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On manifolds with finitely generated homotopy groups

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Dedicated to professor Emil C. Popa for his sixtieth birthday

Abstract

Let G be an infinite group which is finitely presented. Let X be a finite $CW-$complex of dimension q whose fundamental group is $\mathbf{Z}^{2q} \times G$. We prove that for some $i \leq q$ the homotopy group $\pi_i(X)$ is not finitely generated. Let M be a manifold of dimension n whose fundamental group is $\mathbf{Z}^{n-2} \times G$. Then the same conclusion holds (for some $i \leq \max\left\{\left[\frac{n}{2}\right], 3\right\}$) unless M is an Eilenberg-McLane space. In particular, if $G = \mathbf{Z} \times H$ and the homotopy groups of M are finitely generated, then M is homotopy equivalent to the n-torus.

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1 Introduction

Overall this paper we will denote by X a finite connected CW complex of dimension q and by M a closed connected manifold of dimension n.

Consider the homotopy groups $\pi_i(X) = [\mathbf{S}^i, X]$. Recall that a presentation of a fundamental group $\pi_1(X)$ is given by the one and two-dimensional cells of the skeleton of X. Others tools for the computation of π_1 such as van Kampenş theorem are available.

At the opposite, there is no general tool and very few general results concerning the (Abelian) groups $\pi_i(X)$ for $i \geq 2$. The general question which was the motivation for this paper is the following :

Q : When are the groups π_i finitely generated for all i ?

If X is simply connected a celebrated theorem of Serre [18] answers affirmatively to the question above. If $\pi_1(X) \neq 1$ the answer may vary as it is shown by the examples below :
- $X = \mathbf{S}^1 \wedge \mathbf{S}^1$: YES ($\pi_i(X) = 0 \; \forall i \geq 2$)
- $X = \mathbf{S}^1 \wedge \mathbf{S}^2$: NO ($\pi_2(X) = \mathbf{Z}^{(\mathbf{N})}$)

Note that in the first case the fundamental group of X is free non-Abelian with two generators and in the second $\pi_1(X) = \mathbf{Z}$. As we will see below the fact that $\pi_1(X)$ is Abelian plays an important role here. Namely in a previous paper [5] the author proved the following :

Theorem 1.1 *Let $q \geq 2$ and let X be a finite connected CW complex of dimension q. Suppose that $\pi_i(X)$ are finitely generated for all i (or, equivalently, for $i = 1, \ldots, q$, by 2.1, see below). We have :*
a) Suppose that $\pi_1(X)$ is Abelian. Then :
1. $\beta_1(X) \leq q$.
2. If $\beta_1(X) \in \{ q-1, q \}$, then $\pi_1(X) \approx \mathbf{Z}^{\beta_1(X)}$ and X has the homotopy type of the torus $\mathbf{T}^{\beta_1(X)}$.

b) If $\pi_1(X) = \mathbf{Z} \times G$ *then* $\chi(X) = 0$.

Therefore, for instance any 2 complex with infinite Abelian fundamental group (like $X = \mathbf{S}^1 \wedge \mathbf{S}^2$ above) which is not a homotopy torus yield a negative answer to the question Q. An analogous statement for manifolds was proved in the paper [5] above :

Theorem 1.2 *a) Let* M^n *be a closed connected manifold whose homotopy groups* $\pi_i(M)$ *are finitely generated for* $i = 1, \ldots, \max\left\{\left[\frac{n}{2}\right], 3\right\}$ *and whose fundamental group is Abelian. We have then :*
1. $\beta_1(M) \leq n$ *and* $\beta_1(M) \neq n - 1$.
2. If $\beta_1(M) = n$, *then* $\pi_1(M) \approx \mathbf{Z}^n$ *and* M *is homeomorphic to* $\mathbf{T}^n$.

b) If the groups $\pi_i(M)$ *are finitely generated for* $i = 2, \ldots, \left[\frac{n}{2}\right]$ *and if* $\pi_1(M) = \mathbf{Z} \times G$, *where* G *is finitely presented, then* $\chi(M) = 0$.

Remark H-spaces and manifolds with non negative scalar curvature satisfy to the hypothesis of 1.1 and 1.2. See [5], [3] for further comments on these examples.

