problem on the 3D ball, and in section 3 we perform the computations needed to reduce it to a family of inverse problems on 2D planar cross-sections of the ball. These 2D problems are in fact inverse branchpoint problems. To approach them in the style of [7], we recall in section 4 a few facts from potential theory and some results from [8, 11, 31, 32]. Numerical examples are presented in section 5.

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2. An inverse source problem. Let $\mathbb{B} \subset \mathbb{R}^3$ be the open unit ball and $\mathbb{S} = \partial\mathbb{B}$ the unit sphere. The electric potential P generated by m_1 pointwise monopolar sources $S_1, \ldots, S_{m_1}$ with intensities $\lambda_1, \ldots, \lambda_{m_1} \in \mathbb{R}$ and m_2 dipolar sources $C_1, \ldots, C_{m_2}$ with moments $p_1, \ldots, p_{m_2} \in \mathbb{R}^3$, is the unique solution that vanishes at infinity of the equation

$$-\Delta P = F,$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is the standard Laplacian, and F is the distribution:

$$F = \sum_{j=1}^{m_1} \lambda_j \, \delta_{S_j} + \sum_{k=1}^{m_2} p_k \, . \nabla \, \delta_{C_k} . \tag{2.1}$$

Assume now that all the sources S_j and C_k are located in $\mathbb{B}$, assumed to be filled with a homogeneous *medium*. From outside $\mathbb{B}$, one can usually observe P only up to the addition of a harmonic function, which is the potential generated by other sources located outside $\overline{\mathbb{B}}$. This additive combination, denoted by u, satisfies an equation of the form:

$$\text{(PS)} \quad \left\{ \begin{array}{l} -\Delta \, u = F \text{ in } \mathbb{B} \\ \dfrac{\partial u}{\partial \nu}_{|\mathbb{S}} = \phi \, , \;\; u_{|\mathbb{S}} = g \end{array} \right\}$$

where ϕ and g are the *boundary conditions* and ν represents the outward unit normal vector to $\mathbb{S}$ (normalized radius, in spherical coordinates).

The *direct* Neumann problem is that of finding u given ϕ and F. By Green's formula, the compatibility condition:

$$\int_{\mathbb{S}} \phi \, ds = - \sum_{j=1}^{m_1} \lambda_j \, . \tag{2.2}$$

must hold for u to exist, and then it is unique up to an additive constant.

The *inverse* Dirichlet-Neumann problem is that of recovering F given ϕ and g. More precisely, *postulating* that ϕ and g are linked by a relation of the form (PS) where F is as in (2.1), we want to recover $m_1, m_2, S_j, C_k, \lambda_j$ and p_k for $1 \leq j \leq m_1$, $1 \leq k \leq m_2$.

Uniqueness of a solution (*i.e.* identifiability) is established in [17], while stability results are available from [28, 33].

For applications to EEG like those mentioned above, one actually needs to handle a more involved situation. In fact, a simplified spherical model of the head [14, 17, 18, 19] assumes that it is a ball $\mathbb{B}_R \subset \mathbb{R}^3$ centered at 0 of *radius* R, which is the union of 3 disjoint homogeneous layers $\Omega_0, \Omega_1, \Omega_2$, namely the scalp, the skull, and the brain. Up to a rescaling, we may assume that $\Omega_0 = \mathbb{B}$, that $\Omega_1 = \mathbb{B}_r \setminus \overline{\mathbb{B}}$, and that $\Omega_2 = \mathbb{B}_R \setminus \overline{\mathbb{B}_r}$ where $1 < r < R$. The conductivity in each Ω_i will be constant and known, equal to σ_i. Then, if we define a piecewise constant function σ on $\mathbb{B}_R$ by $\sigma_{|\Omega_i} = \sigma_i$, we find (again up to the addition of a harmonic function) that the potential created by the monopolar sources (S_j, λ_j) and the dipolar sources (C_k, p_k) located inside Ω_0 is a solution to

$$\left\{ \begin{array}{l} -\nabla \cdot (\sigma \nabla u) = F \text{ in } \mathbb{B}_R \\ \dfrac{\partial u}{\partial \nu}_{|\mathbb{S}_R} = \phi_R \\ u_{|\mathbb{S}_R} = g_R, \end{array} \right.$$

of which (PS) is a sub-equation whose boundary conditions ϕ and g on $\mathbb{S}$ have to be computed from ϕ_R and g_R by solving recursively the Cauchy problem for the Laplace equation in the layers Ω_2 and Ω_1. Moreover, it is often the case that u cannot be measured on the whole of $\mathbb{S}_R$, but rather on some subset $K \subset \mathbb{S}_R$.

The full inverse problem to be solved in this context is then the following:

Given measurements of g_R and ϕ_R on K, find the number and the location of unknown pointwise monopolar and dipolar sources $S_j, C_k \in \mathbb{B}$ together with their intensities $\lambda_j \in \mathbb{R}$ and their moments $p_k \in \mathbb{R}^3$.

