New results in extremal problems with polynomials

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Abstract

In this paper we give some new results in extremal problems with polynomials which are generalization of some results of F. Locher.

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In [6] F. Locher studies some extremal problems for semidefinition polynomials (nenegative or nepositive) with the dominant coefficient equal to 1. For this he uses the quadrature formulae with high algebraical degree of exactness.

Thus, using Gauss - Jacobi؛s quadrature formula, he proves:

Proposition 1. *For any polynomial $p_{2m}(x) \geq 0$, $x \in [-1,1]$, with the dominant coefficient equal to 1, the inequality*

$$\int_{-1}^{1} (1-x)^\alpha (1+x)^\beta p_{2m}(x)dx \geq \frac{2^{\alpha+\beta+2m+1} \cdot m!\Gamma(\alpha+m+1)}{(\alpha+\beta+2m+1)}.$$

$$\cdot \frac{\Gamma(\beta + m + 1) \cdot \Gamma(\alpha + \beta + m + 1)}{(\Gamma(\alpha + \beta + 2m + 1))^2}$$

is true, in which the equal sign is reached only if

$$p_{2m}(x) = \frac{2^m m! \Gamma(\alpha + \beta + m + 1)}{(\Gamma(\alpha + \beta + 2m + 1))^2} \cdot \left(p_m^{(\alpha,\beta)}(x)\right)^2.$$

Proposition 2. *For any polynomial $p_{2m+1}(x) \geq 0$, $x \in [-1, 1]$, of the degree $2m + 1$ and with the dominant coefficient equal to 1, the inequality*

$$\int_{-1}^{1} (1 - x)^\alpha (1 + x)^\beta p_{2m+1}(x) dx \geq$$

$$\geq \frac{2^{\alpha+\beta+2m+2} m! \Gamma(\alpha + m + 1) \Gamma(\beta + m + 2) \Gamma(\alpha + \beta + m + 2)}{(\alpha + \beta + 2m + 2)(\Gamma(\alpha + \beta + 2m + 2))^2}$$

is valid, with the equal sign reached for

$$p_{2m+1}(x) = \left(\frac{2^m m! \Gamma(\alpha + \beta + m + 2)}{\Gamma(\alpha + \beta + 2m + 2)}\right)^2 (x + 1) \left(p_m^{(\alpha,\beta+1)}(x)\right)^2.$$

Proposition 3. *For any polynomial $p_{2m+1}(x) \leq 0$, $x \in [-1, 1]$ of the degree $2m + 1$ and with the dominant coefficient equal to 1, the inequality*

$$\int_{-1}^{1} (1 - x)^\alpha (1 + x)^\beta p_{2m+1}(x) dx \leq$$

$$\leq -\frac{2^{\alpha+\beta+2m+2} m! \Gamma(\alpha + m + 2) \Gamma(\beta + m + 1) \Gamma(\alpha + \beta + m + 2)}{(\alpha + \beta + 2m + 2)(\Gamma(\alpha + \beta + 2m + 2))^2}$$

is true, with the equal sign reached for

$$p_{2m+1}(x) = \left(\frac{2^m m! \Gamma(\alpha + \beta + m + 2)}{\Gamma(\alpha + \beta + 2m + 2)}\right)^2 (x - 1) \left(p_m^{(\alpha+1,\beta)}(x)\right)^2.$$

Proposition 4. *For any polynomial $p_{2m}(x) \leq 0$, $x \in [-1, 1]$, of the degree $2m$ and with the dominant coefficient equal to 1, we have the inequality*

$$\int_{-1}^{1} (1-x)^\alpha (1+x)^\beta p_{2m}(x) dx \leq$$

$$\leq -\frac{2^{\alpha+\beta+2m+1}(m-1)!\Gamma(\alpha+m+1)\Gamma(\beta+m+1)\Gamma(\alpha+\beta+m+2)}{(\alpha+\beta+2m+1)(\Gamma(\alpha+\beta+2m+1))^2}$$

with equality only if

$$p_{2m}(x) = \left(\frac{2^{m-1}(m-1)!\Gamma(\alpha+\beta+m+2)}{\Gamma(\alpha+\beta+2m+1)} \right)^2 (x^2-1) \left(p_{m-1}^{(\alpha+1,\beta+1)}(x) \right)^2.$$

In [2] D. Acu proves:

Proposition 5. *For each polynomial $p_{2m}(x) \geq 0$, $x \geq 0$ of degree $2m$ and with the dominant coefficient equal with 1 the inequality*

$$\int_0^\infty x^\alpha e^{-x} p_{2m}(x) dx \geq m!\Gamma(\alpha+m+1), \ \alpha > -1$$

is valid, with equality only if

$$p_{2m}(x) = (m!)^2 \left(P_m^{(\alpha)}(x) \right)^2,$$

where $P_m^{(\alpha)}(x)$ is the Legendre polynomial.

Proposition 6. *For each polynomial $p_{2m+1}(x) \geq 0$, $x \geq 0$, with the dominant coefficient equal with 1, the inequality*

$$\int_0^\infty x^\alpha e^{-x} p_{2m+1}(x) dx \geq m!\Gamma(\alpha+m+2)$$

is valid, with equality only if

$$p_{2m+1}(x) = (m!)^2 x (P_m^{(\alpha+1)}(x))^2,$$

where $P_m^{(\alpha+1)}(x)$ is the polynomial of degree m from the system of orthogonal polynomials on the interval $[0, \infty)$ referring to the weight $x^{\alpha+1} e^{-x}$.

