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A New Class of Multivalent Harmonic Functions [1]

K. Al Shaqsi and M. Darus

In memoriam of Associate Professor Ph. D. Luciana Lupaş

Abstract

In this paper, we introduce a new class of multivalent harmonic functions. We investigate various properties of functions belonging to this class. Coefficients bounds, distortion bounds and extreme points are given.

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1 Introduction

A continuous functions $f = u + iv$ is a complex valued harmonic function in a complex domain $\mathbb{C}$ if both u and v are real harmonic in $\mathbb{C}$. In any

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simply connected domain $\mathcal{D} \subset \mathbb{C}$ we can write $f = h + \overline{g}$, where h and g are analytic in $\mathcal{D}$. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathcal{D}$ is that $|h'(z)| > |g'(z)|$ in $\mathcal{D}$. See Clunie and Sheil-Small (see [2]).

Denote by $\mathcal{H}(p)$ the class of functions $f = h + \overline{g}$ that are harmonic multivalent and sense-preserving in the unit disk $\mathbb{U} = \{z : |z| < 1\}$. For $f = h + \overline{g} \in \mathcal{H}(p)$ we may express the analytic functions h and g as

$$h(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad g(z) = \sum_{k=p}^{\infty} b_k z^k, \quad |b_p| < 1. \tag{1}$$

Also denote by $\mathcal{T}(p)$, the subclass of $\mathcal{H}(p)$ consisting of all functions $f = h + \overline{g}$ where h and g are given by

$$h(z) = z^p - \sum_{k=p+1}^{\infty} |a_k| z^k, \quad g(z) = -\sum_{k=p}^{\infty} |b_k| z^k, \quad |b_p| < 1. \tag{2}$$

We denote by $\mathcal{H}_\lambda^n(p, \alpha)$ the class of all functions of the form (1.1) that satisfy the condition

$$\Re\left\{ \frac{(D_\lambda^{n+p-1} f(z))'}{p z^{p-1}} \right\} > \alpha, \tag{3}$$

where $0 \leq \alpha < p$, $p \in \mathbb{N}$, $\lambda \geq 0$, $n \in \mathbb{N}_0$ and $D_\lambda^{n+p-1} f(z) = D_\lambda^{n+p-1} h(z) + \overline{D_\lambda^{n+p-1} g(z)}$.

When $p = 1$, D_λ^n denotes the operator introduced by [3]. For h and g given by (1.1) we have

$$D_\lambda^{n+p-1} h(z) = z^p + \sum_{k=p+1}^{\infty} \big[1 + \lambda(k-p)\big] C(n,k,p) a_k z^k,$$

$$D_\lambda^{n+p-1} g(z) = \sum_{k=p}^{\infty} \big[1 + \lambda(k-p)\big] C(n,k,p) b_k z^k$$

where $\lambda \geq 0$, $p \in \mathbb{N}$, $n > -p$ and $C(n,k,p) = \binom{k+n-1}{n+p-1}$.

Note that :

$\mathcal{H}_0^0(1,0) \equiv S_{\mathcal{H}}^*$ studied by Silverman [1],
$\mathcal{H}_\lambda^0(1,0) \equiv H(\lambda)$ studied by Yalçin and Öztürk [7],
$\mathcal{H}_0^0(1,\alpha) \equiv N_H(\alpha)$ studied by Ahuja and Jahangiri [5],
$\mathcal{H}_\lambda^n(1,0) \equiv \mathcal{H}_\lambda^n$ studied by the authors in [4].
Also we note that for the analytic part the class $\mathcal{H}_0^n(p,\alpha)$ was introduced and studied by Goel and Sohi [6].

We further denote by $\mathcal{T}_\lambda^n(p,\alpha)$ the subclass of $\mathcal{H}_\lambda^n(p,\alpha)$, where

$$\mathcal{T}_\lambda^n(p,\alpha) = \mathcal{T}(p) \cap \mathcal{H}_\lambda^n(p,\alpha).$$

2 Coefficients Bounds

Theorem 2.1. *Let $f = h + \overline{g}$ with h and g are given by (1.1). Let*

$$\sum_{k=p}^{\infty} k\big[1 + \lambda(k-p)\big] C(n,k,p)[|a_k| + |b_k|] \leq p(2-\alpha) \tag{4}$$

where $a_p = p$, $\lambda \geq 0$ and $0 \leq \alpha < p$. Then f is harmonic multivalent sense preserving in $\mathbb{U}$ and $f \in \mathcal{H}_\lambda^n(p,\alpha)$.

