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A set of tangential approximation by meromorphic functions

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Dedicated to Professor Dumitru Acu on his 60th anniversary

Abstract

The aim of this paper is to establish a closed subset E of the complex plane $\mathbb{C}$, the interior E^0 of which forms one unbounded Gleason part, nevertheless E is a set of tangential approximation by functions meromorphic in $\mathbb{C}$.

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1 Introduction

To state our main result we need some notations and facts. For arbitrary $A \subset \mathbb{C}$ we denote by $A^0, \partial A, \overline{A}$ and A^c the interior, boundary, closure and complement of A in $\mathbb{C}$, respectively. For a closed subset $E \subset \mathbb{C}$ let $\mathcal{A}(E)$ be the space of all complex valued functions which are continuous on E and holomorphic in the interior E^0 of E. For a compact $K \subset \mathbb{C}$ we denote by $\mathcal{R}(K)$ the set of all functions on K which are uniform limits of functions rational in $\mathbb{C}$ without poles on K. Further, let E be a relatively closed subset of a domain $D \subseteq \mathbb{C}$. Then the space $\mathcal{M}(E)$ denote the set of all functions on E which are uniform limits of functions meromorphic in D without poles on E.

Theorem 1. (*Nersessian* [5]) *$\mathcal{A}(E) = \mathcal{M}(E)$ if and only if $\mathcal{R}(E \cap K) = \mathcal{A}(E \cap K)$ for any closed disk $K \subset D$.*

On the other hand we can base on the following sufficient conditions for the equality $\mathcal{A}(K) = \mathcal{R}(K)$ for any compact $K \subset \mathbb{C}$.

Theorem 2. (*Mergelyan* [4]) *If K^c has a finite number of components then $\mathcal{A}(K) = \mathcal{R}(K)$.*

Theorem 3. (*Vitushkin* [6])) *If the interior boundary of K lies on countable many C^2 curves then $\mathcal{A}(K) = \mathcal{R}(K)$.*

($x \in \partial K$ is said to be an interior boundary point, if $x \notin \partial\Omega$ for any component Ω of K^c).

Theorem 2 is a consequence of Theorem 3, when the interior boundary of K is empty.

Definition 1. *A closed subset $E \subset \mathbb{C}$ is said to be a set of tangential (Carleman) approximation with functions meromorphic in $\mathbb{C}$, if for arbitrary functions f and ε, where $f \in \mathcal{A}(E)$ and $\varepsilon \in C(E), \varepsilon > 0$, there exists a meromorphic function g in $\mathbb{C}$ without poles on E such that*

$$|f(z) - g(z)| < \varepsilon(z) \quad \text{for} \quad z \in E.$$

Definition 2.

(i) *For a compact* $K \subset \mathbb{C}$ *we say that* $x, y \in K$ *are equivalent,* $x \sim y$, *if there exists a* $c > 0$ *such that*

$$\frac{1}{c} < \frac{u(x)}{u(y)} < c$$

for any $u \in \mathrm{Re}\,(R(K)), u > 0$.

(ii) *Any equivalence class of* K *is said to be a Gleason part of* $\mathcal{R}(K)$.

(iii) *For a closed* $E \subset \mathbb{C}$ *a subset* $G \subset E$ *is called to be a Gleason part of* $\mathcal{M}(E)$, *if* $K \cap G$ *is a Gleason part of* $\mathcal{R}(K \cap E)$ *for any closed disk* K.

In the paper [1] the following condition is given for sets to be sets of tangential approximation.

Theorem 4. (*Boivin* [1]) *Let* $E \subset \mathbb{C}$ *be closed. If for any closed disk* K

(i) *there exists a disk* $\tilde{K} \supset K$ *such that any Gleason part of* $\mathcal{M}(E)$ *that has a non empty intersection with* K *lie in* $\tilde{K}$,

(ii) *if* $\mathcal{A}(K \cap E) = \mathcal{R}(K \cap E)$,
then E *is a set of tangential approximation with functions meromorphic in* $\mathbb{C}$.

In this paper we show that there exists a set E, the interior of which forms one unbounded Gleason part of $\mathcal{M}(E)$ (the condition (i) of Theorem 4 is not satisfied), but E is a set of tangential approximation by functions meromorphic in $\mathbb{C}$.

