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Some inequalities concerning starlike and convex functions

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Abstract

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in the complex plane and let A be the set of all analytic functions in Δ, satisfying the conditions of normalization: $f(0) = f'(0) - 1 = 0$. The purpose of this article is to show that the functions in A satisfying the following condition:

$$\Re \frac{zf'(z)}{f(z)} > \left| \frac{zf'(z)}{f(z)} - 1 \right|$$

have the property:

$$\Re \sqrt{\frac{f(z)}{z}} > c$$

where c is any real constant in $(1/2, 2/3)$. As a simpler case it is shown that every starlike function in Δ has the property: $\Re\sqrt{f(z)/z} > 1/2$.

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1 Introduction

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in the complex plane and let A be the set of all analytic functions in Δ, normalized with the conditions $f(0) = 0$ and $f'(0) = 1$. It is well-known that a function in A is starlike (i.e. $f \in \mathbf{ST}$) if and only if

$$\Re \frac{zf'(z)}{z} > 0 \quad in\ \Delta.$$

Also, it is known that a function in A is convex (ie. $f \in \mathbf{CV}$) if and only if:

$$\Re[1 + \frac{zf''(z)}{f'(z)}] > 0 \ \ in\ \Delta$$

In 1991, A.W. Goodman introduced the concept of "uniformly starlike" function and of "uniformly convex" function (in [2]) and proved some properties for such functions (in [2] and [3]). The class of uniformly starlike function is the subclass of A denoted by **UST** and the class of uniformly convex functions in A is denoted by **UCV**. Geometrically, the property of uniform starlikeness (respectively uniform convexity) of a function $f \in A$ means that the image of every circular arc contained in Δ, with center ζ also in Δ, is starlike with respect to $f(\zeta)$ (respectively convex). These properties are expressed using two complex variables, but in the case of uniformly convex functions it exists a characterization using one single complex variable. This was found by Frode Ronning in [7] in 1993 and is:

$$UCV = \{f \in A : \Re[1 + \frac{zf''(z)}{f'(z)}] \geq \left| \frac{zf''(z)}{f'(z)} \right|,\ z \in \Delta\}.$$

Properties of those classes and of other related to these (such the class of the so-called "uniformly starlike functions with respect to symmetrical points") were obtained by Frode Ronning in [8]. In this article we will consider functions in A satisfying the condition:

$$\Re\frac{zf'(z)}{f(z)} > \left|\frac{zf'(z)}{f(z)} - 1\right|$$

which is very similar to the condition for uniform convexity obtained by Frode Ronning. We will denote by **QUST**(quasi-uniformly starlike functions) the subclass of the functions in A which satisfy the above condition. It is well-known that the convex functions in A are starlike of order 1/2, (i.e. $f \in$**CV** implies that $\Re zf'(z) > 1/2$ in Δ). This beautiful result was first obtained in [4] by A. Marx, using a very complicated method. But another result of A. Marx (in [1]) claims that the convex functions in the unit disc have also the property that the square root of their derivative has the real part greater than 1/2 (the square root is considered with the principal part). The two theorems of Marx can be easily obtained by using the so-called method of the differential subordinations developed by S.S. Miller and P.T. Mocanu in the 80'th (see [5] and [6]). In this article we use this method to show that every function in **QUST** has the property that the real part of the square root (considered with its principal determination, which takes the value 1 at the origin) of $f(z)/z$ is greater than c, where c can be every real constant in the interval (1/2,2/3) (that means that $\Re\sqrt{f(z)/z} \geq 2/3$ in Δ).

2 Preliminaries

For proving our principal result we will need the following definitions and results:

Definition 1. *A function is said to be in the class* **ST**(α) *if and only if* f *is in* A *and* $\Re zf'(z)/z > \alpha$ *in* Δ.

Lemma 1. ([4]) **CV** $\subset$ **ST(1/2)** *and also:*

$$\Re\left[1+\frac{zf''(z)}{f'(z)}\right]>0 \ \textit{ implies that } \ \Re\sqrt{f'(z)}>\frac{1}{2}$$

where the square root is considered with its principal determination.

Lemma 2. ([1]) *A function $f \in A$ is convex in Δ if and only if the function $zf'(z)$ is starlike in Δ.*

Lemma 2 is well-known as "Alexander's duality theorem" and has a very simple proof based on the characterization of starlike and convex functions in the unit disc.

Lemma 3. ([5]) *Let a be a complex number with $\Re a > 0$ and let $\psi : \mathbb{C} \times \Delta \longrightarrow \mathbb{C}$ a function satisfying:*

$$\Re\psi(\mathrm{i}x, y; z) \leq 0 \ \textit{ in } \Delta \ \textit{ and for all } x \textit{ and } y, \ \textit{ with } \ y \leq -\frac{|a-\mathrm{i}x|^2}{2\Re a}$$

If

$$p(z) = a + p_1 z + p_2 z^2 + \cdots \ \textit{ is analytic in } \Delta, \ \textit{ then}:$$

$$[\Re\psi(p(z), zp'(z); z) > 0 \ \textit{ for all } z \in \Delta] \ \textit{ implies } \ \Re p(z) > 0 \ \textit{ in } \Delta$$

Proofs of more general forms of **Lemma 3** can be found in [5] and in [6].

3 Main result

Theorem 1. *If $f \in A$ is starlike, then:*

$$\Re\sqrt{\frac{f(z)}{z}} > \frac{1}{2}$$

where the determination of the square root is the principal one.