Proof. Letting $w(z) = \frac{(D_\lambda^{n+p-1} f(z))'}{pz^{p-1}}$. Using the fact $\Re\{w\} \geq \alpha$ if and only if $|p-\alpha+w(z)| \geq |p+\alpha-w(z)|$, it suffices to show that

$$\left|p-\alpha+\frac{(D_\lambda^{n+p-1}f(z))'}{pz^{p-1}}\right|-\left|p+\alpha-\frac{(D_\lambda^{n+p-1}f(z))'}{pz^{p-1}}\right| \geq 0. \tag{5}$$

Substituting for h and g in (2.2) yields

$$\left|p-\alpha+\frac{(D_\lambda^{n+p-1}h(z))'+\overline{(D_\lambda^{n+p-1}g(z))'}}{pz^{p-1}}\right|-$$

$$-\left|p+\alpha-\frac{(D_\lambda^{n+p-1}h(z))'+\overline{(D_\lambda^{n+p-1}g(z))'}}{pz^{p-1}}\right| =$$

$$=\left|p+1-\alpha+\sum_{k=p+1}^{\infty}\frac{k}{p}\big[1+\lambda(k-p)\big]C(n,k,p)a_kz^{k-p}+\right.$$

$$\left.+\sum_{k=p}^{\infty}\frac{k}{p}\big[1+\lambda(k-p)\big]C(n,k,p)b_kz^{k-p}\right|-$$

$$-\left|p-1+\alpha-\sum_{k=p+1}^{\infty}\frac{k}{p}\big[1+\lambda(k-p)\big]C(n,k,p)a_kz^{k-p}-\right.$$

$$\left.-\sum_{k=p}^{\infty}\frac{k}{p}\big[1+\lambda(k-p)\big]C(n,k,p)b_kz^{k-p}\right| \geq$$

$$\geq 2\left\{(1-\alpha)-\left[\sum_{k=p+1}^{\infty}\frac{k}{p}\big[1+\lambda(k-p)\big]C(n,k,p)|a_k||z^{k-p}|+\right.\right.$$

$$\left.\left.+\sum_{k=p}^{\infty}\frac{k}{p}\big[1+\lambda(k-p)\big]C(n,k,p)|b_k||z^{k-p}|\right]\right\} >$$

$$> 2\left\{p(1-\alpha)-\left[\sum_{k=p+1}^{\infty}k\big[1+\lambda(k-p)\big]C(n,k,p)|a_k|+\right.\right.$$

$$\left.\left.+\sum_{k=p}^{\infty}k\big[1+\lambda(k-p)\big]C(n,k,p)|b_k|\right]\right\} > 0.$$

The Harmonic mappings

$$f(z) = z^p + \sum_{k=p+1}^{\infty} \frac{x_k}{k\big[1+\lambda(k-p)\big]C(n,k,p)} z^k +$$

$$+ \sum_{k=p}^{\infty} \frac{\overline{y}_k}{k\big[1+\lambda(k-p)\big]C(n,k,p)} \overline{z}^k$$

where $\sum\limits_{k=p+1}^{\infty} |x_k| + \sum\limits_{k=p}^{\infty} |y_k| = p(1-\alpha)$, show that the coefficient bound given by (2.1) is sharp.

The functions of the form (2.3) are in $\mathcal{H}_\lambda^n(p,\alpha)$ because

$$\sum_{k=p+1}^{\infty} k\big[1+\lambda(k-p)\big]C(n,k,p)\Big(|a_k|+|b_k|\Big) =$$

$$= p + \sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = p(2-\alpha).$$

The restriction placed in Theorem 2.1 on the moduli of the coefficients of $f = h + \overline{g}$ enables us to conclude for arbitrary rotation of the coefficients of f that the resulting functions would still be harmonic multivalent and $f \in H_\lambda^n(p,\alpha)$.

We next show that the condition (2.1) is also necessary for functions in $\mathcal{T}_\lambda^n(p,\alpha)$.