2 (L)-type sets

Let us set

$$D(a,r) := \{z \in \mathbb{C} : |z-a| < r\}, \ \ \mathbb{D} := D(0,1), \ \ C := \partial\mathbb{D}.$$

Definition 3. *A closed domain* $\mathcal{L} = \mathcal{L}(\{z_i\}_{i=1}^{\infty}, \{r_i\}_{i=1}^{\infty}) := \overline{\mathbb{D}} \setminus \bigcup_{i=1}^{\infty} D(z_i, r_i)$ *is said to be an* (L)*-type set, if the sequences* $\{z_i\}_{i=1}^{\infty}$ *and* $\{r_i\}_{i=1}^{\infty}$ *satisfy the following conditions:*

(i) $|z_i| < 1, r_i < 1 - |z_i|, i = 1, 2, \ldots,$

(ii) $(\{z_i\}_{i=1}^{\infty})' = C,$

(iii) $r_i + r_j < |z_i - z_j|$ *for* $i \neq j$.

(*In* (ii) *"′" means the set of all cluster points*).

Definition 4. *An* (L)*-type set* $\mathcal{L}$ *is called a uniqueness set if* $f \in \mathcal{A}(\mathcal{L})$ *and* $f(z) = 0$ *on* C *imply* $f(z) \equiv 0$ *on* $\mathcal{L}$.

In [2] A.A. Gonchar has shown that there are (L)-type non-uniqueness sets. More precisely, the following proposition is true.

Proposition 1. *For every* $\beta > 2$ *and* $\varepsilon > 0$ *there are*

(i) (L)*-type set* $\mathcal{L}$ *with the property*

$$\sum_{i=1}^{\infty} \left(\frac{1}{\ln \frac{1}{r_i}} \right)^{\beta} < \varepsilon, \tag{1}$$

(ii) *a function* μ *of the form*

$$\mu(z) = \sum_{i=0}^{\infty} \frac{A_i}{z - z_i} \tag{2}$$

(*the serie converges uniform on* $\mathcal{L}$)

such that $\mu(z) = 0$ *on* C *and* $\mu(z) \not\equiv 0$ *on* $\mathcal{L}$.

Corollary 1. *For arbitrary* $\alpha > 0$ *there exists an* (L) *type set with a function* μ *satisfying the condition* (ii) *of Proposition* 1 *so that*

$$\sum_{i=1}^{\infty} r_i^{\alpha} < \infty.$$

In fact, for any $\alpha > 0, \beta > 2$ we have

$$r_i^{\alpha} \left(\ln \frac{1}{r_i} \right)^{\beta} \to 0 \quad \text{when} \quad r_i \to 0.$$

Hence, $\sum_{i=1}^{\infty} \frac{1}{(\ln \frac{1}{r_i})^{\beta}} < \varepsilon$ implies $\sum_{i=1}^{\infty} r_i^{\alpha} < \infty$.

Remark 1. *The function* μ *is meromorphic in the unit disk.*

In fact, on the circle $C(z_i, r_i), i = 0, 1, 2, \ldots,$ the series $\sum_{k=0}^{\infty} \frac{A_k}{z - z_k}$ converges uniformly, and from the maximum principle follows that the series $\sum_{k=0}^{i-1} \frac{A_k}{z-z_k} + \sum_{k=i+1}^{\infty} \frac{A_k}{z-z_k}$ uniformly converges in the circle $D(z_i, r_i)$. Hence, μ is analytic in $D(z_i, r_i)$ except at the point $z = z_i$, where μ has a simple pole. Resuming, μ is meromorphic in unit circle with the simple poles $\{z_i\}_{i=0}^{\infty}$.

Below $\mathcal{L}$ denotes an (L)-type non-uniqueness set with the property $\sum_{i=1}^{\infty} r_i < \infty$, and μ is the function from Proposition 1.

