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Order of certain classes of analytic and univalent functions using Ruscheweyh derivative

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Abstract

Let $D^{\alpha}f(z)$ be the Ruscheweyh derivative defined by using the Hadamard product of $f(z)$ and $z/(1-z)^{\alpha+1}$. The object of this paper is to find the order for certain analytic and univalent functions using the Ruscheweyh derivative $D^{\alpha}f(z)$.

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1 Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$. A function $f(z)$ belonging to $\mathcal{A}$ is said to be starlike in $\mathcal{U}$ if it satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \tag{1.2}$$

for all $z \in \mathcal{U}$. We denote by $\mathcal{S}^*$ the subclass of $\mathcal{A}$ consisting of functions which are starlike in $\mathcal{U}$. Also, a function $f(z)$ belonging to $\mathcal{A}$ is said to be convex in $\mathcal{U}$ if it satisfies

$$\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \tag{1.3}$$

for all $z \in \mathcal{U}$. We denote by $\mathcal{K}$ the subclass of $\mathcal{A}$ consisting of functions which are convex in $\mathcal{U}$. A function $f(z)$ in $\mathcal{A}$ is said to be close-to-convex of order δ if there exists a function $g(z)$ belonging to $\mathcal{S}^*$ such that

$$\text{Re}\left(\frac{zf'(z)}{g(z)}\right) > \delta \tag{1.4}$$

for some $\delta (0 \leq \delta < 1)$, and for all $z \in \mathcal{U}$. We denote by $\mathcal{C}(\delta)$ the subclass of $\mathcal{A}$ consisting of functions which are close-to-convex of order δ in $\mathcal{U}$. It is well known that $\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{C} \equiv \mathcal{C}(0) \subset \mathcal{S}$. A function $f(z)$ belonging to $\mathcal{A}$ is said to be quasi-convex of order $\delta (0 \leq \delta < 1)$ if there exists a function $g(z)$ belonging to $\mathcal{C}$ such that

$$\text{Re}\left(\frac{(zf'(z))'}{g'(z)}\right) > \delta \tag{1.5}$$

for all $z \in \mathcal{U}$. Denote the class of quasi-convex of order δ by $\mathcal{C}^*(\delta)$. The class $\mathcal{C}^*(0)$ was introduced and studied by Noor [1]. We note that every quasi-convex function is close-to-convex and hence univalent in $\mathcal{U}$.

Let the function $f(z)$ be defined by (1.1) and the function $g(z)$ be defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \tag{1.6}$$

then the Hadamard product (or convolution) of the functions $f(z)$ and $g(z)$ is defined by

$$f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{1.7}$$

Using the convolution (1.5), Ruscheweyh [3] introduced what is now referred to as the Ruscheweyh derivative $D^{\alpha} f(z)$ of order α of $f(z) \in \mathcal{A}$ by

(1.8) $$D^{\alpha} f(z) = \frac{z}{(1-z)^{\alpha+1}} * f(z) \qquad (\alpha \geq -1).$$

We note that $D^0 f(z) = f(z)$ and $D^1 f(z) = zf'(z)$.

Owa *et al.* [2] have introduced and studied the following classes:

(1.9) $$\mathcal{S}^*_{\alpha} = \{f(z) \in \mathcal{A}\colon\ D^{\alpha} f(z) \in \mathcal{S}^*,\ \alpha \geq -1\}$$

and

(1.10) $$\mathcal{K}_{\alpha} = \{f(z) \in \mathcal{A}\colon\ D^{\alpha} f(z) \in \mathcal{K},\ \alpha \geq -1\}.$$

Note that $\mathcal{S}^*_0 \equiv \mathcal{S}^*$and $\mathcal{S}^*_1 \equiv \mathcal{K}_0 \equiv \mathcal{K}$.

The aim of this paper is to find the order for certain analytic and univalent functions using the Ruscheweyh derivative $D^{\alpha} f(z)$.

In order to show our results, we shall need the following lemmas due to Owa *et al.* [2].

Lemma 1 . *Let the function $f(z)$ be in the class $\mathcal{S}^*_{\alpha}$ with $\alpha \geq -1$. Then*

(1.11) $$\operatorname{Re}\left(\frac{D^{\alpha} f(z)}{z}\right)^{\beta-1} > \frac{1}{2\beta-1}\ , \qquad z \in \mathcal{U},$$

where $1 < \beta \leq 3/2$.

Lemma 2. *Let the function $f(z)$ be in the class $\mathcal{K}_{\alpha}$ with $\alpha \geq -1$. Then*

(1.12) $$\operatorname{Re}\Big((D^{\alpha} f(z))'\Big)^{\beta-1} > \frac{1}{2\beta-1}\ , \qquad z \in \mathcal{U},$$

where $1 < \beta \leq 3/2$.