Actually, classical EEG models assume that there are only dipolar sources so that $m_1 = 0$, and that the system is isolated so that $\phi_R = 0$. These simplifications do not essentially modify the complexity of the problem.

Hereafter, we shall not consider the more general formulation given above that involves several layers Ω_j, and we shall stick to (PS) by simply assuming that ϕ and g have been computed on $\mathbb{S}$ from the knowledge of ϕ_R and g_R. Let us merely point out that such a computation is nontrivial, because the Cauchy problem for the Laplace equation is the prototype of an ill-posed problem. Regularisation schemes have been proposed in [24], see also [2] for an approach based on extremal problems for harmonic fields.

3. About the solution u. The radial fundamental solution of Laplace's equation in $\mathbb{R}^3$ is $(4\pi\|X\|)^{-1}$ [15, 21] hence the solution to (PS) assumes the form:

$$u(X) = h(X) - \sum_{j=1}^{m_1} \frac{\lambda_j}{4\pi \, \|X - S_j\|} + \sum_{k=1}^{m_2} \frac{< p_k \, , \, X - C_k >}{4\pi \, \|X - C_k\|^3} \tag{3.1}$$

where h is harmonic.

The first step that we take is to recover $h_{|\mathbb{S}}$ from the knowledge of $u_{|\mathbb{S}} = g$ and $\frac{\partial u}{\partial \nu}_{|\mathbb{S}} = \phi$. For this we use the well-known fact [3, 15] that u, which is harmonic in a neighborhood of $\mathbb{S}$ by assumption, can be uniquely written there in spherical coordinates as:

$$u(r,\sigma) = \sum_{k=0}^{\infty} \left(r^k \sum_{i=1}^{2k+1} \alpha_k^i Y_k^i(\sigma) \; + \; r^{-(k+1)} \sum_{i=1}^{2k+1} \beta_k^i Y_k^i(\sigma) \right) \; , \;\; r > 0, \; \sigma \in \mathbb{S},$$

where $(Y_k^i)_{1 \le i \le 2k+1}$ is an orthonormal basis of the spherical harmonics of order k in $\mathbb{R}^3$. Thus if the boundary data on $\mathbb{S}$ are

$$g(\sigma) = \sum_{k=0}^{\infty} \sum_{i=1}^{2k+1} \gamma_k^i Y_k^i(\sigma) \quad \text{and} \quad \phi(\sigma) = \sum_{k=0}^{\infty} \sum_{i=1}^{2k+1} \phi_k^i Y_k^i(\sigma) \, ,$$

it follows from Euler's equation for homogeneous functions, and the fact that the normal to the sphere at X is X itself, that the boundary relations in (PS) become:

$$\alpha_k^i \; + \; \beta_k^i \; = \; \gamma_k^i \qquad \text{and} \qquad k\alpha_k^i \; - \; (k+1)\beta_k^i \; = \; \phi_k^i \, ,$$

$$\text{hence} \qquad \alpha_k^i \; = \; \frac{(k+1)\gamma_k^i + \phi_k^i}{2k+1} \, .$$

This allows one to compute h and $u - h$ on $\mathbb{S}$; indeed, because h is harmonic in $\mathbb{B}$, it has vanishing coefficients of $r^{-(k+1)} Y_k^i$ for $k \geq 0$, and similarly for $u - h$, harmonic in $\mathbb{R}^3 \setminus \overline{\mathbb{B}}$, with vanishing coefficients of $r^k Y_k^i$. Thus:

$$h(\sigma) = \sum_{k=0}^{\infty} \sum_{i=1}^{2k+1} \alpha_k^i Y_k^i(\sigma), \qquad (u-h)(\sigma) = \sum_{k=0}^{\infty} \sum_{i=1}^{2k+1} \beta_k^i Y_k^i(\sigma). \tag{3.2}$$

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Subsequently, on replacing u by $u-h$, we assume throughout that $h=0$.

Next, we put:

$$(x_l, y_l, z_l)^T = \begin{cases} S_l \text{ for } l = j = 1, \ldots, m_1, \\ C_l \text{ for } l = k = m_1+1, \ldots, m_1+m_2, \end{cases}$$

and, for $l = 1, \ldots, m_1+m_2$, we let

$$\xi_l = x_l + iy_l \in D_l \subset \mathbb{C} \tag{3.3}$$

be the complex coordinate of the source S_l or C_l in the disk D_l centered at 0 of radius $r_l = (1-z_l^2)^{1/2}$. We also denote the dipolar moments in Euclidean coordinates by:

$$p_k = (p_{k,x}, p_{k,y}, p_{k,z}).$$

When we slice the ball $\mathbb{B}$ along a generic horizontal plane $\{z = z_p\}$, the intersection with $\mathbb{S}$ is a circle $\mathcal{C}_p$ of radius $r_p = (1-z_p^2)^{1/2}$. Throughout, the square-root is understood to be positive for positive arguments. We further let $h_{l,p} = z_p - z_l$ for $l \in \{j,k\} = \{1, \ldots, m_1+m_2\}$, and we put $\xi = x+iy$ to mean the complex variable in the generic plane $\{z = z_p\}$.