Let us set

$$C_1 := \{z = x + iy : |z| = 1, -1 \leq x < 0\},$$

$$C_2 := \{z = x + iy : |z| = 1, 0 < x \leq 1\}.$$

Lemma 1. *For any $\mathcal{L}$ set there exists a meromorphic function $\nu(z)$ in the unit disk such that*

$$\lim_{\mathcal{L}\ni z\to l}\nu(z)=\begin{cases}0, & l\in C_1\\ 1, & l\in C_2\end{cases}.$$

Proof. Let us set

$$A_1:=\mathcal{L}\setminus(D(1,\sqrt{2})\cup C),\ \ A_2:=\mathcal{L}\setminus(D(-1,\sqrt{2})\cup C),\ \ F:=A_1\cup A_2,$$

and take

$$f(z)=\begin{cases}0, & z\in A_1\\ 1, & z\in A_2\end{cases}.$$

We have $f(z)\in\mathcal{A}(F)$. The complement of the intersection $F\cap K$ for any closed disk $K\subset\mathbb{D}$ consists of a finite number of components (since $\{z_i\}'=C$); hence, $F\cap K$ is a set of uniform approximation with rational functions (Theorem 2) which implies that F is a set of uniform approximation with functions meromorphic in $\mathbb{D}$ (Theorem 1). Let us take the function f/μ. The zeros of μ that lie in A_2 are denoted by $\xi_1,\ldots,\xi_n,\ldots$. Applying Mittag-Leffler's theorem there is a meromorphic function $h(z)$ with poles at $\xi_1,\ldots,\xi_n,\ldots$ (and only this points), and with principal parts of the Laurent expansions coinciding with the corresponding principal parts of the function $1/\mu$. In that case we have $(f/\mu-h)\in\mathcal{A}(F)$. In $\mathbb{D}$ there exists a meromorphic function v without poles on F such that

$$\left|\frac{f}{\mu}-h-v\right|<1\quad\text{on}\quad F,$$

which gives us $|f-\mu(h+v)|<|\mu|$ on F. Because of $|\mu|\to 0$ when $z\to l\in C$, the in $\mathbb{D}$ meromorphic function $\nu:=\mu(h+v)$ satisfies the assumption of the lemma.

Remark 2. *For an $\mathcal{L}$ set for every point $z_0 \in C$ the condition*

$$\lim_{\delta \to 0} \frac{\sum_{D(z_i,r_i) \subset D(z_0,\delta)} r_i}{\delta} = 0 \tag{3}$$

is satisfied (cf. [2]); hence according to a result from [2] we get that the interior X^0 of $X := \overline{D(0,2)} \setminus \cup_{i=0}^{\infty} D(z_i, r_i)$ forms a Gleason part of $R(X)$.

3 The main result

Theorem 5. *There exists a closed subset $E \subset \mathbb{C}$ so that E^0 forms an unbounded Gleason part of $\mathcal{M}(E)$ and E is a set of tangential approximation by functions meromorphic in $\mathbb{C}$.*

Proof. Consider the strip

$$\Pi := \{z = x + iy : -1 \le y \le 1\}$$

and the $\mathcal{L}$ set

$$\overline{\mathbb{D}} \setminus \bigcup_{i=1}^{\infty} D(z_i, r_i)$$

and set

$$\begin{aligned} E &:= (\Pi \setminus \cup_{n=-\infty}^{\infty} \cup_{i=1}^{\infty} D(z_i + 3n, r_i)) \setminus \cup_{n=-\infty}^{\infty} D(3n \pm i, \frac{1}{4}), \\ D_n' &:= \overline{D(0,3n)} \setminus (D(3m,1) \cup D(3m \pm i, \frac{1}{4})), m = \pm n, n = 1, \ldots, \\ D_n'' &:= (\overline{D(0,3n)} \cup \overline{D(3m,1)}) \setminus D(3m \pm i, \frac{1}{4}), m = \pm n, n = 1, \ldots . \end{aligned}$$

Because of Remark 2 the interior E^0 forms one Gleason part of $\mathcal{M}(E)$. Let $f \in \mathcal{A}(E)$ be arbitrary, and $\varepsilon \in C(E), \varepsilon > 0$, tends to 0 if $|z| \to \infty$. There exists a rational function $R_2(z)$ (Theorem 3) so that

$$|f(z) - R_2(z)| < \frac{\varepsilon(7)}{4(c+1)}, z \in D_2'' \cap E,$$

where $c = ||\nu||_{\mathcal{L}}$ (ν is the function from Lemma 1).