In the present paper we generalize the statements 1.1.a and 1.2.a above for CW-complexes (or manifolds) whose fundamental group is of the form $\mathbf{Z}^m \times G$. Namely we prove :

Theorem 1.3 *1. Let* M^n *be a closed connected manifold whose homotopy groups* $\pi_i(M)$ *are finitely generated for* $i = 1, \ldots, \max\left\{\left[\frac{n}{2}\right], 3\right\}$ *and whose fundamental group is of the form* $\mathbf{Z}^{n-2} \times G$, *where* G *is an infinite group which is finitely presented . Then* M *is an Eilenberg-McLane space* $K(\pi_1(M))$.
2. If $\pi_1(M)$ *equals* $\mathbf{Z}^{n-1} \times G$ *for some finitely presented (but not necessarily*

infinite) group G then $G = \mathbf{Z}$ and M is homotopy equivalent to the $n-$torus $\mathbf{T}^n$.

Remark In the hypothesis of 1.3.2 a theorem of T. Farell and L. Jones [7] asserts that for $n \geq 5$ that M^n is actually homeomorphic to $\mathbf{T}^n$.

For CW-complexes we prove the following :

Theorem 1.4 *1. Let X be a finite $CW-$complex of dimension q whose fundamental group is of the form $\mathbf{Z}^{2q} \times G$, where G is an infinite group which is finitely presented. Then for some $i \leq q$ the homotopy group $\pi_i(X)$ is not finitely generated.*

Here is a straightforward corollary of 1.3 :

Corollary 1.5 *Let M be a closed connected manifold satisfying the hypothesis of 1.3.1. Then the homological dimension of G satisfies $hd(G) \leq 2$. In other words, if $hd(G) \geq 3$ then M provides a negative answer to the question Q.*

Remark In the hypothesis of 1.3.1 one may expect G to be the fundamental group of a 2-manifold. However the author was not able to prove this stronger restriction.

The paper is organized as follows. In Section 2 we present the idea of the proof and we state the main tool, theorem 2.3. We show that 2.3 implies our main results 1.3 and 1.4. In Section 3 we recall the definition and the main properties of Novikov homology. Finally in Section 4 we give the proof of 2.3, completing thus the proof of 1.3 and 1.4.

2 The ingredients of the proof

2.1 Hurewicz-Serre morphisms

The first ingredient is a Hurewicz type relation between homotopy and homology. Recall that the classical Hurewiczş theorem asserts that for $q \geq 2$ the cannonical morphism $I_q : \pi_q(M) \to H_q(M)$ is an isomorphism provided that M is $(q-1)$-connected.

In [18] J-P. Serre generalized this theorem (see also [20], p. 504) : For some "admissible" classes of groups $\mathcal{C}$, he showed that, if X is simply connected such that $\pi_i(X) \in \mathcal{C}$ for $i = 1, \ldots, q-1$, where $q \geq 2$, then I_q is an isomorphisme mod $\mathcal{C}$: This means that $Ker(I_q)$ and $Coker(I_q)$ are in $\mathcal{C}$.

The class of finitely generated Abelian groups is such an admissible class. In particular we have :

Theorem 2.1 *Let X be a simply connected space. Then $\pi_i(X)$ is finitely generated for $i \leq q$ iff $H_i(X)$ is finitely generated for $i \leq q$.*

In particular any closed, simply connected CW-complex has finitely generated homotopy groups.

The following corollary is straightforward :

Corollary 2.2 *Let X be a CW-complex of dimension q and $\widetilde{X}$ its universal cover. The following are equivalent :*
a) The homotopy groups $\pi_i(X)$ are finitely generated for all $i \leq q$.
b) The homology groups $H_i(\widetilde{X})$ are finitely generated for all $i \leq q$.
c) The homotopy groups $\pi_i(X)$ are finitely generated for all $i \in \mathbf{N}$.

