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On the Structure of Some Irrotational Vector Fields

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Dedicated to Associate Professor Emil C. Popa on his 60th anniversary

Abstract

In this article is given a simple method to describe the structure of irrotational vector fields defined on some domains in the Euclidean 3-space and which appear often in both pure or applied mathematics.

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1 Introduction

Let us consider a domain $\Omega \subseteq \mathbb{R}^3$ and $\mathbf{F} := (F_x, F_y, F_z) : \Omega \longrightarrow \mathbb{R}^3$ a vector field of class C^1 on Ω.

The *curl of* $\mathbf{F}$ or *rotor of* $\mathbf{F}$ is denoted as $\mathrm{curl}(\mathbf{F})$ or $\mathrm{rot}(\mathbf{F})$ and it is defined as the formal cross product of ∇ with $\mathbf{F}$:

$$\text{curl}(\mathbf{F})=\text{rot}(\mathbf{F}) = \nabla \times \mathbf{F} = (\imath\frac{\partial}{\partial x} + \jmath\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}) \times (F_x, F_y, F_z) :=$$

$$:= \imath\,(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}) + \jmath\,(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}) + \mathbf{k}\,(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}) := \begin{vmatrix} \imath & \jmath & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}.$$

The last formal determinant is useful to keep in mind the definition of the vector field rot($\mathbf{F}$) and to compute it.

It is well known that if the vector field $\mathbf{F}$ is conservative and if the scalar field $G : \Omega \longrightarrow \mathbb{R}$ is a potential for $\mathbf{F}$ then the equality $\mathbf{F} = \text{grad}(G)$ implies $\text{rot}(\mathbf{F}) = \mathbf{0}$ i.e. the vector field $\mathbf{F}$ is **irrotational**.

The converse of this property is true **if and only if** the domain Ω is **simply connected** i.e. its fundamental group is the trivial group. For multiple connected domains the class of irrotational vector fields is strictly larger as the class of conservative ones.

In this paper we shall give a canonical method to evaluate the deviation of an irrotational vector field from being conservative, for a large class of multiple connected domains Ω of the Euclidean 3-space $\mathbb{R}^3$.
The content of the paper is the natural completion of the content of [1].

2 Elementary Irrotational Vector Fields

Let us consider the line $L \subset \mathbb{R}^3$ given by the following Cartesian equations:

$$L : \begin{cases} f_1 := a_1x + b_1y + c_1z + d_1 = 0 \\ \\ f_2 := a_2x + b_2y + c_2z + d_2 = 0 \end{cases}$$

where the vectors $(a_1, b_1, c_1), (a_2, b_2, c_2) \in \mathbb{R}^3$ are linearly independent.

A nice property of the function arctan permits the construction of an irrotational vector field

$$\mathbf{E} : \Omega := \mathbb{R}^3 \setminus L \longrightarrow \mathbb{R}^3$$

arriving in the description of the deviation from being conservative of any other irrotational vector field on Ω.

We shall denote by $P_k, k = 1, 2$, the planes in $\mathbb{R}^3$ given by the equations $f_k = 0$ and by Ω_k the two components (half-planes) open sets $\mathbb{R}^3 \setminus P_k$.

Let us consider the scalar fields $G_k : \Omega_k \longrightarrow \mathbb{R}$ defined by:

$$G_1(x, y, z) := \arctan \frac{f_2}{f_1} \ \text{ and } G_2(x, y, z) := -\arctan \frac{f_1}{f_2}.$$

One check at once that the partial derivatives of order 1 of G_1 and G_2 are given by **the same** analytical expressions (but they have **different** domains of definition !).

Let us define the vector field $\mathbf{E}$ by:

$$(1)\ \mathbf{E}(x, y, z) = \frac{1}{f_1^2 + f_2^2} \left[f_2 \frac{\partial f_1}{\partial x} - f_1 \frac{\partial f_2}{\partial x}; f_2 \frac{\partial f_1}{\partial y} - f_1 \frac{\partial f_2}{\partial y}; f_2 \frac{\partial f_1}{\partial z} - f_1 \frac{\partial f_2}{\partial z} \right].$$