Choose a rational function $Q_3(z)$ so that

$$|f(z) - R_2(z) - Q_3(z)| < \frac{\varepsilon(10)}{4(c+1)}, z \in D_3'' \cap E.$$

Set

$$\mu_3(z) := \begin{cases} 0, \ \ z \in D_2', \\ \nu(z), \ \ z \in \overline{D(6,1)} \setminus D(6 \pm i, \frac{1}{4}), \\ 1 - \nu(z), z \in \overline{D(-6,1)} \setminus D(-6 \pm i, \frac{1}{4}), \\ 1, \ \ z \in (D_3'' \setminus (D_2'')^0) \cap E. \end{cases}$$

Clearly $\mu_3 \in \mathcal{A}(D_2' \cup (D_3'' \cap E))$. According to Theorem 3 for any given $\delta > 0$ there exists a rational function $\tilde{\rho}_3$ so that

$$\begin{aligned} &i) \quad |\tilde{\rho}_3|_{D_2'} < \delta, \\ (4) \qquad &ii) \quad |\tilde{\rho}_3 - 1|_{(D_3'' \setminus (D_2'')^0) \cap E} < \delta, \\ &iii) \quad |\tilde{\rho}_3|_{(D_3'' \cap E) \cup D_2'} < c + 1. \end{aligned}$$

Let $z_1^{(3)}, \ldots, z_{m_3}^{(3)}$ be the poles of Q_3 in D_1' with multiplicities $\alpha_1^{(3)}, \ldots, \alpha_{m_3}^{(3)}$, respectively. According to the Cauchy integral formula for derivatives and (4) i), from the arbitrariness of δ we can assume that

$$(5) \qquad |\tilde{\rho}_3^{(s)}(z_j^{(3)})| < \delta',$$

for arbitrary $\delta' > 0, s = 0, \ldots, \alpha_j^{(3)} - 1, j = 1, \ldots, m_3$. It is well-known that there exists a unique polynom p_3 of order $\sum_{j=1}^{m_3} \alpha_j^{(3)} - 1$, satisfying the conditions

$$p_3^{(s)}(z_j^{(3)}) = \tilde{\rho}_3^{(s)}(z_j^{(3)}), j = 1, \ldots, m_s, s = 0, \ldots, \alpha_j^{(3)} - 1.$$

In this connection the polynom has the form

$$p_3(z) = \sum_{j=1}^{m_3} \frac{\omega(z)}{(z - z_j^{(3)})^{\alpha_j^{(3)}}} \sum_{s=0}^{\alpha_j^{(3)}-1} A_{j,s}(z - z_j^{(3)})^s,$$

where

$$\omega(z) = \prod_{j=1}^{m_3} (z - z_j^{(3)})^{\alpha_j^{(3)}},$$

$$A_{j,s} = \sum_{\nu=0}^{s} \frac{1}{\nu!(s-\nu)!} \tilde{\rho}_3^{(\nu)}(z_j^{(3)}) \Big[\frac{d^{s-\nu}}{dz^{s-\nu}} \frac{(z - z_j^{(3)})^{\alpha_j^{(3)}}}{\omega(z)} \Big]_{z=z_j}.$$

¿From (5) it follows that $|A_{j,s}|$ and hence also $||p_3||_K$ for any compact $K \subset \mathbb{C}$ can be assumed arbitrarily small. Summing up, it can be assumed that the rational function $\rho_3 = \tilde{\rho}_3 - p_3$ satisfies the conditions

$$\begin{aligned} & \rho_3^{(s)}(z_j^{(3)}) = 0, s = 0, 1, \ldots, \alpha_j^{(3)} - 1, j = 1, \ldots, m_j, \\ & |\rho_3|_{D_2'} < \varepsilon, \\ & |\rho_3 - 1|_{(D_3'' \setminus (D_2'')^0) \cap E} < \varepsilon, \\ & |\rho_3|_{(D_3'' \cap E) \cup D_2'} < c + 1 \end{aligned} \tag{6}$$

for any $\varepsilon > 0$. Observe that the function ρ_3 is taken so that the rational function $R_3 = \rho_3 Q_3$ has no poles in D_1'. Taking ε in (6) sufficiently small, we can assume that the rational function R_3 satisfies the conditions

$$\begin{aligned} & |R_3| < \frac{1}{2^3}, z \in D_1', \\ & |f(z) - R_2(z) - R_3(z)| < \varepsilon(4), z \in D_1'' \cap E, \\ & |f(z) - R_2(z) - R_3(z)| < \frac{\varepsilon(10)}{4(c+1)}, z \in (D_3'' \setminus D_2'') \cap E, \\ & |f(z) - R_2(z) - R_3(z)| \leq |f(z) - R_2(z)| + |R_3(z)| \\ < \; & \frac{\varepsilon(7)}{4(c+1)} + (c+1)|Q_3(z)| < \frac{\varepsilon(7)}{4(c+1)} + (c+1)\Big(\frac{\varepsilon(7)}{4(c+1)} + \frac{\varepsilon(10)}{4(c+1)}\Big) \\ < \; & \varepsilon(7), z \in (D_2'' \setminus D_1'') \cap E. \end{aligned}$$

