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Certain Divisible Hypergroups

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Dedicated to Professor Dumitru Acu on his 60th anniversary

Abstract

A group G is said to be *divisible* if for every $x \in G$ and every $n \in \mathbb{N}$, $x = y^n$ for some $y \in G$ where $\mathbb{N}$ is the set of all positive integers. More generally, we call a hypergroup $(A, \circ)$ a *divisible hypergroup* if for every $x \in A$ and every $n \in \mathbb{N}$, $x \in (y, \circ)^n$ for some $y \in A$ where $(y, \circ)^n$ denotes $y \circ y \circ .... \circ y$ (n copies). If G is any group and $H < G$, let G/H and $G|H$ be respectively the sets $\{xH | x \in G\}$ and $\{HxH | x \in G\}$. It is known that $(G/H, \circ)$ and $(G|H, \diamond)$ are hypergroups where $xH \circ yH = \{tH | t \in xHy\}$ and $HxH \diamond HyH = \{HtH | t \in xHy\}$. These hypergroups will be shown to be divisible if the group G is divisible. Let $U_n(\mathbb{R})$ be the group under multiplication of all nonsingular upper triangular $n \times n$ matrices over $\mathbb{R}$. Then the group $U_n(\mathbb{R})$ is not divisible. However, it is known that the group $U_n^+(\mathbb{R}) = \{A \in U_n(\mathbb{R}) | A_{ii} > 0 \text{ for all } i \in \{1, ..., n\}\}$ is divisible. Based on this result, we show that there are infinitely many subgroups H of $U_n(\mathbb{R})$ such that the hypergroups $(U_n(\mathbb{R})/H, \circ)$ and $(U_n(\mathbb{R})|H, \diamond)$ are divisible.

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1 Introduction

The cardinality of a set X will be denoted by $|X|$. Let $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ denote respectively the set of positive integers, the set of integers, the set of rational numbers and the set of real numbers. For any subfield F of the field $\mathbb{R}$, let $F^* = F \backslash \{0\}$ and $F^+ = \{x \in F | x > 0\}$.

We call a group G a *divisible group* if for every $x \in G$ and every $n \in \mathbb{N}$, $x = y^n$ for some $y \in G$. The the additive group $(\mathbb{Q}, +)$ is divisible while the multiplicative group $(\mathbb{Q}^+, \dot{})$ is not divisible. The group $(\mathbb{R}^+, \dot{})$ is clearly divisible. Divisible abelian groups have been characterized in terms of $\mathbb{Z}$-injectively. This can be seen in [2], page 195. It is also known that every nonzero finite abelian group is not divisible ([2], page 198). In fact, a more general result is obatined from [5] as follows:

Proposition 1.*([5]) If G is a nontrivial finite group, then G is not divisible.*

Let $M_n(\mathbb{R})$ be the semigroup of all $n \times n$ matrices over $\mathbb{R}$ under matrix multiplication. Then the unit group of the semigroup $M_n(\mathbb{R})$ is

$$G_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) | \det A \neq 0\}$$

For each $A \in M_n(\mathbb{R})$, the entry of A in the i^{th} row and the j^{th} column will be denoted by $A_{i,j}$. Next, let

$$U_n(\mathbb{R}) = \{A \in G_n(\mathbb{R}) |\ A \text{ is upper triangular}\}.$$

Then $U_n(\mathbb{R})$ is a subgroup of $G_n(\mathbb{R})$ ([3], page 410). For convenience, let

$$U_n^+(\mathbb{R}) = \{A \in G_n(\mathbb{R}) | A_{ii} > 0 \text{ for all } i \in \{1, ..., n\}\}.$$

If $A, B \in U_n^+(\mathbb{R})$, then for every $i \in \{1, ..., n\}$, $(AB)_{ii} = A_{ii}B_{ii} > 0$ and $(A^{-1})_{ii} = \dfrac{1}{A_{ii}} > 0$, so $U_n^+(\mathbb{R})$ is a subgroup of $U_n(\mathbb{R})$ and $G_n(\mathbb{R})$. The groups $G_n(\mathbb{R})$ and $U_n(\mathbb{R})$ are clearly not divisible. An interesting result for the group $U_n^+(\mathbb{R})$ was given by N. Triphop and A. Wasanawichit [4] as follows:

Theorem 1. *([4]) For every $n \in \mathbb{N}$, $U_n^+(\mathbb{R})$ is a divisible group.*

The notation of divisibility is defined more extensively in this paper. Divisible hypergroups will be defined. Let us recall some hyperstructures which will be used. A *hyperoperation* on a nonempty set A is a mapping $\circ : A \times A \to P^*(A)$ where $P(A)$ is the power set of A and $P^*(A) = P(A)\backslash\{\emptyset\}$, and $(A, \circ)$ is called a *hypergroupoid*. If X and Y are nonempty subsets of A, let

$$X \circ Y = \bigcup_{\substack{x \in X \\ y \in Y}} (x \circ y).$$

A *semihypergroups* is a hypergroupoid $(A, \circ)$ such that $x \circ (y \circ z) = (x \circ y) \circ z$ for all $x, y, z \in A$. A semihypergroup $(A, \circ)$ with $A \circ x = x \circ A = A$ for all $x \in A$ is called a *hypergroup*. A hypergroup $(A, \circ)$ is said to be *divisible* if for any $x \in A$ and every $n \in \mathbb{N}$, $x \in (y, \circ)^n$ for some $y \in A$ where $(y, \circ)^n$ denotes the set $y \circ y \circ ... \circ y$ (n copies). Then a total hypergroup, that is, a hypergroup $(A, \circ)$ with $x \circ y = A$ for all $x, y \in A$, is clearly divisible.

Let G be a group and H a subgroup of G. It is well-known that the relation $\sim$ defined on G by $a \sim b \Leftrightarrow a = bx$ for some $x \in H$ is an equivalence relation on G and the $\sim$-class of G containing $a \in G$ is aH and $aH = H \Leftrightarrow a \in H$. Similarly, it is easy to verify the relation $\approx$ defined on G by $a \approx b \Leftrightarrow a = xby$ for some $x, y \in H$ is an equivalence relation on G and the $\approx$-class of G containing $a \in G$ is HaH. Moreover, $HaH = H \Leftrightarrow a \in H$. The notation G/H denotes the set of all left cosets of H in G, that is,

$$G/H = \{xH |\ x \in G\}.$$

Define the hyperoperation $\circ$ on G/H by

$$xH \circ yH = \{tH \mid t \in xHy\} \text{ for all } x, y \in G.$$

Also, let $G|H$ and $\diamond$ the hyperoperation defined on $G|H$ as follows:

$$G|H = \{HxH \mid x \in G\},$$

$$HxH \diamond HyH = \{HtH \mid t \in xHy\} \text{ for all } x, y \in G.$$

Then $(G/H, \circ)$ and $(G|H, \diamond)$ are both hypergroups ([1], page 11). Notice that if H is normal in G, then $(G/H, \circ) = (G|H, \diamond)$ which is the quotient group of G by H. Moreover, if H_1 and H_2 are subgroups of G such that $H_1 \neq H_2$, then $G/H_1 \neq G/H_2$ and $G|H_1 \neq G|H_2$.

Our main purpose is to show that there are infinite many subgroups H of $U_n(\mathbb{R})$ such that the hypergroups $(U_n(\mathbb{R})/H, \circ)$ and $(U_n(\mathbb{R})|H, \diamond)$ are divisible. Theorem 1 is helpful for our work.

2 Basic Properties

Throughout this section, let G be any group, H a subgroup of G. Also, $(G/H, \circ)$ and $(G|H, \diamond)$ are hypergroups defined previously.

Lemma 1. *For $x \in G$ and $n \in \mathbb{N}\backslash\{1\}$,*

$$(xH, \circ)^n = \{tH | t \in (xH)^{n-1}x\}$$

and

$$(HxH, \diamond)^n = \{HtH | t \in (xH)^{n-1}x\}$$

Hence $x^nH \in (xH, \circ)^n$ and $Hx^nH \in (HxH, \diamond)^n$ for all $n \in \mathbb{N}$. In particular, $(H, \circ)^n = \{H\} = (H, \diamond)^n$.