LEMMA 3.1. *If $\xi_j, \xi_k \neq 0$ for $1 \leq j \leq m_1$ and $1 \leq k \leq m_2$, then the restriction $u_{|\mathcal{C}_p}$ of u given by (3.1) is the trace on $\mathcal{C}_p$ of the function*

$$f_p(\xi) = \frac{i}{4\pi} \times \left[\sum_{j=1}^{m_1} \frac{\Lambda_{j,p}(\xi)}{(\xi - \xi_{j,p}^-)^{1/2}} - \sum_{k=1}^{m_2} \frac{R_{k,p}(\xi)}{(\xi - \xi_{k,p}^-)^{3/2}} \right], \quad \xi \in \mathbb{C} \setminus \{\xi_{l,p}^\pm\}, \tag{3.4}$$

where

$$\xi_{l,p}^\pm = \frac{\xi_l}{2\,|\xi_l|^2} \left\{ |\xi_l|^2 + r_p^2 + h_{l,p}^2 \pm \sqrt{[(|\xi_l| + r_p)^2 + h_{l,p}^2]\,[(|\xi_l| - r_p)^2 + h_{l,p}^2]} \right\}, \quad l \in \{j,k\},$$

and

$$\Lambda_{j,p}(\xi) = \frac{\lambda_j \sqrt{\xi}}{\sqrt{\overline{\xi_j}(\xi - \xi_{j,p}^+)}}, \tag{3.5}$$

while

$$R_{k,p}(\xi) = \frac{\sqrt{\xi}\,\left[\tilde{p}_k \xi^2 + 2(p_{k,z} h_{k,p} - Re\,\{\tilde{p}_k \xi_k\})\xi + \overline{\tilde{p}_k}\, r_p^2\right]}{2\left[\overline{\xi_k}(\xi - \xi_{k,p}^+)\right]^{3/2}} \tag{3.6}$$

with

$$\tilde{p}_k = p_{k,x} - ip_{k,y}\,.$$

If $\xi_j = 0$, that is if the monopolar source S_j lies on the vertical axis, then the corresponding term of the first sum appearing in the bracket on the right hand side of (3.4) becomes independent of ξ and given by

$$\frac{i\lambda_j}{\sqrt{r_p^2 + h_{j,p}^2}}\,. \tag{3.7}$$

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If $\xi_k = 0$, that is if the dipolar source C_k lies on the vertical axis, then the corresponding term of the second sum appearing in the bracket on the right hand side of (3.4) becomes rational in ξ and is given by

$$i\frac{\tilde{p}_k \xi^2 + 2p_{k,z}h_{k,p}\xi + \overline{\tilde{p}}_k r_p^2}{2\xi(r_p^2 + h_{k,p}^2)^{3/2}} . \tag{3.8}$$

Proof. Put

$$\tilde{Q}_{l,p}(x,y) = (x - x_l)^2 + (y - y_l)^2 + h_{l,p}^2 \ , \ \ l \in \{j,k\},$$

for the trace of $\|X - S_l\|^2$ or $\|X - C_l\|^2$ at $z = z_p$, and

$$\tilde{P}_{k,p}(x,y) = p_{k,x}\,(x - x_k) + p_{k,y}\,(y - y_k) + p_{k,z}\,h_{p,k} \ ,$$

From (3.1), we get

$$u(x,y,z_p) = \frac{1}{4\pi} \times \left[-\sum_{j=1}^{m_1} \frac{\lambda_j}{\sqrt{\tilde{Q}_{j,p}(x,y)}} + \sum_{k=1}^{m_2} \frac{\tilde{P}_{k,p}(x,y)}{\sqrt{\tilde{Q}_{k,p}^3(x,y)}} \right] . \tag{3.9}$$

Now, for $\xi \in \mathcal{C}_p$ we have that $\overline{\xi} = r_p^2/\xi$, hence

$$\tilde{Q}_{l,p}(x,y) = |\xi - \xi_l|^2 + h_{l,p}^2 \triangleq Q_{l,p}(\xi), \ \ l \in \{j,k\}, \tag{3.10}$$

and if $\xi_l \neq 0$ this gives us

$$Q_{l,p}(\xi) = -\frac{\overline{\xi_l}}{\xi}\left(\xi^2 - \left[1 + \frac{r_p^2 + h_{l,p}^2}{|\xi_l|^2}\right]\xi_l\xi + \frac{r_p^2\,\xi_l^2}{|\xi_l|^2}\right) = -\frac{\overline{\xi_l}}{\xi}\,(\xi - \xi_{l,p}^-)\,(\xi - \xi_{l,p}^+) \tag{3.11}$$

where $\xi_{l,p}^{\pm}$ is as stated above. Similarly, we observe that for $\xi \in \mathcal{C}_p$

$$\tilde{P}_{k,p}(x,y) = \frac{1}{2\xi}\left[\tilde{p}_k\xi^2 + 2(p_{k,z}h_{k,p} - \mathrm{Re}\,\{\tilde{p}_k\xi_k\})\xi + \overline{\tilde{p}_k}\,r_p^2\right] \triangleq P_{k,p}(\xi) \ ,$$

and if we put

$$R_{k,p}(\xi) = \frac{P_{k,p}(\xi)}{(\xi - \xi_{j,p}^+)^{3/2}}\left(\frac{\xi}{\overline{\xi_k}}\right)^{3/2} \tag{3.12}$$

we see that (3.6) holds. Then (3.4) follows upon substituting (3.11) and (3.12) in (3.9).