Theorem 2.2. *Let $f = h + \overline{g}$ with h and g are given by (1.2). Then $f \in \mathcal{T}_\lambda^n(p,\alpha)$ if and only if*

$$\sum_{k=p}^{\infty} k\big[1+\lambda(k-p)\big]C(n,k,p)[|a_k|+|b_k|] \leq p(2-\alpha) \tag{6}$$

where $a_p = p$, $\lambda \geq 0$ and $0 \leq \alpha < p$.

Proof. The "*if*" part follows from Theorem 2.1 upon noting $\mathcal{T}^n_\lambda(p,\alpha) \subset \mathcal{H}^n_\lambda(p,\alpha)$. For the "*only if*" part, assume that $f \in \mathcal{T}^n_\lambda(p,\alpha)$. Then by (1.3) we have

$$\Re\left\{\frac{(D^n_\lambda h(z))' + \overline{(D^n_\lambda g(z))'}}{pz^{p-1}}\right\} =$$

$$= \Re\left\{1 - \sum_{k=p+1}^{\infty} \frac{k}{p}\big[1+\lambda(k-1)\big]C(n,k)|a_k|z^{k-p} - \right.$$

$$\left. - \sum_{k=p}^{\infty} \frac{k}{p}\big[1+\lambda(k-1)\big]C(n,k)|b_k|\overline{z}^{k-p}\right\} > \alpha.$$

If we choose z to be real and let $z \to 1^-$, we get

$$1 - \sum_{k=p+1}^{\infty} \frac{k}{p}\big[1+\lambda(k-1)\big]C(n,k)|a_k|z^{k-p} -$$

$$- \sum_{k=p}^{\infty} \frac{k}{p}\big[1+\lambda(k-1)\big]C(n,k)|b_k|\overline{z}^{k-p} \geq \alpha,$$

which is precisely the assertion (2.4) of Theorem 2.2.

3 Distortion Bounds and Extreme Points.

In this section, we shall obtain distortion bounds for functions in $\mathcal{T}^n_\lambda(p,\alpha)$ and also provide extreme points for the class $\mathcal{T}^n_\lambda(p,\alpha)$.

Theorem 3.1. *If $f \in \mathcal{T}^n_\lambda(p,\alpha)$, for $\lambda \geq 0$, $p \in \mathbb{N}$, $n \in \mathbb{N}_0$ and $|z| = r > 1$, then*

$$|f(z)| \leq (1+b_p)r^p + \frac{p(1-\alpha) - |b_p|}{(p+1)(1+\lambda)(n+p)}r^{p+1},$$

and

$$|f(z)| \geq (1-b_p)r^p - \frac{p(1-\alpha) - |b_p|}{(p+1)(1+\lambda)(n+p)}r^{p+1}.$$

Proof. We only prove the second inequality. The argument for first inequality is similar and will be omitted. Let $f \in \mathcal{T}_\lambda^n(p,\alpha)$. Taking the absolute value of f, we obtain

$$|f(z)| \geq (1-b_p)r^p - \sum_{k=p+1}^{\infty}(|a_k|+|b_k|)r^k \geq (1-b_p)r^p -$$

$$-\sum_{k=p+1}^{\infty}(|a_k|+|b_k|)r^{p+1} =$$

$$= (1-b_p)r^p - \frac{1}{(p+1)(1+\lambda)(n+p)} \cdot$$

$$\cdot \sum_{k=p+1}^{\infty}(p+1)(1+\lambda)(n+p)(|a_k|+|b_k|)r^{p+1} \geq$$

$$\geq (1-b_p)r^p - \frac{1}{(p+1)(1+\lambda)(n+p)} \cdot$$

$$\cdot \sum_{k=p+1}^{\infty} k\big[1+\lambda(k-p)\big]C(n,k,p)(|a_k|+|b_k|)r^{p+1} \geq$$

$$\geq (1-b_p)r^p - \frac{1}{(p+1)(1+\lambda)(n+p)}\Big[p(1-\alpha)-|b_p|\Big]r^{p+1}.$$

The bounds given in Theorem 3.1 for the functions $f = h+\overline{g}$ of the form (1.2) also hold for functions of the form (1.1) if the coefficient condition (2.1) is satisfied. The functions

$$f(z) = z^p + |b_p|\overline{z}^p - \frac{p(1-\alpha)-|b_p|}{(p+1)(1+\lambda)(n+p)}\overline{z}^{p+1}$$

and

$$f(z) = (1-|b_p|)z^p - \frac{p(1-\alpha)-|b_p|}{(p+1)(1+\lambda)(n+p)}z^{p+1}$$

for $|b_p| < 1$ show that the bounds given Theorem 3.1 are sharp.