Proof. This is clear for $n = 2$. If $k \geq 2$ is such that $(xH, \circ)^k = \{tH | t \in (xH)^{k-1}x\}$ and $(HxH, \diamond)^k = \{HtH | t \in (xH)^{k-1}x\}$. Hence

$$(xH, \circ)^{k+1} = xH \circ \{tH | t \in (xH)^{k-1}x\} =$$

$$= \{rH | r \in xHt \text{ for some } t \in (xH)^{k-1}x\} = \{rH | r \in xH(xH)^{k-1}x\} =$$

$$= \{tH | t \in (xH)^k x\},$$

and

$$(HxH, \diamond)^{k+1} = HxH \diamond \{HtH | t \in (xH)^{k-1}x\} =$$

$$= \{HrH | r \in xHt \text{ for some } t \in (xH)^{k-1}x\} =$$

$$= \{HrH | r \in xH(xH)^{k-1}x\} = \{HtH | t \in (xH)^k x\}.$$

If $x, y \in G$ and $n \in \mathbb{N}$ are such that $x = y^n$, then $xH = y^n H \in (yH, \circ)^n$ and $HxH = Hy^n H \in (HyH, \diamond)^n$ by Lemma 1. Hence we have:

Proposition 2. *If G is a divisible group, then both $(G/H, \circ)$ and $(G|H, \diamond)$ are divisible hypergroups.*

For any group G if $H = G$, then $|G/H| = 1 = |G|H|$, so $(G/H, \circ)$ and $(G|H, \diamond)$ are divisible hypergroups. Hence the converse of Proposition 2 is not generally true. A nontrivial example is as follows:

Example 1 *By Proposition 1, S_3 is not a divisible group. Let H be the subgroup of S_3 generated by the cycle $(1\ 2)$, that is, $H = \{(1), (1\ 2)\}$. Since $|S_3/H| = \frac{6}{2} = 3$ and $(1\ 3)^{-1}(2\ 3) = (1\ 3)(2\ 3) = (1\ 2\ 3) \notin H$, it follows that $H \notin (1\ 3)H \notin (2\ 3)H \notin H$. Thus*

$$S_3/H = \{H, (1\ 3)H, (2\ 3)H\}.$$

Since $(1\ 3) \notin H, (2\ 3) \notin H$,

$$(1\ 3) \in H(1\ 3)H = (1\ 3)H \cup (1\ 2)(1\ 3)H =$$

$= (1\ 3)H \cup (1\ 2\ 3)H = (1\ 3)H \cup (2\ 3)H$ *since* $(1\ 2\ 3) = (2\ 3)(1\ 2)$

and

$$S_3 = H \cup (1\ 3)H \cup (2\ 3)H,$$

it follows that

$$S_3|H = \{H, H(1\ 3)H\} \text{ and } H(2\ 3)H = H(1\ 3)H.$$

We know that $(1\ 3) = (2\ 3)(1\ 2)(2\ 3) \in (2\ 3)H(2\ 3)$ *and* $(1\ 3) = (1\ 3)^3$. *By Lemma 2,*

$$(1\ 3)H \in ((2\ 3)H, \circ)^2, (1\ 3)H = (1\ 3)^3H \in ((1\ 3)H, \circ)^3,$$

$$H(1\ 3)H \in (H(2\ 3)H, \diamond)^2 = (H(1\ 3)H, \diamond)^2,$$

$$H(1\ 3)H = H(1\ 3)^3H \in (H(1\ 3)H, \diamond)^3.$$

Next, let $n \in \mathbb{N}$ *be such that* $n \geq 3$. *If* n *is odd, then* $(1\ 3) = (1\ 3)^n$, *so by Lemma 2*

$$(1\ 3)H = (1\ 3)^nH \in ((1\ 3)H, \circ)^n$$

and

$$H(1\ 3)H = H(1\ 3)H = H(1\ 3)^nH \in (H(1\ 3)H, \diamond)^n.$$

If n *is even, then*

$$(1\,3) = (2\,3)^{n-2}(2\,3)(1\,2)(2\,3) \in ((2\,3)H)^{n-2}(2\,3)H(2\,3) = ((2\,3)H, \circ)^{n-1}(2\,3),$$

thus by Lemma 2.