If $\xi_l = 0$, then $Q_{l,p}(\xi) = |\xi|^2 + h_{l,p}^2 = r_p^2 + h_{l,p}^2$ for $\xi \in \mathcal{C}_p$, and the conclusion in this case follows from a similar computation. $\square$

When there is a *single* source, that is if $m_1 + m_2 = 1$, then we see from Lemma 3.1 that f_p^2 is a rational function of ξ. When there are several sources it is no longer the case, and this is why we introduce a class of analytic functions as follows.

Let $\mathbb{D}$ be the unit disk and, for $\varepsilon > 0$, define $\mathcal{A}_\varepsilon$ to be the annulus:

$$\mathcal{A}_\varepsilon = \{z;\ 1 - \varepsilon < |z| < 1\}. \tag{3.13}$$

Consider the class of functions:

$$\mathcal{BP} \triangleq \{F \text{ continuous in } \overline{\mathcal{A}_\varepsilon} \text{ and holomorphic in } \mathcal{A}_\varepsilon \text{ for some } \varepsilon > 0, \text{ and } F \text{ can be continued analytically in } \mathbb{D} \text{ except for finitely many poles and branchpoints}\} .$$

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PROPOSITION 3.2. *For each $z_p \in (-1,1)$, the function f_p in (3.4) is such that $f_p^2(\xi/r_p)$ belongs to $\mathcal{BP}$, with poles and branchpoints at the points $\xi_{l,p}^-/r_p$ only.*

Proof. The hypothesis that $z_p \in (-1,1)$ is to the effect that $r_p \neq 0$. Now, by (3.11), we see that for $l \in \{j,k\}$

$$\left|\xi_{l,p}^+ \xi_{l,p}^-\right| = \left|\frac{r_p^2\,\xi_l^2}{|\xi_l|^2}\right| = r_p^2,$$

and since $\xi_{l,p}^+$ and $\xi_{l,p}^-$ are zeros of $Q_{l,p}(\xi)$ it follows from (3.10) that none of them has modulus r_p because this would imply $|\xi_l| = r_p$, a contradiction since the S_j and the C_k lie inside $\mathbb{B}$ by assumption. Consequently one of the two complex numbers $\xi_{l,p}^\pm$ must have modulus bigger than r_p, and the other modulus smaller than r_p; by the formula for $\xi_{l,p}^\pm$ in Lemma 3.1, the biggest modulus is in fact that of $\xi_{l,p}^+$ hence

$$\left|\xi_{l,p}^-\right| < r_p < \left|\xi_{l,p}^+\right|. \tag{3.14}$$

From this and (3.5)–(3.6), it follows when no ξ_j nor ξ_k is zero that, for any quadruple of indices $j,j' \in \{1,\ldots,m_1\}$ and $k,k' \in \{1,\ldots,m_2\}$, the three functions $\Lambda_{j,p}\Lambda_{j',p}$, $R_{k,p}R_{k',p}$, and $\Lambda_{j,p}\,R_{k,p}$ are holomorphic in the disk of radius r_p, because the terms $\sqrt{\xi}$ in the numerators of (3.5)-(3.6) get multiplied, while $\xi - \xi_{j,p}^+$ does not vanish in $|\xi| < r_p$ and therefore has a well-defined square-root there. Expanding f_p^2 using (3.4) now shows that $f_p^2(\xi/r_p) \in \mathcal{BP}$ with branchpoints in $\mathbb{D}$ at the $\xi_{l,p}^-$ only. Finally, if some of the ξ_j or ξ_k are zero, the corresponding terms in the right hand-side of (3.4) have to be replaced by (3.7) or (3.8). The previous reasoning still applies, except this time that multiplying (3.7) (resp. (3.8)) by terms like (3.5) or (3.6) will generate a branchpoint at 0 of order $1/2$ (resp. $-1/2$), and that multiplying together terms like (3.7) and (3.8) will generate poles at 0. $\square$

The next proposition connects the location of the singularities $\xi_{l,p}^-$, that the analytic function $f_p^2(\xi)$ has in the disk of radius r_p, with the original sources S_j and C_k:

PROPOSITION 3.3. *Assume that $\xi_l \neq 0$. Then, for $z_p \in (-1,1)$, the argument of the complex number $\xi_{l,p}^-$ defined in the statement of Lemma 3.1 is independent of z_p, and equal to the argument of ξ_l defined in (3.3). The modulus of $\xi_{l,p}^-$ increases monotonically for $z_p < z_l$, decreases monotonically for $z_l < z_p$, and attains a maximum when $z_p = z_l$ in which case one has $\xi_{l,p}^- = \xi_l$.*