Let w a positive integrable weight function, $x \in [-1, 1]$.

In [3] a part of above results are generalized thus:

Proposition 7. *For each polynomial $p_{2m}(x) \geq 0$, $x \in [-1, 1]$, degree $2m$ and with the dominant coefficient equal 1, the inequality*

$$\int_{-1}^{1} w(x) p_{2m}(x) dx \geq \frac{1}{a_m^2} \int_{-1}^{1} w(x) Q_m^2(x) dx$$

is valid, with equality only if

$$p_{2m}(x) = \frac{1}{a_m^2} Q_m^2(x)$$

where $Q_m(x)$ is the polynomial of degree m with dominant coefficient a_m, from the system of orthogonal polynomials on the interval $[-1, 1]$ referring to the weight $w(x)$.

In this paper we shall give some new generalizations for the above results.

Proposition 8. *For each polynomial $p_{2m+1}(x) \geq 0$, $x \in [-1, 1]$, of the degree $2m + 1$ and with the dominant coefficient equal to 1, the inequality*

$$\int_{-1}^{1} w(x) p_{2m+1}(x) dx \geq \frac{1}{a_m^2} ||Q_m(x)||^2$$

is valid, with equality only if

$$p_{2m+1}(x) = \frac{1}{a_m^2}(x+1)Q_m^2(x)$$

where $Q_m(x)$ is the polynomial of degree m, with the dominant coefficient a_m, from the system of orthogonal polynomials on the interval $[-1,1]$ referring to the weight $w(x)(x+1)$.

Proof. The proof of this proposition results of the Gauss - Radau quadrature formula ([4]):

$$\int\limits_{-1}^{1} f(x)w(x)dx = Bf(-1) + \sum_{i=1}^{m} A_i f(x_i) + R_{2m+1}(f),$$

where $B > 0$, $A_i > 0$, $i = \overline{1,m}$ and

$$R_{2m+1}(x) = \frac{f^{(2m+1)}(\xi)}{(2m+1)!a_m^2}\int_{-1}^{1} w(x)(x+1)Q_m^2(x)dx > 0, \ \xi \in (-1,1).$$

Remark 1. *For $w(x) = (1-x)^\alpha(1+x)^\beta$, $x \in (-1,1)$, $\xi \in (-1,1)$, $\alpha > -1$, $\beta > -1$, from the Proposition 8 it results the inequality from the Proposition 2.*

Proposition 9. *For each polynomial $p_{2m+1}(x) \leq 0$, $x \in [-1,1]$, of the degree $2m+1$ and with the dominant coefficient equal 1, the inequality*

$$\int\limits_{-1}^{1} w(x)p_{2m+1}(x)dx \leq -\frac{1}{a_m^2}||Q_m(x)||^2,$$

is valid, with equality only if

$$p_{2m+1}(x) = -\frac{1}{a_m^2}(1-x)Q_m^2(x),$$

where $Q_m(x)$ is the polynomial of degree m, with the dominant coefficient a_m, from the system of orthogonal polynomials on the interval $[-1, 1]$ referring to the weight $(1 - x)w(x)$.

Proof. Using the results from [4], we construct the Gauss - Radau quadrature formula

$$\int\limits_{-1}^{1} f(x)w(x)dx = Bf(1) + \sum_{i=1}^{m} A_i f(x_i) + R_{2m+1}(f)$$

with $B > 0$, $A_i > 0$, $i = \overline{1, m}$ and

$$R_{2m+1}(f) = -\frac{f^{(2m+1)}(\xi)}{(2m+1)!a_m^2} \int\limits_{-1}^{1} w(x)(1 - x)Q_m^2(x)dx, \ \xi \in (-1, 1).$$

Remark 2. *For $w(x) = (1 - x)^\alpha(1 + x)^\beta$, $\alpha > -1$, $\beta > -1$, $x \in (-1, 1)$ from the Proposition 9 we obtain the Proposition 3.*

Proposition 10. *For each polynomial $p_{2m+2}(x) \leq 0$, $x \in [-1, 1]$, of the degree $2m + 2$ and with the dominant coefficient equal 1, the inequality*

$$\int\limits_{-1}^{1} w(x)p_{2m+2}(x)dx \leq -\frac{1}{a_m^2}||Q_m(x)||^2$$

is valid, with equality only if

$$p_{2m+2}(x) = \frac{(x^2 - 1)}{a_m^2}Q_m^2(x)$$

where $Q_m(x)$ is the polynomial of degree m with the dominant coefficient a_m, from the system of orthogonal polynomial on the interval $[-1, 1]$ referring to the weight $(x + 1)(1 - x)w(x)$.

Proof. For the proof, using the results from [4], we construct the Gauss - Lobatto quadrature formula ([4])

$$\int_{-1}^{1} f(x)w(x)dx = Bf(-1) + Df(1) + \sum_{i=1}^{m} A_i f(x_i) + R_{2m+2}(f)$$

with $B > 0, D > 0, A_i > 0, i = \overline{1,m}$ and

$$R_{2m+2}(f) = -\frac{f^{(2m+2)}(\xi)}{(2m+2)!a_m^2} \int_{-1}^{1} w(x)(1-x^2)Q_m^2(x)dx, \ \xi \in (-1,1).$$

Remark 3. *For $w(x) = (1-x)^\alpha(1+x)^\beta$, $x \in (-1,1)$, $\alpha > -1$, $\beta > -1$, from the Proposition 10 we obtain Proposition 4.*

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