$$(1\ 3)H \in ((2\ 3)H, \circ)^n$$

and

$$H(1\ 3)H \in (H(2\ 3)H, \diamond)^n = (H(1\ 3)H, \diamond)^n.$$

This shows that for every $n \in \mathbb{N}$, $(1\ 3)H \in ((1\ 3)H, \circ)^n$ *or* $(1\ 3)H \in ((2\ 3)H, \circ)^n$ *and* $H(1\ 3)H \in (H(1\ 3)H, \diamond)^n$. *We can show similarly that for every* $n \in \mathbb{N}$, $(2\ 3)H \in ((2\ 3)H, \circ)^n$ *or* $(2\ 3)H \in ((1\ 3)H, \circ)^n$.

Hence we have that S_3 is not a divisible group and $H \neq S_3$, but $(S_3/H, \circ)$ and $(S_3|H, \diamond)$ are divisible hypergroups.

3 The Hypergroups $(U_n(\mathbb{R})/H, \circ)$ and $(U_n(\mathbb{R})|H, \diamond)$

For each prime $p, \mathbb{Q}(\sqrt{p})$ is a subfield of $\mathbb{R}$ and if p_1 and p_2 are distinct primes, then $\mathbb{Q}(\sqrt{p_1}) \neq \mathbb{Q}(\sqrt{p_2})$. Hence there are infinitely many subfields of $\mathbb{R}$. For each subfield F of $\mathbb{R}$, let

$$H_F = \{A \in U_n(\mathbb{R}) | A_{ii} \in F^* \text{ for all } i \in \{1, ..., n\}\}.$$

Clearly, for distinct F_1 and F_2 of $\mathbb{R}$, $H_{F_1} \neq H_{F_2}$.

Lemma 2. *For every subfield F of $\mathbb{R}$, H_F is a subgroup of the group $U_n(\mathbb{R})$.*

Proof. Since for $A, B \in H_F$, $(AB)_{ii} = A_{ii}B_{ii} \in F^*$ and $(A^{-1})_{ii} = \dfrac{1}{A_{ii}} \in F^*$ for all $i \in \{1, ..., n\}$, it follows that H_F is a subgroup of $U_n(\mathbb{R})$.

Theorem 2. *If F is a subfield of $\mathbb{R}$, then $(U_n(\mathbb{R})/H_F, \circ)$ and $(U_n(\mathbb{R})|H_F, \diamond)$ are both divisible hypergroups.*

Proof. Let $A \in U_n(\mathbb{R})$ and $m \in \mathbb{N}$. Define the diagonal matrix $B \in U_n(\mathbb{R})$ by $B_{ii} = 1$ if $A_{ii} = -1$ if $A_{ii} < 0$. Then B is clearly an element of H_F and $AB = C^m$. Thus $A = C^m B^{-1}$ and hence $AH_F = C^m B^{-1} H_F = C^m H_F$ and $H_F A H_F = H_F C^m B^{-1} H_F = H_F C^m H_F$. But $C^m H_F \in (CH_F, \circ)^m$ and $H_F C^m H_F \in (H_F C H_F, \diamond)^m$ by Lemma 1, so $AH_F \in (CH_F, \circ)^m$ and $H_F A H_F \in (H_F C H_F, \diamond)^m$.

Hence the theoem is proved.

Remark 1. *If F_1 and F_2 are distinct subfields of $\mathbb{R}$, then $H_{F_1} \neq H_{F_2}$ which implies that $U_n(\mathbb{R})/H_{F_1} \neq U_n(\mathbb{R})/H_{F_2}$ and $U_n(\mathbb{R})|H_{F_1} \neq U_n(\mathbb{R})|H_{F_2}$.*

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