Proof. Put $\xi = t\,\xi_l/|\xi_l|$ in (3.11); this gives us

$$Q_{l,p}(\xi) = -\frac{\xi_l}{\xi} q_{l,p}(t)\ , \quad \text{with}\ \ q_{l,p}(t) \triangleq t^2 - \left[1 + \frac{r_p^2 + h_{l,p}^2}{|\xi_l|^2}\right] |\xi_l|\, t + r_p^2. \tag{3.15}$$

Let $t_{l,p}^\pm$ denote the roots of $q_{l,p}(t)$. Obviously $t_{l,p}^\pm = \xi_{l,p}^\pm |\xi_l|/\xi_l$, thus by Lemma 3.1

$$t_{l,p}^\pm = \frac{1}{2\,|\xi_l|}\left\{|\xi_l|^2 + r_p^2 + h_{l,p}^2 \pm \sqrt{[(|\xi_l| + r_p)^2 + h_{l,p}^2]\,[(|\xi_l| - r_p)^2 + h_{l,p}^2]}\right\}, \tag{3.16}$$

and by (3.14),

$$\left|t_{l,p}^-\right| < r_p < \left|t_{l,p}^+\right|. \tag{3.17}$$

But the polynomial $q_{l,p}$ has real coefficients while (3.17) indicates that its roots have distinct *moduli*, hence they must be real as well. Also, they are positive as their sum and their product is positive. By the very definition of $t_{l,p}^\pm$ this implies

$$|\xi_{l,p}^-| = t_{l,p}^- \ \text{ and } \ \xi_{l,p}^\pm/\xi_l \in \mathbb{R}^+,$$

which entails that the argument of $\xi_{l,p}^-$ is the same as the argument of ξ_l. Moreover, since $q_{l,p}(t)$ is monic while

$$q_{l,p}(|\xi_l|) = -h_{l,p}^2 \leq 0, \tag{3.18}$$

we necessarily have that $t_{l,p}^- \leq |\xi_l| \leq t_{l,p}^+$, and it is easily seen from (3.16) that equality holds on the left if, and only if $h_{l,p} = 0$. Therefore $t_{l,p}^- = |\xi_l|$ is a maximum value for $t_{l,p}^-$, and it is attained for $z_p = z_l$.

Finally we observe that $t_{l,p}^-$ is differentiable with respect to z_p because the roots of a polynomial ($q_{l,p}(t)$ in our case) are smooth functions of the coefficients away from multiple roots. Differentiating with respect to z_p the equality

$$q_{l,p}(t_{l,p}^-) = 0,$$

we obtain in view of (3.15) on taking into account the relation $r_p^2 + z_p^2 = 1$ that:

$$q_{l,p}'(t_{l,p}^-)\,\frac{dt_{l,p}^-}{dz_p} + \frac{2z_l}{|\xi_l|}t_{l,p}^- - 2z_p = 0. \tag{3.19}$$

Because $q_{l,p}(t)$ is monic and $t_{l,p}^-$ is its smallest root, it holds that $q_{l,p}'(t_{l,p}^-) < 0$, and since we saw that $0 < t_{l,p}^-/|\xi_l| \leq 1$ it follows from (3.19) that $dt_{l,p}^-/dz_p < 0$ when $z_p > z_l$. Then, on changing z_p into $-z_p$ and z_l into $-z_l$ (*i.e.* reflecting the geometry across the (x, y)-plane), the same argument shows that $dt_{l,p}^-/dz_p > 0$ when $z_p < z_l$. Hence $t_{l,p}^-$ increases with z_p for $z_p < z_l$ and decreases for $z_p > z_l$, as desired. $\square$

When no source lies on the vertical axis, Propositions 3.2 and 3.3 show that the inverse source problem stated in section 2 can be traded for a family of inverse 2D problems, each of which consists in recovering the branchpoints of an analytic function in a disk. Indeed, if we were able to locate the $\xi_{l,p}^-$ given f_p and z_p, then we could use dichotomy on the latter to estimate z_l as a maximal place for $|\xi_{l,p}^-|$, and for $z_p = z_l$ we would compute $\xi_l = \xi_{l,p}^-$. Of course, by spherical symmetry, the vertical axis can be chosen arbitrarily, so the preceding situation is generic with respect to that choice unless there is a source at the origin.

If some of the sources lie on the vertical axis, we saw in the proof of Proposition 3.2 that f_p^2 has branchpoints and/or poles at 0. In this case the previous approach remains valid to locate the other sources, and the occurrence of a branchpoint or a pole at 0 indicates that there is at least one source on the vertical axis, but it gives no direct clue about its exact location. Chasing the sources located on the vertical axis, if any, is outside the scope of the present paper, as it would require slightly more refined tools in singularity detection than we intend to use. Thus, we stick to the generic case which amounts to a family of inverse branchpoint problems.