2.2 Fibrations over the circle

Let M^n be a closed manifold of dimension $n \geq 6$ such that $\pi_i(M)$ are finitely generated for $i = 2, \ldots, \left[\frac{n}{2}\right]$ and $\pi_1(M) = \mathbf{Z} \times H$. It was established in [5], th. 2.1 that if the Whitehead group $Wh(\pi_1(M)$ vanishes, then there exists a fibration $f : M \to \mathbf{S}^1$, which induces the projection on the first factor at the level of fundamental groups. If we drop the assumption on the Whitehead group we have the following weaker statement :

Theorem 2.3 *Let M^n be a closed connected manifold of dimension n and H a finitely presented group. Suppose that $\pi_i(M)$ is finitely generated for $i = 2, \ldots, \left[\frac{n}{2}\right]$ and $\pi_1(M) = \mathbf{Z} \times H$.*

Then, if $n \geq 6$ there exists a closed connected $(n-1)-$dimensional submanifold $i : F \hookrightarrow M$ such that :
a) $\pi_1(F) = H$.
b) i induces an isomorphism in π_i for all $i \geq 2$.

For M of arbitrary dimension n, the same conclusion holds if we replace M by $M \times \mathbf{S}^p$ for all $p \geq 6 - n$ (the dimension of F will be $n + p - 1$ in this case).

The proof of 2.3 will be given in Section 4. Let us show now how this theorem implies the statement a) of our main results 1.3 and 1.4.

<u>Proof of 2.3 $\Longrightarrow$ 1.3</u>

Without restricting the generality of our statements, we may suppose that $n \geq 3$.

1. Suppose first that $n \geq 6$. Using 2.3, we get a $(n-1)$-dimensional manifold F_1 as above. Then F_1 is connected (and closed) with fundamental group $\pi_1(F_1) = \mathbf{Z}^{n-3} \times G$ and $\pi_i(F_1) \sim \pi_i(M)\ \forall\, i \geq 2$. So, if $n - 3 > 0$ and $n - 1 \geq 6$, the fiber F_1 still satisfies the hypothesis of 2.3. We may then

apply successively 2.3 and get a sequence of closed connected manifolds

$$(1) \qquad F_{n-5} \hookrightarrow F_{n-6} \hookrightarrow \cdots \hookrightarrow F_1 \hookrightarrow M$$

such that for $k = 1, \ldots, n-1$:

1. $\pi_1(F_k) = \mathbf{Z}^{n-2-k} \times G$.
2. $dim(F_k) = n - k$.
3. The inclusion $F_k \hookrightarrow M$ induces isomorphisms in π_i for $i \geq 2$.

In particular the manifold F_{n-5} is of dimension 5, its fundamental group is $\mathbf{Z}^3 \times G$ and its higher homotopy groups π_i are finitely generated for $i = 1, \ldots, \max\left\{\left[\frac{n}{2}\right], 3\right\}$.

In order to apply 2.3 to this manifold, we consider the product $F = F_{n-5} \times \mathbf{S}^3$. By 2.3, we get a submanifold K_0 which is a closed connected manifold of dimension 7 with finitely generated π_i for $i = 1, \ldots, \max\left\{\left[\frac{n}{2}\right], 3\right\}$ (since $\mathbf{S}^3$ has the same property by 2.1). Since $[7/2] = 3$, we may apply 2.3 to K_0 and then again to its submanifold K_1, given by 2.3 to obtain a sequence as above, where the maps are inclusions which induce isomorphisms at the level of π_i for $i \geq 2$:

$$K_2 \hookrightarrow K_1 \hookrightarrow K_0 \hookrightarrow F.$$

It follows that the universal covers of G_2 and F are homotopically equivalent, in particular

$$(2) \qquad H_*(\widetilde{K}_2) \sim H_*(\widetilde{F}_{n-5} \times \mathbf{S}^3).$$

Now K_2 is a closed connected 5-dimensional manifold whose fundamental group is G. Since G is an infinite group it follows that $H_5(\widetilde{K}_2) = 0$ (see [1], p.346), we have that $H_i(\widetilde{K}_2)$ vanishes for $i > 4$.

Using the Kunneth formula we infer from (2) that $H_i(\widetilde{F}_{n-5})$ vanishes for $i > 0$. Therefore $\widetilde{F}_{n-5}$ is contractible and, using the sequence (1) $\widetilde{M}$ is contractible, too. So M is an Eilenberg-Mc Lane space $K(\pi_1(M))$.