We see that $\mathbf{E} : \mathbb{R}^3 \setminus (P_1 \bigcap P_2) \longrightarrow \mathbb{R}^3$ i.e. $\mathbf{E} : \mathbb{R}^3 \setminus L \longrightarrow \mathbb{R}^3$ and for every $(x, y, z) \in \mathbb{R}^3 \setminus L$ one has:

$$(2) \qquad \begin{cases} \mathbf{E}(x, y, z) = \operatorname{grad} G_1(x, y, z) \quad \text{for} \quad (x, y, z) \in \mathbb{R}^3 \setminus P_1 \\ \\ \mathbf{E}(x, y, z) = \operatorname{grad} G_2(x, y, z) \quad \text{for} \quad (x, y, z) \in \mathbb{R}^3 \setminus P_2. \end{cases}$$

$$\mathbf{E}(x, y, z) = \operatorname{grad} G_1(x, y, z) = \operatorname{grad} G_2(x, y, z)$$

for every $(x, y, z) \in \mathbb{R}^3 \setminus P_1 \bigcup P_2$.

The equations (2) assures that the vector field $\mathbf{E}$ is irrotational.
If we consider a circle C in a plane perpendicular on L, with radius equal to 1 (for example),with the center on L and which surrounds L once, one sees that $\displaystyle\int_C \mathbf{E}\cdot d\mathbf{r} = \pm 2\pi$.
Thus the vector field $\mathbf{E}$ is **not** conservative.
A vector field $\mathbf{E}$ of the previous type will be called *an elementary irrotational vector field of the second type.*

We formulate now the precise definitions.
Let $f_1, f_2 : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be two scalar fields which are supposed, for simplicity, to be of class C^∞. Moreover suppose that for every $(x,y,z)\in\mathbb{R}^3$ for which $[f_1(x,y,z)]^2+[f_2(x,y,z)]^2>0$ the Jacobi matrix

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} & \dfrac{\partial f_1}{\partial z} \\ \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} & \dfrac{\partial f_2}{\partial z} \end{bmatrix} \tag{3}$$

in the point (x,y,z) has the rank 2. Particularly the gradients of f_1 and f_2 are different from zero in these points and the implicit-function theorem is applicable around every point satisfying the previous restrictions.

Suppose that the sets $S_k := \{(x,y,z)\in\mathbb{R}^3 |\ f_k(x,y,z)=0\}$, $k=1,2$ are non-empty. Then the implicit-function theorem assures that each S_k is a surface with one or several connected component(s).

Suppose that $C := S_1 \bigcap S_2$ is **non-empty**, consists of precisely one connected component and the fundamental group of $\mathbb{R}^3 \setminus C$ is a free group with one generator (i.e. it is isomorphic to the additive group of rational integers

$(\mathbb{Z},+)$). (The implicit-function theorem assures that C is a curve of class C^∞).

It is well known from topology that the curve C is either compact and diffeomorphic with a circle or non-compact and diffeomorphic with a straight line.

Let us consider the scalar fields $G_k : \mathbb{R}^3 \setminus S_k \longrightarrow \mathbb{R}$ defined by:

$$(4)\quad G_1(x,y,z) := \arctan \frac{f_2(x,y,z)}{f_1(x,y,z)} \text{ and } G_2(x,y,z) := -\arctan \frac{f_1(x,y,z)}{f_2(x,y,z)}.$$

To simplify the text in the future we shall write G_k, f_k etc. for $G_k(x,y,z)$, $f_k(x,y,z)$ any time when this abbreviation does not produce ambiguities.

One check at once that gradG_1 and gradG_2 are given by

$$(5)\qquad \frac{1}{f_1^2+f_2^2}\left[f_1\frac{\partial f_2}{\partial x} - f_2\frac{\partial f_1}{\partial x}, f_1\frac{\partial f_2}{\partial y} - f_2\frac{\partial f_1}{\partial y}, f_1\frac{\partial f_2}{\partial z} - f_2\frac{\partial f_1}{\partial z}\right]$$

in their domains of definition which are those of G_1 respectively G_2.
We see that the expression (5) is defined on $\mathbb{R}^3 \setminus C$ i.e. on a domain which is **strictly larger** as $\mathbb{R}^3 \setminus S_k$, $k = 1, 2$.
We define the vector field $\mathbf{E} : \mathbb{R}^3 \setminus C \longrightarrow \mathbb{R}^3$ by the expression (5).
Since the restrictions of $\mathbf{E}$ to $\mathbb{R}^3 \setminus S_k$, $k = 1, 2$ coincide with the gradients of G_k, the vector field $\mathbf{E}$ is **irrotational**. If one takes a small circle γ with the center on C and lying in the normal plane to C in that point, such that the homotopy class of γ is a generator of the fundamental group of $\mathbb{R}^3 \setminus C$ one can check that $\displaystyle\int_C \mathbf{E}\cdot d\mathbf{r} = \pm 2\pi$.