To the authors knowledge, this inverse branchpoint problem has received little attention so far. In the next section we develop an approach to it which is based on meromorphic approximation.

4. Best meromorphic approximation and the behaviour of poles. In this section, we discuss without proofs some results from meromorphic approximation that we use to approach the inverse branchpoint problem in the style of [7]. We begin with standard facts from potential theory for which the reader may want to consult [29].

4.1. Some potential theory. Recall that $\mathbb{D}$ stands for the unit disk and let $\mathbb{T}$ be the unit circle. The *Green capacity* of a compact set $K \subset \mathbb{D}$ is the non-negative real number $C_{\mathbb{T},K}$

such that

$$\frac{1}{C_{\mathbb{T},K}} = \inf_{\mu \in \mathcal{P}_K} \int\int \log \left| \frac{1-\bar{t}z}{z-t} \right| \, d\mu(t) d\mu(x), \tag{4.1}$$

where $\mathcal{P}_K$ stands for the set of all probability measures supported on K. Some sets are so thin that $C_{\mathbb{T},K} = 0$, and they are referred to as *polar sets*. A countable set is polar, but there are also uncountable polar sets. A polar set has to look bad from the topological viewpoint: for instance it is totally disconnected, and no positive measure of finite energy (*i.e.* no measure $\mu \geq 0$ such that the integral in the right-hand side of (4.1) is finite) can have nonzero mass on it. In particular polar sets have zero planar Lebesgue measure.

If K is non-polar, that is if $C_{\mathbb{T},K} > 0$, then there is a unique measure $\omega_K \in \mathcal{P}_K$ to meet the above infimum, and it is called the *Green equilibrium measure of* K. For example, the Green equilibrium measure of a segment $[a,b] \subset (0,1)$ is given by

$$d\omega_{[a,b]}(t) = \frac{C\,dt}{\sqrt{(1-bt)(b-t)(1-at)(t-a)}} \tag{4.2}$$

where C is a normalizing constant. The measure ω_K is difficult to compute in general, but it charges more the edges of K, in particular it charges the endpoints of a system of arcs (compare (4.2) where the density is infinite at the endpoints a and b).

We shall need the notion of extremal domain, which is specialized below to the case of a disk. To interpret the statement, one should recall definition (3.13).

THEOREM 4.1 ([31]). *Let F be holomorphic in $\mathcal{A}_\varepsilon$ and continuous in $\overline{\mathcal{A}_\varepsilon}$. Set*

$$\mathcal{V}_F = \{V;\ V \text{ connected open in } \overline{\mathbb{D}} \text{ with } \overline{\mathcal{A}_\varepsilon} \subset V,\ \ F \text{ extends holomorphicaly to } V\}.$$

Then, there is a unique $V_F \in \mathcal{V}_F$ satisfying

$$C_{\mathbb{T},\mathbb{D}\setminus V_F} = \inf_{V \in \mathcal{V}_F} C_{\mathbb{T},\mathbb{D}\setminus V}$$

and containing every other member of $\mathcal{V}_F$ with this property.

For functions in $\mathcal{BP}$ (see the definition before Proposition 3.2), more is known on the structure of extremal domains.

THEOREM 4.2 ([32]). *If $F \in \mathcal{BP}$, then $\overline{\mathbb{D}} \setminus V_F$ consists of its poles, its branchpoints, and finitely many analytic cuts. A cut ends up either at a branchpoint or at an end of another cut. The diagram thus formed has no loop.*

When F has only two branchpoints and possibly poles, $\overline{\mathbb{D}} \setminus V_F$ consists of the poles and of a hyperbolic geodesic arc joining the branchpoints. For more than two branchpoints, $\overline{\mathbb{D}} \setminus V_F$ is, apart from the poles, a trajectory of a rational quadratic differential, but there is no easy computation. The situation is similar to that in problems of Tchebotarev-Lavrentiev type, where one must find the continuum of minimal capacity that connects prescribed groups of points [25, 26]. The difference here is that the connectivity is not known *a priori*, but rather induced by the function F.

4.2. Best meromorphic approximation. We need some more notations.

Let $\mathcal{P}_n$ be the space of algebraic polynomials of degree $\leq n$ with complex coefficients, H^2 and H^∞ the familiar Hardy spaces of the disk. We introduce a set of meromorphic function with n poles in $\mathbb{D}$ by setting:

$$H^p_n = \{h/q_n;\ h \in H^p,\ q_n \in \mathcal{P}_n\}, \quad p = 2, \infty.$$

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By a best meromorphic approximant with at most n poles of F in the $L^p(\mathbb{T})$-sense, we mean some $g_n \in H_n^p$ such that :

$$\|F - g_n\|_{L^p(\mathbb{T})} = \inf_{g \in H_n^p} \|F - g\|_{L^p(\mathbb{T})}.$$

Such approximants are studied more generally in [5] for every $p \geq 2$.