2. We treat now the cases $n = 3, 4, 5$. Consider the product $F = M_0 \times \mathbf{S}^3$. As above, we can successively apply 2.3 to get a sequence

$$K_{n-2} \hookrightarrow K_{n-3} \hookrightarrow \cdots \hookrightarrow K_1 \hookrightarrow F.$$

The manifold K_{n-2} is closed connected of dimension 5 and its fundamental group is G. We use it to finish the proof using the same argument as above. This proves 1.3.1.

The proof of corollary 1.5 is immediate : $hd(G)$ (the maximal degree r such that $H_r(G, \mathbf{Z})$ does not vanish) is obviously less than equal than the dimension of any finite $K(G, 1)$. Since by Kuneth formula $hd(\mathbf{Z}^{n-2} \times G) = hd(\mathbf{Z}^{n-2}) + hd(G)$ we obtain :

$$hd(G) = hd(\mathbf{Z}^{n-2} \times G) - hd(\mathbf{Z}^{n-2}) \leq dim(M) - (n-2) = 2.$$

Let us now prove 1.3.2. Suppose that $\pi_1(M) = \mathbf{Z}^{n-1} \times G$. From point 1 we infer that M is an Eilenberg-McLane space. As in the estimation above, we obtain that $hd(G) \leq 1$. By a celebrated theorem of J.R. Stallings [21] and Swan [22] we know that G is a free group. Denote by l the number of generators of G. The homology of G is the homology of a wedge of l circles. In particular, using the Kunneth formula we get :

$$dim(\, H_1(\mathbf{Z}^{n-1} \times G, \mathbf{Z}/2\mathbf{Z})\,) \; = \; l + (n-1),$$

and

$$dim(\, H_{n-1}(\mathbf{Z}^{n-1} \times G, \mathbf{Z}/2\mathbf{Z})\,) \; = \; 1 + l(n-1).$$

This means that the Betti numbers of M satisfy $\beta_1(M) = l + (n-1)$ and $\beta_{n-1}(M) = 1 + l(n-1)$. By Poincare duality these numbers coincide, therefore $l = 1$, and $G = \mathbf{Z}$. So M is a $K(\mathbf{Z}^n, 1)$, i.e. a homotopy $n-$torus.

Using again the theorem 2.3 (whose proof is postponed in Section 4), we give the :

Proof of theorem 1.4

Obviously $q \geq 2$. Suppose that $\pi_i(X)$ is finitely generated for all $i \leq q$. By 2.2 the same is true for all the homotopy groups of X.

Embed X in the Euclidean space $\mathbf{R}^{2q+3}$. Let W be a tubular neighbourhood of X and denote by M^{2q+2} the smooth manifold ∂W. Since M is a deformation retract of $W \setminus X$ and X is a deformation retract of X, we look at the sequence of applications

$$M \hookrightarrow W \setminus X \hookrightarrow W \to X,$$

and use a general position argument in the middle arrow to get isomorphisms between $\pi_i(X)$ and $\pi_i(M)$ for $i \leq q+1$. In particular, the homotopy groups $\pi_i(M)$ are finitely generated for $1 \leq i \leq q+1$, since, by 2.2, all the homotopy groups of X are finitely generated. Therefore M satisfies the assumptions of 1.3 and is thus an Eilenberg-McLane space. Since the first $q+1$ homotopy groups of X and of M are the same it follows that X is an Eilenberg-McLane space, too. But this implies that $hd(\mathbf{Z}^{2q} \times G) \leq q$, contradiction. This proves 1.4.

3 Novikov homology and fibrations over the circle

If M is the total space of a fibration f over the circle, then $f^*d\theta$ is a non-vanishing closed one form on M. An analogue of Morse theory for circle valued functions was established by S. P. Novikov in [12]. In the subsequent Morse-Novikov inequalities usual homology is replaced by the Novikov homology of M associated to the cohomology class $u := [f^*d\theta]$. So, if $f : M \to \mathbf{S}^1$ is a fibration, the Novikov homology $H_*(M, [f^*d\theta])$ will vanish.

Conversely, F. Latour [9] and A. Pajitnov [?] [?] proved that under some additional hypothesis the vanishing of the Novikov homology implies the existence of a fibration of M over the circle. We will use this result (actually its proof) in the proof of 2.3. It will be stated in Section 3.2.