Thus the vector field $\mathbf{E}$ is irrotational but **not conservative**.

Now we can formulate the following definition:

Definition 1. *In the previous context, the irrotational vector field* $\mathbf{E}$ *is called elementary irrotational vector field of the first type if* C *is diffeomorphic with a circle and elementary vector field of the second type if it is diffeomorphic with a straight line.*

Examples.

1. The circle $C : z = 0;\ x^2 + y^2 = 1$ generates the elementary irrotational vector field of the first type $\mathbf{E} : \mathbb{R}^3 \setminus C \longrightarrow \mathbb{R}^3$ given by:

$$\mathbf{E}(x, y, z) = \frac{1}{z^2 + [x^2 + y^2 - 1]^2}\ [2xz, 2yz, 1 - x^2 - y^2].$$

2. The graph of the function $\sin : \mathbb{R} \longrightarrow \mathbb{R}$ viewed as a curve C in $\mathbb{R}^3$ with the equations $z = 0$ and $y = \sin x$ generates the elementary irrotational vector field of the second type $\mathbf{E}$ given by:

$$\mathbf{E}(x, y, z) = \frac{1}{z^2 + (y - \sin x)^2}\ [-z \cos x, z, -y + \sin x].$$

Remark. For a given curve C correspond an infinity of elementary irrotational vector fields. For example, the curve C from the second example can be written by means of the equations $z = 0$ and $ay = a \sin x$ where $a \neq 0$ is an arbitrary constant. With these equations the elementary irrotational vector field generated by C is given by:

$$\mathbf{E}(x, y, z) = \frac{1}{z^2 + a^2(y - \sin x)^2}\ [-az \cos x, az, a(-y + \sin x)].$$

Now we can formulate the main result of this paper.

3 The Main Result

Let $C_1, C_2, \ldots, C_n$ be n curves of the type considered before, such that they are pairwise disjoint: $C_i \bigcap C_j = \emptyset$ for every $i \neq j$. We recall that each domain $\mathbb{R}^3 \setminus C_i$ has its fundamental group generated by one generator.
Suppose that C_i is given by the equations $f_{i1}(x, y, z) = 0$; $f_{i2}(x, y, z) = 0$. Let $\mathbf{E}_i$ be the elementary vector field corresponding to C_i and defined by the previous equations of C_i according to (5). We shall denote also by $\mathbf{E}_i$ the **restrictions** of $\mathbf{E}_i$ to the domain $\Omega := \mathbb{R}^3 \setminus \bigcup_{i=1}^{n} C_i$.
In this context we can formulate:

Theorem 1. *If* $\mathbf{E} : \Omega \longrightarrow \mathbb{R}^3$ *is an arbitrary irrotational vector field on* Ω*, there exist constants* $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ *uniquely determined by* $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_n$ *and* $\mathbf{E}$ *such that the vector field* $\mathbf{F} := \mathbf{E} - \Sigma_{i=1}^{n} \alpha_i \mathbf{E}_i$ *is conservative.*

Proof. For every curve C_i we choose a circle γ_i such that its homotopy class $[\gamma_i]$ is a generator of the fundamental group $\Pi^1(\mathbb{R}^3 \setminus C_i)$; γ_i can be a small circle with the center in a point of C_i and which lies in the normal plane to C_i at that point.
We identify the constants $\alpha_i \in \mathbb{R}$ by imposing to $\mathbf{F}$ to satisfy the conditions:

$$(6) \qquad \int_{\gamma_i} \mathbf{F} \cdot d\mathbf{r} = 0 \quad \text{for all } i,\ 1 \leq i \leq n.$$

The equalities (6) give:

$$(7) \qquad \alpha_i = \frac{\displaystyle\int_{\gamma_i} \mathbf{E} \cdot d\mathbf{r}}{\displaystyle\int_{\gamma_i} \mathbf{E}_i \cdot d\mathbf{r}} \quad \text{for all } i,\ 1 \leq i \leq n.$$

From now on we keep for α_i the values given by (7).