For $p = \infty$, by the Adamjan-Arov-Krein theory [1], a best meromorphic approximant with at most n poles uniquely exists provided that $F \in C(\mathbb{T})$. Moreover, it can be computed from the singular value decomposition of the Hankel operator with symbol F.

For $p = 2$, existence and non-uniqueness are discussed in [6, 4, 12, 13] for $F \in L^2(\mathbb{T})$. Concerning constructive aspects, algorithms to generate local minima can be obtained using Schur parameterization [27].

If g_n is the sequence of best meromorphic approximants with at most n poles of F in the $L^p(\mathbb{T})$-sense, whose poles are d_n in number, $d_n \leq n$, and denoted by $\zeta_{j,n}$ for $1 \leq j \leq d_n$, we form the sequence of counting probability measures

$$\mu_n = \sum_{j=1}^{d_n} \delta_{\zeta_{j,n}} / d_n.$$

The next theorem refers to weak* convergence of the sequence μ_n. Recall that a sequence ν_n of (generally complex) measures converges weak* to the measure ν if, and only if

$$\int h \, d\nu_n \overset{n\to\infty}{\longrightarrow} \int h \, d\nu$$

for each complex continuous function h with compact support. When it comes to the sequence of counting probability measures μ_n introduced above, this definition is equivalent to the fact that, for any open subset $O \subset \mathbb{C}$, the proportion of $\zeta_{j,n}$ contained in O converges to $\mu(O)$, where μ is the weak* limit (which is necessarily a probability measure as well).

THEOREM 4.3 ([11]). *If $F \in \mathcal{BP}$ is not single-valued, then with the above notations the measure μ_n converges weak* to the Green equilibrium distribution of $\overline{\mathbb{D}} \setminus V_F$. Moreover, only finitely many poles can remain in a fixed compact subset of V_F as $n \to \infty$.*

This result can be used to approach the inverse branchpoint problem in that the counting measure μ_n can be computed for increasing values of n, and will asymptotically charge the endpoints of $\overline{\mathbb{D}} \setminus V_F$ that include the branchpoints by Theorem 4.2, for the equilibrium density is infinite there.

In fact, it can be shown that the equilibrium density is infinite exactly at the branchpoints of F. When $p = 2$ the result remains valid for *local best approximants* and not just for best approximants, which is important from the computational viewpoint since local minima is all one can guarantee in general from a numerical search. Convergence results of the same type, together with sharper estimates, are carried out in [8, 10] for the case of two branchpoints.

The weak spot of the method is of course that one can only compute meromorphic approximants for limited values of n, so the efficiency of the approach depends crucially on how fast the poles do converge. When there are two branchpoints, non-asymptotic bounds to be found in [8] remedy this situation to some extent, by bounding from below the angles under which the poles "see" the set $\overline{\mathbb{D}} \setminus V_F$. In the general case, however, no estimates of this kind are available at present.

5. Algorithms and numerical experiments.

The following examples numerically illustrate the results of sections 3 and 4.

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Sources are only dipoles inside the unit ball, defined by their support $C_k \in \mathbb{R}^3$ and their moment $p_k \in \mathbb{R}^3$ (see expression (2.1) with $m_1 = 0$).

The unit ball $\mathbb{B}$ is intersected by 39 equidistant parallel planes, orthogonal to the z-axis, so that we handle a family of 2D-disks $D_p, p = -19, \ldots, 19$.

Each of the following figures shows two views of $\mathbb{B}$ and of the circles $\mathcal{C}_p$ (dashed lines): the left hand side one is seen from a distant point of the (x, y)-plane, while the right hand side one is taken from above, at a point of the z axis.

For the examples of figures 5.1, 5.2, 5.3, we computed the solution u from its expression (3.1) with $h = 0$ (and $m_1 = 0$) so that on each circle $\mathcal{C}_p$, we get $u_{|\mathcal{C}_p} = f_p = f$ in (3.4).

In figures 5.4 and 5.5, the solution u of the Neumann problem associated to (PS) with $\phi = 0$ is numerically computed using an external solver for Laplace equation [23]. The trace of u on $\mathbb{S}$ is projected on a finite basis of spherical harmonics ($degree = 20, dimension = 441$), and the harmonic part h is removed accordingly to equations (3.2) so that we get $(u-h)_{|\mathcal{C}_p} = f_p = f$ as in previous examples.

In all cases, we then compute for each p a best meromorphic approximant to the function f^2 on $\mathcal{C}_p$. More precisely, we use here local best approximants with n poles for the L^2-norm on the circle that are computed using a descent algorithm[1] [27].

For a number m_2 of dipoles, $m_2 = 1, 2, 4$, figures 5.1, 5.2, 5.3 show the sources positions (big dots) and their (exactly known for these simulations) associated branching lines (linking the $\xi^-_{k,p}$ for $p = -19, \ldots, 19$. The poles (small dots) of the rational approximants accumulate near the branching lines when $m_2 = 2, 4$, while they exactly match the branching line in the case $m_2 = 1$, where f^2 is rational.