Let us start by recalling the definition and some properties of Novikov homology :

3.1 Novikov homology

Let $u \in H^1(M; \mathbf{R})$. Denote by Λ the ring $\mathbf{Z}[\pi_1(M)]$ and by $\hat{\Lambda}$ the ring of formal series $\mathbf{Z}[[\pi_1(M)]]$. Consider a $\mathcal{C}^1$-triangulation of M which we lift it to the universal cover $\widetilde{M}$. We get a Λ-free complex $C_\bullet(M)$ spanned by (fixed lifts of) the cells of the triangulation of M.

We define now *the completed ring* Λ_u :

$$\Lambda_u := \left\{ \lambda = \sum n_i g_i \in \hat{\Lambda} \mid g_i \in \pi_1(M),\ n_i \in \mathbf{Z},\ u(g_i) \to +\infty \right\}$$

The convergence to $+\infty$ means here that for all $A > 0$, $u(g_i) < A$ only for a finite number of g_i which appear with a non-zero coefficient in the sum λ.

Remark Let $\lambda = 1 + \sum n_i g_i$ where $u(g_i) > 0)$ for all i. Then λ is invertible in Λ_u. Indeed, if we denote by $\lambda_0 = \sum n_i g_i$ then it is easy to check that $\sum_{k \geq 0} (-\lambda_0)^k$ is an element of Λ_u and it is obvious that it is the inverse of λ.

Definition *Let $C_\bullet(M, u)$ be the Λ_u-free complex $\Lambda_u \otimes_\Lambda C_\bullet(M)$. The Novikov homology $H_*(M, u)$ is the homology of the complex $C_\bullet(M, u)$.*

A purely algebraic consequence of the previous definition is the following version of the universal coefficients theorem ([8], p.102, Th 5.5.1) :

Theorem 3.1 *There is a spectral sequence E^r_{pq} which converges to $H_*(M,u)$ and such that*

$$E^2_{pq} = Tor^{\Lambda}_p(\Lambda_u, H_q(\widetilde{M})).$$

We will use this result in Section 4 to prove that in the hypothesis of 2.3, the Novikov homology associated to some class u vanishes.

3.2 Morse-Novikov theory

Let α be a closed generic one form in the class u. Let ξ be the gradient of α with respect to some generic metric on M. For every critical point c of α we fix a point $\tilde{c}$ above c in the universal cover $\widetilde{M}$. We can define then a complex $C_\bullet(\alpha,\xi)$ spanned by the zeros of α : the incidence number $[d,c]$ for two zeros of consecutive indices is the (possibly infinite) sum $\sum n_i g_i$ where n_i is the algebraic number of flow lines which join c and d and which are covered by a path in $\widetilde{M}$ joining $g_i\tilde{c}$ and $\tilde{d}$. It turns out that this incidence number belongs to Λ_u, so $C_\bullet(\alpha,\xi)$ is actually a Λ_u-free complex.

The fundamental property of the Novikov homology is that it is isomorphic to the homology of the complex $C_\bullet(\alpha,\xi)$ above for any couple (α,ξ).

By comparing the complexes $C_\bullet(\alpha,\xi)$ and $C_\bullet(-\alpha,-\xi)$ we get the following duality property (see prop.2.8 in [4] and 2.30 in [9]) :

Theorem 3.2 *Let M^n be a closed connected manifold, $u \in H^1(M;\mathbf{R})$ and let l be an integer. If $H_i(M,-u) = 0$ for $i \leq l$, then $H_i(M,u) = 0$ for $i \geq n-l$.*

If the form α has no zeroes then $C_\bullet(\alpha,\xi)$ vanishes and therefore we have $H_*(M,[\alpha]) = 0$. Conversely, one can ask if the vanishing of $H_*(M,u)$ implies the existence of a nowhere vanishing closed 1-form belonging to the class $u \in H^1(M)$. For $n \geq 6$ this problem was independently solved by F. Latour [9] and A. Pajitnov [**?**], [**?**]. The statement is ([9], Th.1 :

Theorem 3.3 *For $dim(M) \geq 6$) the following set of conditions is equivalent to the existence of a nowhere vanishing closed* 1*-form in* $u \in H^1(M,\mathbf{Z})$ *:*

1. *Vanishing Novikov homology $H_*(M,u)$.*
2. *Vanishing Whitehead torsion $\tau(M,u) \in Wh(M,u)$.*
3. *Finitely presented $Ker(u) \subset \pi_1(M)$.*

Remarks

1. The definition of the generalized Whitehead group $Wh(M,u)$ and of the Whitehead torsion is given in [9].
2. In the statement of [**?**] the first two conditions are replaced by :

   1 $C_\bullet(M,u)$ is simply equivalent to zero.