Let now for any $n > 3$ the functions $R_2, \ldots, R_n$ are taken so that

$$\begin{aligned}
&i)\quad |R_k(z)| < \frac{1}{2^k}, z \in D'_{k-2}, k = 3, \ldots, n,\\
(7)\quad &ii)\quad |f(z) - R_2(z) - \ldots - R_n(z)| < \varepsilon(3k+1), z \in (D''_k \setminus D''_{k-1}) \cap E,\\
&\quad k = 1, \ldots, n-1, D''_0 = \emptyset,\\
&iii)\quad |f(z) - R_2(z) - \ldots - R_n(z)| < \frac{\varepsilon(3n+1)}{4(c+1)}, z \in (D''_n \setminus D''_{n-1}) \cap E.
\end{aligned}$$

According to Theorem 3 there exists a rational function Q_{n+1} satisfying the condition

$$(8)\quad |f(z) - R_2(z) - \ldots - R_n(z) - Q_{n+1}(z)| < \frac{\varepsilon(3(n+1)+1)}{4(c+1)}, z \in D''_{n+1} \cap E.$$

Arguing as for the construction of the function ρ_3 we get a rational function ρ_{n+1} satisfying the conditions

$$\begin{aligned}
&\rho^{(s)}_{n+1}(z^{(n+1)}_j) = 0, s = 0, \ldots, \alpha^{(n+1)}_j - 1, j = 1, \ldots, m_{n+1},\\
&|\rho_{n+1}|_{D'_n} < \varepsilon,\\
(9)\quad &|\rho_{n+1} - 1|_{(D''_{n+1} \setminus (D''_n)^0) \cap E} < \varepsilon,\\
&|\rho_{n+1}|_{(D''_{n+1} \cap E) \cup D'_n} < c + 1,
\end{aligned}$$

where ε can be arbitrarily small. In particular, we can assume ε so small that the rational function $R_{n+1} = \rho_{n+1} Q_{n+1}$ satisfies the conditions

$i)\quad |R_{n+1}(z)| < \frac{1}{2^{n+1}}, z \in D'_{n-1},$

$$(10)\quad ii)\ |f(z) - R_2(z) - \ldots - R_{n+1}(z)| < \varepsilon(3k+1), z \in (D''_k \setminus D''_{k-1}) \cap E,$$

$$k = 1, \ldots, n-1, D''_0 = \emptyset,$$

$$iii)\ |f(z) - R_2(z) - \ldots - R_{n+1}(z)| < \tfrac{\varepsilon(3(n+1)+1)}{4(c+1)}, z \in (D''_{n+1} \setminus D''_n) \cap E.$$

According to the relations (7) iii), (8) and (9), we have

$$(11)\quad |f(z) - R_2(z) - \ldots - R_{n+1}(z)| \leq |f(z) - \ldots - R_n(z)| + |R_{n+1}(z)| <$$
$$< \frac{\varepsilon(3n+1)}{4(c+1)} + (c+1)\Big(\frac{\varepsilon(3(n+1)+1)}{4(c+1)} + \frac{\varepsilon(3n+1)}{4(c+1)}\Big) < \varepsilon(3n+1).$$

¿From (10) and (11) it follows that the relations (7) are true if n is replaced by $n+1$. By induction, we can assume that there exists a sequence $\{R_n\}_{n=2}^{\infty}$ of rational functions satisfying the conditions

$$|R_k(z)| < \frac{1}{2^k}, z \in D'_{k-2}, k = 3, 4, \ldots.$$

Thus it follows that the serie $G = \sum_{n=2}^{\infty} R_n$ uniformly converges on any compact subset of $\mathbb{C}$ after dropping a finite number of summands. Since all summands are ratioanl functions, G is meromorphic in $\mathbb{C}$. On the other hand, since for any number $k = 1, 2, \ldots$ the relation (7) ii) is valid for all numbers $n, n > k$, passing to the limit when $n \to \infty$, for any $z \in E$ we get

$$|f(z) - G(z)| < \varepsilon(z).$$

The theorem is proved.

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