2 Main Results

With the aid of Lemma 1, we can prove the following

Theorem 1 . *If the function $f(z)$ in $\mathcal{A}$ satisfies the condition*

(2.1) $$\operatorname{Re}\left[\frac{z(D^{\alpha} f(z))''}{(D^{\alpha} f(z))'}\right] > -\beta\ , \qquad z \in \mathcal{U}$$

then

(2.2) $$\mathrm{Re}\left[\frac{z(D^{\alpha}f(z))'}{D^{\alpha}g(z)}\right] > \frac{1}{2\beta-1}\,, \qquad z\in\mathcal{U},$$

where $\alpha \geq -1, \quad 1 < \beta \leq 3/2$ *and*

(2.3) $$D^{\alpha}g(z) = z\left[(D^{\alpha}f(z))'\right]^{\frac{1}{\beta}}\,, \qquad z\in\mathcal{U}.$$

Proof. From (2.3) by differentiating, we obtain

(2.4) $$\frac{z[D^{\alpha}g(z)]'}{D^{\alpha}g(z)} = 1 + \frac{1}{\beta}\frac{z[D^{\alpha}f(z)]''}{[D^{\alpha}f(z)]'}\,, \qquad z\in\mathcal{U}.$$

Using (2.1) in (2.4) we have

$$\mathrm{Re}\left[\frac{z(D^{\alpha}g(z))'}{D^{\alpha}g(z)}\right] = \mathrm{Re}\left[1+\frac{1}{\beta}\frac{z(D^{\alpha}f(z))''}{(D^{\alpha}f(z))'}\right] > 1+\frac{1}{\beta}(-\beta) > 0,$$

from which we deduce $g(z)\in\mathcal{S}^{*}_{\alpha}$, $z\in\mathcal{U}$.

From (2.3) we obtain

$$[D^{\alpha}f(z)]' = \left[\frac{D^{\alpha}g(z)}{z}\right]^{\beta-1}\cdot\frac{D^{\alpha}g(z)}{z}\,, \quad z\in\mathcal{U}, \quad z\neq 0$$

and we have

(2.5) $$\frac{z[D^{\alpha}f(z)]'}{D^{\alpha}g(z)} = \left[\frac{D^{\alpha}g(z)}{z}\right]^{\beta-1}\,, \qquad z\in\mathcal{U}, \quad z\neq 0.$$

Applying Lemma 1 to (2.5) we obtain

$$\mathrm{Re}\left[\frac{z(D^{\alpha}f(z))'}{D^{\alpha}g(z)}\right] = \mathrm{Re}\left[\frac{D^{\alpha}g(z)}{z}\right]^{\beta-1} > \frac{1}{2\beta-1}\,, \quad z\in\mathcal{U}, \quad z\neq 0.$$

Letting $\alpha = 0$ in Theorem 1, we obtain:

Corollary 1. *If the function* $f(z)$ *in* $\mathcal{A}$ *satisfies the condition*

(2.6) $$\mathrm{Re}\left(\frac{zf''(z)}{f'(z)}\right) > -\beta\,, \qquad z\in\mathcal{U}$$

then

$$\mathrm{Re}\left(\frac{zf'(z)}{g(z)}\right) > \frac{1}{2\beta-1}\ , \quad z \in \mathcal{U}.$$

Function $f(z)$ belongs to the class $\mathcal{C}(\delta)$, where $\delta = 1/(2\beta - 1)$ and $1 < \beta \leq 3/2$. Therefore $f(z)$ is close-to-convex of order δ.

Letting $\beta = 3/2$ in Corollary 1, we have:

Corollary 2. *If the function $f(z)$ in $\mathcal{A}$ satisfies the condition*

$$\mathrm{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > -1/2\ , \qquad z \in \mathcal{U} \tag{2.7}$$

then

$$\mathrm{Re}\left(\frac{zf'(z)}{g(z)}\right) > \frac{1}{2}\ ,\ z \in \mathcal{U},\ \text{i.e. } f(z) \text{ is in } \mathcal{C}(1/2).$$

Letting $\alpha = 1$ in Theorem 1, we obtain:

Corollary 3. *If the function $f(z)$ in $\mathcal{A}$ satisfies the condition*

$$\mathrm{Re}\left[\frac{z(zf'(z))''}{(zf'(z))'}\right] > -\beta\ , \qquad z \in \mathcal{U} \tag{2.8}$$

then

$$\mathrm{Re}\left[\frac{(zf'(z))'}{g'(z)}\right] > \frac{1}{2\beta-1}\ , \qquad z \in \mathcal{U}, \tag{2.9}$$

where $1 < \beta \leq 3/2$. Therefore $f(z)$ is in $\mathcal{C}^*(\frac{1}{2\beta-1})$.