   Actually, one can show (see [10]) that 1îs equivalent to "1 and 2".
3. In earlier works on the subject as those of F.T. Farell [6] and L. Siebenmann [19] the algebraic conditions which are equivalent to the existence of a nowhere vanishing closed 1-form in a rational cohomology class u were stated in the hypothesis that the infinite cyclic cover X associated to u is finitely dominated. ($X \to M$ is defined to be the pull-back of the universal covering $\mathbf{R} \to \mathbf{S}^1$ defined by a function $f : M \to \mathbf{S}^1$ such that $[f^*d\theta] = u$). The relation between the finite domination of X and vanishing of the Novikov homology (as well as between the Whitehead "fibering obstruction" from [6] and [19] and the condition 2 above) was first established by A. Ranicki in [16], [17] (see also [15]).

The hypothesis on $\pi_i(M)$ in our theorem 2.3 can be seen in this framework as a sufficient condition for the *algebraic* finite domination of the $\mathbf{Z}[t,t^{-1}]$-free complex $C_{\bullet \leq [\frac{n}{2}]}(X)$.

If we drop the condition (2) in the the statement 3.3 about, we get the following :

Theorem 3.4 *Suppose that M^n is of dimension $n \geq 6$ and that $u \in H^1(M,\mathbf{Z})$ fulfilles the conditions (1) and (3) of 3.3. Fix $2 \leq k \leq n-3$. Then there exists $f : M \to \mathbf{S}^1$ such that $[f^*d\theta] = u$ and whose critical points have indices equal to k or $k+1$.*

This can be seen by following the proof of F. Latourş theorem in [9]. The idea is that under the assumptions (1) and (3), the critical points of small index and co-index can be successively cancelled, as in the theorem of s-cobordism.

4 Proof of 2.3

In order to prove 2.3 we will use a consequence of 3.4, namely :

Theorem 4.1 *Let M^n be a closed connected manifold of dimension $n \geq 6$. Suppose that $\pi_1(M) = \mathbf{Z} \times H$.*

Let u be the cohomology class which corresponds to the projection $\mathbf{Z} \times H \to \mathbf{Z}$ in the natural isomorphism $H^1(M;\mathbf{Z}) \sim Hom(\pi_1(M),\mathbf{Z})$. Suppose that $H_*(M,u)$ vanishes.

Then there exist $f : M \to \mathbf{S}^1$ and $\gamma \in \mathbf{S}^1$ such that $[f^*d\theta] = u$ and $F = f^{-1}(\gamma)$ satisfies to the requirements of 2.3.

<u>Proof</u>

First remark that the third condition of 3.3 is obviously fulfilled since H is finitely presented and the first is granted by hypothesis. So we get $f : M \to \mathbf{S}^1$ with the properties of 3.4. The complex associated to $f^*d\theta$ and a generic gradient will be of the form :

$$0 \to \Lambda_u^p \stackrel{\partial_k}{\to} \Lambda_u^p \to 0,$$

where ∂_k is an isomorphism, since the Novikov homology vanishes.

Let γ be a regular value of f and let $F = f^{-1}(\gamma)$. Cut M along F to get a cobordism W between F and F. Note that, since f has no critical point of index and co-index less or equal than two, the fundamental group of F is H and the inclusion $F \hookrightarrow W$ induces an isomorphism in π_1 (A disk in W can be pushed into F along the flow lines of a gradient by general position).

Remark also that W can be seen as a fundamental domain of the Abelian covering $\bar{M} \to M$ which corresponds to the normal subgroup $H \subset \mathbf{Z} \times H$. In other words $\bar{M}$ is obtain by pasting countable many copies of W. Now for every critical point c of f consider a lift $\tilde{c} \in \widetilde{M}$ such that all the projections of these lifts in $\bar{M}$ lie in the same fundamental domain.