Remark. By using formulae (4) and (5), the patient reader can see that

$$\int_{\gamma_i} \mathbf{E}_i \cdot d\mathbf{r} = \pm 2\pi. \tag{8}$$

Let $(x_0, y_0, z_0) \in \Omega$ be a point which will be kept fixed in all that follows. Let (x, y, z) be a variable point in Ω and Γ an arbitrary piecewise smooth curve in Ω having the start point in (x_0, y_0, z_0) and the end point in (x, y, z). Since the set of homotopy classes $\{[\gamma_1], [\gamma_2, \ldots, [\gamma_n]]\}$ generates the fundamental group $\Pi^1(\Omega)$ the equations (6) implies that $\int_\gamma \mathbf{F} \cdot d\mathbf{r} = 0$ for every piecewise smooth **closed** curve in Ω(See [2],Ch.15). This last property of $\mathbf{F}$ implies that $\int_\Gamma \mathbf{F} \cdot d\mathbf{r}$ does not depend on Γ but only on (x, y, z). Thus one can define unambiguously the scalar field $G : \Omega \longrightarrow \mathbb{R}$ by

$$G(x, y, z) := \int_\Gamma \mathbf{F} \cdot d\mathbf{r}. \tag{9}$$

Now it is well-known from calculus in several variables that the scalar field G is a potential to $\mathbf{F}$. Since Ω is connected, G is the **single** potential to $\mathbf{F}$ satisfying the condition $G(x_0, y_0, z_0) = 0$.

Finally, we get the following formula concerning **the structure of the irrotational vector fields on** Ω:

$$\mathbf{E} = \Sigma_{i=1}^n \alpha_i \mathbf{E}_i + \mathrm{grad}(G). \tag{10}$$

In formula (10) the vector fields $\mathbf{E}_i$ are given by (5) applied to the functions f_{i1} and f_{i2}, the scalars α_i are given by (7) and the scalar field G is given by (9).

Example. Let us consider the sphere S_1 and the cylinder S_2 given by the equations:

$$\begin{cases} S_1: \ f_1(x,y,z) = x^2 + y^2 + z^2 - 25 = 0 \\ \\ S_2: \ f_2(x,y,z) = x^2 + y^2 - 9 = 0 \end{cases} \tag{11}$$

The functions $G_1 := \arctan \dfrac{f_2}{f_1}$ and $G_2 := -\arctan \dfrac{f_1}{f_2}$ generate via formula (5) the vector field $\mathbf{E}$ given by

$$\mathbf{E}(x,y,z) = \frac{[2x(z^2-16); 2y(z^2-16); -2z(x^2+y^2-9)]}{[x^2+y^2+z^2-25]^2 + [x^2+y^2-9]^2}. \tag{12}$$

This vector field is irrotational but it is **not** an elementary irrotational vector field since $S_1 \bigcap S_2$ is not connected. This set consists of the two circles C_1 and C_2 given by:

$$C_1: \begin{cases} x^2 + y^2 - 9 = 0 \\ \\ z - 4 = 0 \end{cases} \quad \text{and } C_2: \begin{cases} x^2 + y^2 - 9 = 0 \\ \\ z + 4 = 0 \end{cases}. \tag{13}$$

The functions appearing in the equations of these two circles define via the functions G_{ij}, $1 \leq i, j \leq 2$, given by

$$\begin{cases} G_{11}(x,y,z) = \arctan \dfrac{x^2+y^2-9}{z-4} \text{ and} \\ G_{21}(x,y,z) = -\arctan \dfrac{z-4}{x^2+y^2-9} \quad \text{respectively,} \\ G_{12}(x,y,z) = \arctan \dfrac{x^2+y^2-9}{z+4} \text{ and} \\ G_{22}(x,y,z) = -\arctan \dfrac{z+4}{x^2+y^2-9} \end{cases} \tag{14}$$

the elementary irrotational vector fields $\mathbf{E}_1$ and $\mathbf{E}_2$ given by:

$$(15)\qquad \begin{cases} \mathbf{E}_1(x,y,z) = \dfrac{[2x(z-4); 2y(z-4); 9-x^2-y^2]}{(z-4)^2+[x^2+y^2-9]^2} \\ \\ \mathbf{E}_2(x,y,z) = \dfrac{[2x(z+4); 2y(z+4); 9-x^2-y^2]}{(z+4)^2+[x^2+y^2-9]^2} \end{cases}.$$

Let us consider the circles γ_1 and γ_2 in the plane $y=0$ having the centers $(3,0,4)$ respectively $(3,0,-4)$ and the same radius 1. We take the parametric representations $\mathbf{r}_1, \mathbf{r}_2 : [0, 2\pi] \longrightarrow \mathbb{R}^3$ given by:

$$(16)\qquad \begin{cases} \mathbf{r}_1(t) = [3+\cos t, 0, 4+\sin t] \\ \\ \mathbf{r}_2(t) = [3+\cos t, 0, -4+\sin t] \end{cases}$$

The homotopy classes of γ_1 and γ_2 generate the fundamental group $\Pi^1(\Omega)$ where $\Omega = \mathbb{R}^3 \setminus (S_1 \bigcap S_2) = \mathbb{R}^3 \setminus (C_1 \bigcup C_2)$.