In figures 5.4 and 5.5, because $m_2 = 1$, f^2 is again a rational function of degree 3, and the localisation of the source can be performed "automatically", using Proposition 3.3, pretty efficiently. Figure 5.4 shows the 3 poles of each 2D-approximant, when figure 5.5, for the same example, shows only one point in each disk, which is the barycenter of the poles.

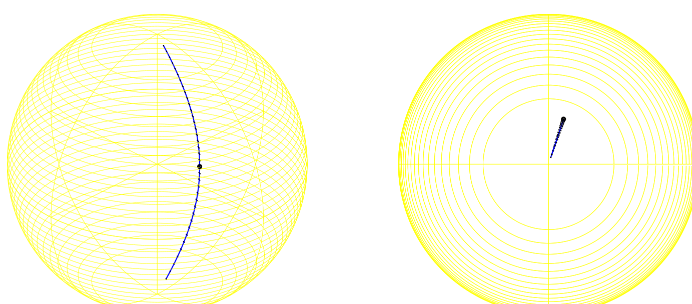

FIG. 5.1. *One source - exact potential - $n = 3$ rational approximant - all poles*

6. Conclusion. In this paper, we have shown and numerically illustrated how the inverse branchpoint problem in 2D can be used to approach the inverse source problem in 3D for a sphere.

It would be quite interesting to single out other classes of 3D domains for which the same approach may work. A natural extension is to domains whose slices along some axis

[1]A brief description of the numerical procedure is available at http://www-sop.inria.fr/apics/RARL2/fichetechnique.html.

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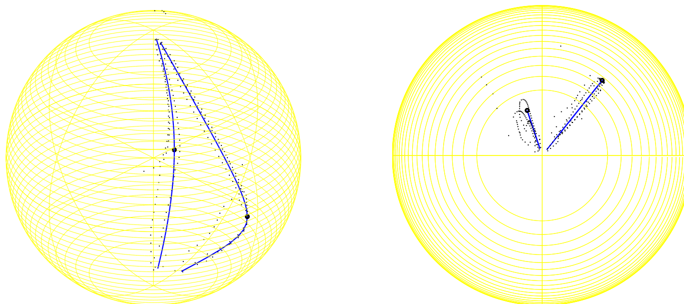

FIG. 5.2. *Two sources - exact potential -* $n = 6$ *rational approximant - all poles*

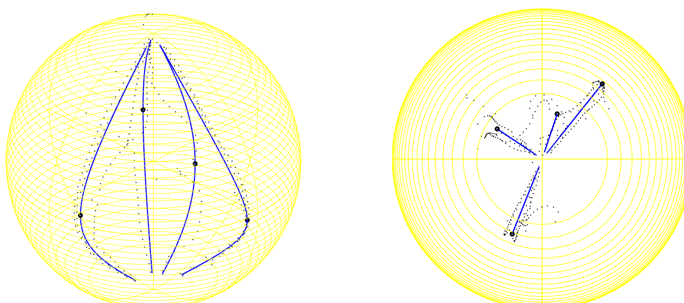

FIG. 5.3. *Four sources - exact potential -* $n = 8$ *rational approximant - all poles*

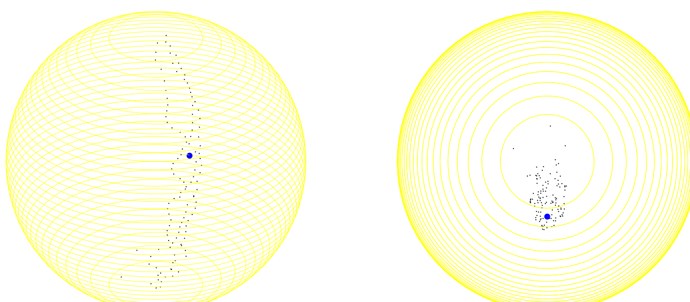

FIG. 5.4. *One source - PDE simulated data -* $n = 3$ *rational approximant - all poles*

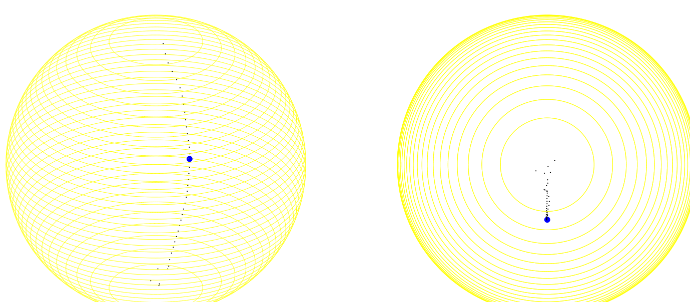

FIG. 5.5. *One source - PDE simulated data -* $n = 3$ *rational approximant - poles barycenters*

are 2D domains, the boundary of which are Jordan curves with algebraic Schwarz function [16, 30]. Actually, the reader can check this is the crucial property which is used (in the case of a disk the Schwarz function is rational). In this connection, one may hope that the theory of quadrature domains can be applied efficiently to 3D inverse source problems, although the theoretical and computational difficulties facing the approach are yet far from being solved [20].

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