Denote by A the matrix corresponding to ∂_k with respect to this basis. If we denote by t the generator of $\mathbf{Z}$ it is easy to see that A can be written as :

$$A \;=\; A_0 + A_1 t + \cdots + A_i t^i + \cdots,$$

where A_i are $p \times p$ matrices with entries in $\mathbf{Z}[H]$. Note that the entries of A_0 are computed using the flow lines of the gradient whose lifts on $\bar{M}$ stay in the fundamental domain W.

Since A is invertible, A_0 is invertible, too ; This means that the complex associated to the cobordism (W, F, F) (endowed with a lift of f (which is a real Morse function) and a lift of the gradient of f) is acyclic. This implies that $H_i(W,F) = 0$ for all i, and using excision and Mayer-Vietoris, that $H_i(\bar{M},F) = 0$ for all i. But $F \hookrightarrow \bar{M}$ yields an isomorphism in π_1 so, by Whiteheadş theorem, it is an isomorphism in π_i for all i. As the projection $\bar{M} \to M$ induces isomorphisms in π_i for $i \geq 2$ the theorem is proved. $\diamond$

Taking into account the preceeding theorem it suffices to prove the following proposition :

Proposition 4.2 *Let M^n be a closed connected manifold. Suppose that $H_i(\widetilde{M})$ are finitely generated for $i = 2, \ldots, \left[\frac{n}{2}\right]$ and $\pi_1(M) = \mathbf{Z} \times H$. Then $H_*(M,u) = 0$, where $u = pr_{\mathbf{Z}}$.*

Remark There is a theorem of A. Ranicki [17] (see also [15], ch.8) which establishes an equivalence between the algebraic finite domination of the infinite cyclic covering X and the vanishing of the Novikov-type homology $H_*(C_\bullet(X) \otimes_{\mathbf{Z}[t,t^{-1}]} \mathbf{Z}(t)[t^{-1}]$. As our hypothesis on $\pi_i(M)$ has an influence on the algebraic finite domination of X this result is strongly related to 4.2.

<u>Proof of 4.2</u>

This proposition was proved in [5] (Prop.4.1). We will only give an outline of the proof. By 3.2 it suffices to prove that $H_i(M, \pm u) = 0$ for $i \leq \left[\frac{n}{2}\right]$. The proof is based on a Cayley-Hamilton argument as in the result of [15] mentioned above. The idea is to find a endomorphism $\bar{\Phi}$ of $H_i(M,u)$ which is both invertible and nilpotent for $i \leq \left[\frac{n}{2}\right]$. The morphism $\bar{\Phi}$ will be induced by a chain morphism

$$\Phi = Id \otimes \psi : \Lambda_u \otimes_\Lambda C_\bullet(M) \to \Lambda_u \otimes_\Lambda C_\bullet(M).$$

Denote by g the element $t \times 1 \in \pi_1(M)$, where t is the generator of $\mathbf{Z}$. the morphism $\psi : C_\bullet(M) \to C_\bullet(M)$ will be the left multiplication by $P(g)$ where P is a polynomial with integer coefficients.

As $g \in Z(\pi_1(M))$ the left multiplication by g induces a isomorphism of Λ-complexes

$$g : C_\bullet(M) \to C_\bullet(M),$$

which yields a Λ-isomorphism in homology

$$\bar{g} : \oplus_{i=0}^{\left[\frac{n}{2}\right]} H_i(\widetilde{M}) \to \oplus_{i=0}^{\left[\frac{n}{2}\right]} H_i(\widetilde{M}).$$

Now if $P \in \mathbf{Z}[X]$ is a polynomial of the form $\pm 1 + XQ(X)$, then $P(g)$ is invertible in Λ_u and therefore Φ is invertible.