Proof. We will use Lemma 1 and Lemma 2 for proving this result. Denote by $g(z)$ the following function:

$$g(z) = \int_0^z \frac{f(z)}{z}$$

It is easy to see that $f(z) = zg'(z)$. By Lemma 2 it follows that f is starlike if and only if g is convex. Then, by Lemma 1 it follows that

$$\Re\sqrt{g'(z)} > \frac{1}{2}$$

Since $g'(z) = f(z)/z$ we have that $\Re\sqrt{f(z)/z} > 1/2$ for $f\in$**ST** and the theorem is proved.

Theorem 2. *Let* $f\in$**QUST**. *Then we have:*

$$\Re\sqrt{\frac{f(z)}{z}} > c \;\; in \;\; \Delta$$

and c is any real number situated in the interval (1/2,2/3) and the square root is considered with its principal branch.

Proof. Because any function in **QUST** is starlike, it follows by Theorem 1 that

$$\Re\sqrt{\frac{f(z)}{z}} > \frac{1}{2}$$

and thus it is clear that c is greater than $1/2$. Let $p : \Delta \to \mathbb{C}$ defined by:

$$p(z) = \sqrt{\frac{f(z)}{z}} - c \;\; with \;\; c \;\; is\, real\, and\, greater\, than\; 1/2.$$

Because $c < 2/3 < 1$ it follows that $p(0) = 1 - c > 0$. It is clear, after a simple calculation, that:

$$\frac{zf'(z)}{z} = 1 + \frac{2zp'(z)}{c + p(z)}$$

and that $f \in$**QUST** is equivalent to:

$$\Re\left[\frac{2zp'(z)}{c+p(z)}+1-\left|\frac{2zp'(z)}{c+p(z)}\right|\right]\geq 0$$

In order to apply **Lemma 3**, we have to prove that

$$\Re\psi(\mathrm{i}x,y;z)\leq 0 \ \ for\ all\ real\ x\ and\ y\ with\ y\leq -\frac{|p(0)-\mathrm{i}x|^2}{2\Re p(0)}$$

Because $p(0)=\Re p(0)=1-c$, the above mentioned inequality have to be proved for all real x and y with $y\leq -[(1-c)^2+x^2]/2(1-c)$. Then, $|y|\leq -[(1-c)^2+x^2]/2(1-c)$ and

$$\Re\psi(\mathrm{i}x,y;z)\leq \frac{2cy}{c^2+x^2}+1-\frac{2|y|}{\sqrt{c^2+x^2}}\leq$$

$$\leq -\frac{c}{1-c}\frac{(1-c)^2+x^2}{c^2+x^2}-\frac{(1-c)^2+x^2}{(1-c)\sqrt{c^2+x^2}}+1$$

We have to find $c\in(1/2,1)$ so that

$$-\frac{c}{1-c}\frac{(1-c)^2+x^2}{c^2+x^2}-\frac{(1-c)^2+x^2}{(1-c)\sqrt{c^2+x^2}}+1\leq 0 \ \text{ for all real } x$$

Let now

$$h(x)=-\frac{c}{1-c}\frac{(1-c)^2+x^2}{c^2+x^2}-\frac{(1-c)^2+x^2}{(1-c)\sqrt{c^2+x^2}}+1$$

A simple calculation shows that

$$h'(x)=-x[h_1(x)+h_2(x)]$$

where

$$h_1(x)=\frac{2c}{1-c}\frac{2c-1}{(c^2+x^2)^2}\ \ and\ \ h_2(x)=\frac{1}{1-c}\frac{c^2+2c-1+x^2}{(c^2+x^2)^2\sqrt{c^2+x^2}}.$$

Because $2c-1>0$ and $c^2+2c-1>0$ for all $c\in(1/2,2/3)$ we have that $h_1(x)+h_2(x)\geq 0$ for all real x, and thus, $h'(x)\geq 0$ for $x<0$ and $h'(x)\leq 0$

for $x \geq 0$. Since $\lim_{|x|\to\infty} h(x) = -\infty$ we have that $h(0) = \max_{x\in\mathbb{R}} h(x)$. It follows immediately that $h(x) \leq h(0)$ for all real x.

But $h(0) = (3c-2)/c < 0$ for all $c \in (1/2, 2/3)$ and thus, $h(x) < 0$ for all real x.

This means that $\Re\psi(\mathrm{i}x, y; z) \leq 0$ for real x and $y \leq -|p(0) - \mathrm{i}x|^2/2\Re p(0)$. By **Lemma 3** we conclude that in these conditions we have $\Re p(z) > 0$ in Δ, and thus: $\Re\sqrt{f(z)/z} > c$ for all $c \in (1/2, 2/3)$ and the theorem is proved.

Remark 1. *The theorem shows that*

$$\Re\sqrt{\frac{f(z)}{z}} \geq \frac{2}{3}$$

for every starlike function f in A.

4 A particular case

If we consider in Theorem 2, $f(z) = zg'(z)$, then the starlikeness of f is equivalent (by Lemma 2) with the convexity of g and a simple calculation shows also that $f \in$**QUST** if and only if $g \in$**UCV**. We can then apply Theorem 2 to the function $zg'(z)$ and obtain the following result:

Corollary 1. *If $g\in$*__UCV__*, then we have:*

$$\Re\sqrt{g'(z)} \geq \frac{2}{3}$$

where the square root is taken with its principal value.

It is easy to see that this last corollary shows that the real part of the derivative of an uniformly convex function is greater than 2/3, while it is only greater than 1/2 if the function is only convex (from the result of A. Marx in [4]). An open question remains to find the greatest constant c so that $\Re\sqrt{g'(z)} > c$ for all uniformly convex functions g.

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