We shall compute now the integrals appearing in formula (7). We shall give details for the computation of $\displaystyle\int_{\gamma_1} \mathbf{E}_1 \cdot d\mathbf{r}$.

We shall use **essentially** the information that $\mathbf{E}_1(x,y,z) = \mathrm{grad} G_{11}(x,y,z)$ in any point (x,y,z) outside the plane $z=4$ and $\mathbf{E}_1(x,y,z) = \mathrm{grad} G_{21}(x,y,z)$ in any point (x,y,z) outside the cylindrical surface $x^2+y^2-9=0$.

Let us consider the square $ABCD$ in the plane $y=0$, where its vertexes A, B, C, D are the points with the coordinates:

$$A(4,0,5);\ B(2,0,5);\ C(2,0,3);\ D(4,0,3).$$

This square endowed with the orientation $A \to B \to C \to D \to A$ is a piecewise smooth path δ homotopic with γ_1. Since the vector field $\mathbf{E}_1$ is irrotational,

$$(17)\qquad \int_{\gamma_1} \mathbf{E}_1 \cdot d\mathbf{r} = \int_{\delta} \mathbf{E}_1 \cdot d\mathbf{r}.$$

$$(18)\ \int_{\delta} \mathbf{E}_1 \cdot d\mathbf{r} = \int_{AB} \mathbf{E}_1 \cdot d\mathbf{r} + \int_{BC} \mathbf{E}_1 \cdot d\mathbf{r} + \int_{CD} \mathbf{E}_1 \cdot d\mathbf{r} + \int_{DA} \mathbf{E}_1 \cdot d\mathbf{r}.$$

The integrals appearing in (18) are computed according to Leibniz-Newton formula, as follows:

$$\int_{AB} \mathbf{E}_1 \cdot d\mathbf{r} = G_{11}(B) - G_{11}(A) = -[\arctan 5 + \arctan 7];$$
$$\int_{BC} \mathbf{E}_1 \cdot d\mathbf{r} = G_{21}(C) - G_{21}(B) = -2\arctan \frac{1}{5};$$
$$\int_{CD} \mathbf{E}_1 \cdot d\mathbf{r} = G_{11}(D) - G_{11}(C) = -[\arctan 5 + \arctan 7];$$
$$\int_{DA} \mathbf{E}_1 \cdot d\mathbf{r} = G_{21}(A) - G_{21}(D) = -2\arctan \frac{1}{7}.$$

These equalities together with (17) and (18) give:

$$(19) \int_{\gamma_1} \mathbf{E}_1 \cdot d\mathbf{r} = -2[\arctan 5 + \arctan \frac{1}{5} + \arctan 7 + \arctan \frac{1}{7}] = -2\pi.$$

(See formula (8)).

In the same way one gets:

$$\int_{\gamma_2} \mathbf{E}_2 \cdot d\mathbf{r} = -2\pi; \int_{\gamma_1} \mathbf{E} \cdot d\mathbf{r} = -2\pi \text{ and } \int_{\gamma_2} \mathbf{E} \cdot d\mathbf{r} = 2\pi.$$

Thus, the numbers α_i from (7) are:

$$\alpha_1 = 1 \text{ and } \alpha_2 = -1.$$

According to formula (10), there exists a scalar field $G : \mathbb{R}^3 \backslash (C_1 \bigcup C_2) \longrightarrow \mathbb{R}$ such that:

$$(20) \qquad \mathbf{E} = \mathbf{E}_1 - \mathbf{E}_2 + \mathrm{grad}\,(G).$$

Remark. The computation of the potential G of $\mathbf{E} - \mathbf{E}_1 + \mathbf{E}_2$ which accomplishes (20) and for which $G(0,0,0) = 0$ (for example), is a completely elementary task and the computation is omitted. One uses only the information that the local potentials of this vector field are constants added to linear combinations of restrictions of the (local) potentials G_i and G_{ij}, $1 \leq i, j \leq 2$, of the vector fields $\mathbf{E}$, $\mathbf{E}_1$ and $\mathbf{E}_2$.

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