On the other hand we are able to choose P such that $P(g)$ induces the zero morphism $H_i(\widetilde{M})$ for $i \leq \left[\frac{n}{2}\right]$: Denote by A the sum $\oplus_{i=0}^{\left[\frac{n}{2}\right]} H_i(\widetilde{M})$. Because of 2.1 we have that A is finitely generated as a $\mathbf{Z}$-module. Therefore A is isomorphic to $\mathbf{Z}^r \oplus T$, where T is a direct sum of modules $\mathbf{Z}/k\mathbf{Z}$, $(k \in \mathbf{N}^*)$.

The restriction $\bar{g} : \mathbf{Z}^r \to \mathbf{Z}^r$ is an isomorphism and therefore its characteristic polynomial is of the form

$$R(X) = \pm 1 + X\, R_0(X).$$

The restriction $\bar{g} : T \to T$ is again an isomorphism and, since $Aut(T)$ is finite, there is an integer s such that $(\bar{g}^s)|_T = Id$.

It easily follows that the polynomial $P(X) = [(1 - X^s)R(X)]^2$ satisfies $P(\bar{g}) = 0$, as claimed.

Now we use the spectral secquence 3.1 which converges to $H_*(M; u)$. The chain morphism $P(g) : C_\bullet(M) \to C_\bullet(M)$ naturally induces at the page

E^2_{pq} a morphism

$$Tor_p^{\Lambda}(Id, P(\bar{g})_q) \,:\, Tor_p^{\Lambda}(\Lambda_u, H_q(\widetilde{M})) \,\to\, Tor_p^{\Lambda}(\Lambda_u, H_q(\widetilde{M}))$$

which, is zero for $q \leq \left[\frac{n}{2}\right]$. By naturality in the spectral sequence E^r_{pq} we obtain that the induced endomorphism $\bar{\Phi}^{\infty}_{pq} : E^{\infty}_{pq} \to E^{\infty}_{pq}$ vanishes for $q \leq \left[\frac{n}{2}\right]$.

The spectral secquence E^r_{pq} yields a filtration

$$0 = A_{-1} \hookrightarrow A_0 \hookrightarrow A_1 \hookrightarrow \cdots \hookrightarrow A_k \hookrightarrow \cdots$$

of the Novikov homology groups $H_*(M, u)$ such that for every couple of integers (i, p) there is an exact sequence :

$$(4) \qquad 0 \to A_{p-1}(H_i(M,u)) \to A_p(H_i(M,u)) \to E^{\infty}_{p,i-p} \to 0.$$

Actually, the above filtration is stationary for $k \geq n$, as one can infer from the exact sequence (4).

By naturality, the morphism $\bar{\Phi} : H_*(M,u) \to H_*(M,u)$ preserves the filtration and yields, together with Φ^{∞}, a morphism from the exact complex (4) to itself.

Therefore we have a commutative diagram :

$$\begin{array}{ccccccccc}
0 & \to & A_{p-1}(H_i(M,u)) & \to & A_p(H_i(M,u)) & \to & E^{\infty}_{p,i-p} & \to & 0 \\
 & & \downarrow \bar{\Phi} & & \downarrow \bar{\Phi} & & \downarrow \bar{\Phi}^{\infty}_{p,i-p} & & \\
0 & \to & A_{p-1}(H_i(M,u)) & \to & A_p(H_i(M,u)) & \to & E^{\infty}_{p,i-p} & \to & 0
\end{array}$$

Fix some $i \leq \left[\frac{n}{2}\right]$. The vertical arrow $\bar{\Phi}^{\infty}$ vanishes for every p. Proceeding by induction on p, we easily infer from the above diagram that the morphism $\bar{\Phi} : A_p(H_i(M,u)) \to A_p(H_i(M,u))$ is nilpotent.

Finally, for $p = n$, $A_n(H_i(M,u)) = H_i(M,u)$, so the morphism $\bar{\Phi}$ is nilpotent in Novikov homology for $i \leq \left[\frac{n}{2}\right]$. Since the same morphism was proved to be invertible, $H_i(M;u)$ vanishes for $i \leq \left[\frac{n}{2}\right]$.

An analogous proof shows that $H_i(M,-u) = 0$ for $i \leq \left[\frac{n}{2}\right]$. Indeed, the same argument with g^{-1} instead of g yields a polynomial P as above. We get thus an endomorphism of $H_i(M,-u)$ which is both nilpotent and onto.

This completes the proof of 4.2 (using the duality 3.2).

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