Letting $\beta = 3/2$ in Corollary 3, we have:

Corollary 4. *If the function $f(z)$ in $\mathcal{A}$ satisfies the condition*

$$\mathrm{Re}\left[\frac{z(zf'(z))''}{(zf'(z))'} + 1\right] > -1/2\ , \qquad z \in \mathcal{U} \tag{2.10}$$

then

$$\mathrm{Re}\left[\frac{(zf'(z))'}{g'(z)}\right] > 1/2\ , \qquad z \in \mathcal{U}. \tag{2.11}$$

Therefore $f(z)$ is in $\mathcal{C}^*(1/2)$.

Next, we prove:

Theorem 2. *If the function* $f(z)$ *in* $\mathcal{A}$ *satisfies the condition*

$$\text{Re}\left[\frac{z(D^{\alpha}f(z))'}{D^{\alpha}f(z)}\right] > 1-\beta\ , \qquad z\in\mathcal{U} \tag{2.12}$$

then

$$\text{Re}\left[\frac{D^{\alpha}f(z)}{z(D^{\alpha}g(z))'}\right] > \frac{1}{2\beta-1}\ , \qquad z\in\mathcal{U}, \quad z\neq 0 \tag{2.13}$$

where $\alpha\geq -1, \quad 1<\beta\leq 3/2$ *and*

$$[D^{\alpha}g(z)]' = \left[\frac{D^{\alpha}f(z)}{z}\right]^{\frac{1}{\beta}}\ , \qquad z\in\mathcal{U}, \quad z\neq 0. \tag{2.14}$$

Proof. From (2.12) we obtain

$$\text{Re}\ \frac{1}{\beta}\left[\frac{z(D^{\alpha}f(z))'}{D^{\alpha}f(z)}\right] > \frac{1}{\beta}-1\ , \qquad z\in\mathcal{U}$$

which is equivalent to

$$\text{Re}\ \frac{1}{\beta}\left[\frac{z(D^{\alpha}f(z))'}{D^{\alpha}f(z)}-1\right] > -1\ , \qquad z\in\mathcal{U}. \tag{2.15}$$

From (2.14), by differentiating we have

$$\frac{[D^{\alpha}g(z)]''}{[D^{\alpha}g(z)]'} = \frac{1}{\beta}\left[\frac{(D^{\alpha}f(z))'}{D^{\alpha}f(z)}-\frac{1}{z}\right]\ , \qquad z\in\mathcal{U}$$

which is equivalent to

$$\frac{z[D^{\alpha}g(z)]''}{[D^{\alpha}g(z)]'} = \frac{1}{\beta}\left[\frac{z(D^{\alpha}f(z))'}{D^{\alpha}f(z)}-1\right]\ , \qquad z\in\mathcal{U}. \tag{2.16}$$

Using (2.15) in (2.16) we have

$$\text{Re}\left[\frac{z(D^{\alpha}g(z))''}{(D^{\alpha}g(z))'}+1\right] > 0\ , \qquad z\in\mathcal{U}$$

from which $g(z)\in\mathcal{K}_{\alpha}$.

From (2.14) we obtain

$$\frac{D^{\alpha} f(z)}{z} = \left[(D^{\alpha} g(z))'\right]^{\beta} , \qquad z \in \mathcal{U}, \quad z \neq 0$$

from which we obtain

$$\text{(2.17)} \qquad \frac{D^{\alpha} f(z)}{z[D^{\alpha} g(z)]'} = \left[(D^{\alpha} g(z))'\right]^{\beta-1} , \qquad z \in \mathcal{U}, \quad z \neq 0.$$

Applying Lemma 2 in (2.17) we obtain

$$\operatorname{Re}\left[(D^{\alpha} g(z))'\right]^{\beta-1} = \operatorname{Re}\left[\frac{D^{\alpha} f(z)}{z(D^{\alpha} g(z))'}\right] > \frac{1}{2\beta - 1} , \qquad z \in \mathcal{U}, \quad z \neq 0.$$

Letting $\alpha = 0$ in Theorem 2, we obtain:
Corollary 5. *If the function* $f(z)$ *in* $\mathcal{A}$ *satisfies the condition*

$$\text{(2.18)} \qquad \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 1 - \beta , \qquad z \in \mathcal{U}$$

then

$$\operatorname{Re}\left(\frac{f(z)}{zg'(z)}\right) > \frac{1}{2\beta - 1} , \qquad z \in \mathcal{U}, \quad z \neq 0$$

where $1 < \beta \leq 3/2$.

Letting $\beta = 3/2$ in Corollary 5, we have:
Corollary 6. *If the function* $f(z)$ *in* $\mathcal{A}$ *satisfies the condition*

$$\text{(2.19)} \qquad \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > -1/2 , \qquad z \in \mathcal{U}$$

then

$$\operatorname{Re}\left(\frac{zf'(z)}{g'(z)}\right) > 1/2 , \qquad z \in \mathcal{U}.$$

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