The following covering result follows from the second inequality in Theorem 3.1.

Corollary 1 *If $f \in \mathcal{T}_\lambda^n(p,\alpha)$, then*

$$\left\{ w : |w| < (1-|b_p|) - \frac{p(1-\alpha)-|b_p|}{(p+1)(1+\lambda)(n+p)} \right\} \subset f(\mathbb{U}).$$

Theorem 3.2. *$f \in \mathcal{T}_\lambda^n(p,\alpha)$ if and only if f can be expressed as*

$$f(z) = \sum_{k=p}^{\infty} (\gamma_k h_k + \mu_k g_k) \tag{7}$$

where $z \in \mathbb{U}$,

$$h_p(z) = z^p,\ h_k(z) = z^p - \frac{p(1-\alpha)}{k\big[1+\lambda(k-p)\big]C(n,k,p)} z^k,\ (k = p+1, p+2, ...),$$

$$g_k(z) = z^p - \frac{p(1-\alpha)}{k\big[1+\lambda(k-p)\big]C(n,k,p)} \overline{z}^k,\ (k = p, p+1, ...),$$

$\sum\limits_{k=p}^{\infty} (\gamma_k + \mu_k) = 1,\ \gamma_k \geq 0$ *and* $\mu_k \geq 0\,(k = p+1, p+2, ...)$.

In particular, the extreme points of $\mathcal{T}_\lambda^n(p,\alpha)$ are $\{h_k\}$ and $\{g_k\}$.

Proof. Note that for f we may write

$$f(z) = \sum_{k=p}^{\infty} (\gamma_k h_k + \mu_k g_k) =$$

$$= \sum_{k=p}^{\infty} (\gamma_k + \mu_k) z^p - \sum_{k=p+1}^{\infty} \frac{p(1-\alpha)}{k\big[1+\lambda(k-p)\big]C(n,k,p)} \gamma_k z^k -$$

$$-\sum_{k=p}^{\infty}\frac{p(1-\alpha)}{k\big[1+\lambda(k-p)\big]C(n,k,p)}\mu_k\overline{z}^k =$$

$$= z^p - \sum_{k=p+1}^{\infty}\frac{p(1-\alpha)}{k\big[1+\lambda(k-p)\big]C(n,k,p)}\gamma_k z^k -$$

$$-\sum_{k=p}^{\infty}\frac{p(1-\alpha)}{k\big[1+\lambda(k-p)\big]C(n,k,p)}\mu_k\overline{z}^k$$

Then

$$\sum_{k=p+1}^{\infty}\Big[k\big[1+\lambda(k-p)\big]C(n,k,p)\Big]\frac{p(1-\alpha)}{k\big[1+\lambda(k-p)\big]C(n,k,p)}\gamma_k$$
$$-\sum_{k=p}^{\infty}\Big[k\big[1+\lambda(k-p)\big]C(n,k,p)\Big]\frac{p(1-\alpha)}{k\big[1+\lambda(k-p)\big]C(n,k,p)}\mu_k$$
$$= p(1-\alpha)\left(\sum_{k=p}^{\infty}(\gamma_k+\mu_k)-\gamma_p\right) = p(1-\alpha)(1-\gamma_p) \le p(1-\alpha)$$

and so $f \in \mathcal{T}_{\lambda}^{n}(p,\alpha)$.

Conversely, suppose that $f \in \mathcal{T}_{\lambda}^{n}(p,\alpha)$. Setting

$$\gamma_k = \frac{k\big[1+\lambda(k-p)\big]C(n,k,p)}{p(1-\alpha)}|a_k|(k=p+1,p+2,...),$$
$$\mu_k = \frac{k\big[1+\lambda(k-p)\big]C(n,k,p)}{p(1-\alpha)}|b_k|(k=p,p+1,p+2,...),$$

we obtain

$$f(z) = \sum_{k=p}^{\infty}(\gamma_k h_k + \mu_k g_k) \text{ as required.}$$

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School of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
Bangi 43600 Selangor D. Ehsan, Malaysia
E-mail address: *ommath@hotmail.com*
E-mail address: *maslina@pkrisc.cc.ukm.my*