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# An inequality of Ostrowski type via a mean value theorem [1]

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Dedicated to Professor Ph.D. Alexandru Lupaş on his 65th anniversary

**Abstract**

The main of this paper is to establish an Ostrowski type inequality by using a mean value theorem.

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## 1 Introduction

The following result is known in the literature as Ostrowski's inequality [3]:

$$(1)\quad \left|f(x)-\frac{1}{b-a}\int\limits_a^b f(t)\mathrm{d}t\right| \le \left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\cdot\|f'\|_\infty$$

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where $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and differentiable on $(a,b)$ and whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on $(a,b)$, i. e. $\|f'\|=\sup\limits_{t\in(a,b)}|f'(t)|<\infty$.

Many researchers have given considerable attention to the inequality (1) and several generalizations, extensions and related results have appeared in the literature.

The main aim of this paper is to establish an Ostrowski type inequality by using a mean value theorem (see [2] and [5]).

## 2 Statement and results

In [5] the author has proved the following mean value theorem:

**Theorem 2.1** *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on $(a,b)$ and $\xi\in(a,b)$ with*

$$f'(\xi)=\frac{f(b)-f(a)}{b-a}.$$

*For every $M(\alpha,\beta)$, $\alpha\notin[a,b]$,*

$$\min\{y_1,y_2\}\le\beta\le\max\{y_1,y_2\},$$

*there exists a point $c$ in $(a,b)$ such that*

$$f'(c)=\frac{\beta-f(c)}{\alpha-c},$$

*where $y_1=f(a)+(\alpha-a)f'(\xi)$, $y_2=f(\xi)+(\alpha-\xi)f'(\xi)$.*

In the proofs of our results we make use this mean value theorem.

The following result holds.

**Theorem 2.2** *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on $(a,b)$. Then for any $x \in [a,b]$ we have the inequality*

$$\left| \left[ \frac{a+b}{2} - \alpha \right] f(x) + \frac{\alpha - x}{b-a} \int_a^b f(t)\mathrm{d}t \right| \leq \tag{2}$$

$$\leq \left[ \frac{1}{4} + \left( \frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \cdot \|f + lf'\|_\infty ,$$

*where $\alpha \notin [a,b]$ and $l(t) = \alpha - t$, $t \in [a,b]$.*

**Proof.** Applying the theorem 2.1 for $\beta = y_1$, we obtain that for any $x, t \in [a,b]$, there is a $c$ between $x$ and $t$ such that

$$f(x) + (\alpha - x)\frac{f(t) - f(x)}{t - x} = y_1 = \beta = f(c) + (\alpha - c)f'(c).$$

We have

$$\begin{aligned} |f(c) + (\alpha - c)f'(c)| &\leq \sup_{t\in[a,b]} |f(t) + (\alpha - t)f'(t)| \\ &= \|f + lf'\|_\infty \end{aligned}$$

where $l(t) = \alpha - t$, $t \in [a,b]$.

Hence

$$\left| f(x) + (\alpha - x)\frac{f(t) - f(x)}{t - x} \right| \leq \|f + lf'\|_\infty ,$$

$$|(t - x)f(x) + (\alpha - x)[f(t) - f(x)]| \leq |t - x| \cdot \|f + lf'\|_\infty ,$$

$$|(t - \alpha)f(x) + (\alpha - x)f(t)| \leq |t - x| \cdot \|f + lf'\|_\infty .$$

Integrating over $t \in [a,b]$, we get

$$\left| f(x) \int_a^b (t - \alpha)\mathrm{d}t + (\alpha - x) \int_a^b f(t)\mathrm{d}t \right| \leq \|f + lf'\|_\infty \cdot \int_a^b |t - x|\mathrm{d}t. \tag{3}$$

We observe that

$$\int\limits_a^b |t-x|\mathrm{d}t = \begin{cases} \dfrac{b^2-a^2}{2} - (b-a)x, & t \geq x \\ -\dfrac{b^2-a^2}{2} + (b-a)x, & t < x. \end{cases}$$

Next

$$\frac{b^2-a^2}{2} - (b-a)x = \frac{(x-b)^2-(x-a)^2}{2} \leq \frac{(x-a)^2+(x-b)^2}{2},$$

$$-\frac{b^2-a^2}{2} + (b-a)x = \frac{(x-a)^2-(x-b)^2}{2} \leq \frac{(x-a)^2+(x-b)^2}{2}.$$

Hence

$$\int\limits_a^b |t-x|\mathrm{d}t \leq \frac{(x-a)^2+(x-b)^2}{2} = \frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2.$$

Of (3) we deduce

$$\left|\left[\frac{b^2-a^2}{2} - \alpha(b-a)\right] f(x) + (\alpha - x)\int\limits_a^b f(t)\mathrm{d}t\right| \leq$$

$$\leq \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right] \|f + lf'\|_\infty,$$

and finally

$$\left|\left[\frac{a+b}{2} - \alpha\right] f(x) + \frac{\alpha - x}{b-a}\int\limits_a^b f(t)\mathrm{d}t\right| \leq$$

$$\leq \left[\frac{1}{4} + \left(\frac{x - \dfrac{a+b}{2}}{b-a}\right)^2\right] (b-a)\cdot \|f + lf'\|_\infty .$$

**Remark 2.1** *If $0 \notin [a,b]$, then for $\alpha = 0$ we obtain the Dragomir's inequality (see [1]):*

$$\left| \frac{a+b}{2} \cdot \frac{f(x)}{x} - \frac{1}{b-a} \int_a^b f(t)\mathrm{d}t \right| \leq$$

$$\leq \frac{b-a}{|x|} \left[ \frac{1}{4} + \left( \frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \cdot \|f + lf'\|_\infty$$

*where $l(t) = -t$, $t \in [a,b]$.*

The following interesting particular case holds.

**Corollary 2.1** *With the assumptions in Theorem 2.2, we have*

(4) $$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t)\mathrm{d}t \right| \leq$$

$$\leq \frac{b-a}{4\left|\frac{a+b}{2} - \alpha\right|} \|f + lf'\|_\infty$$

*for any $\alpha \notin [a,b]$.*

# 3 AN APPLICATION

Using an idea of [1] we consider the division of the interval $[a,b]$ given by

$$\Delta : a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b.$$

Let $\xi_i$ be a sequence of intermediate points, $\xi_i \in [x_i, x_{i+1}]$, $i = \overline{0, n-1}$. We define the quadrature

(5) $$S_\Delta(f, \xi_i) = \sum_{i=0}^{n-1} \frac{f(\xi_i)}{\xi_i - \alpha} \left[ \frac{x_{i+1}^2 - x_i^2}{2} - \alpha\left(x_{i+1} - x_i\right) \right] =$$

$$= \sum_{i=0}^{n-1} \frac{f(\xi_i)}{\xi_i - \alpha} \left( \frac{x_{i+1} + x_i}{2} - \alpha \right) h_i$$

where $h_i = x_{i+1} - x_i$.

We denote

$$\int_a^b f(t)\mathrm{d}t = S_\Delta(f, \xi_i) + R_\Delta(f, \xi_i) \tag{6}$$

where $S_\Delta(f, \xi_i)$ is as defined in (5). In the following result we obtain an estimate for the remainder $R_\Delta(f, \xi_i)$.

**Theorem 3.1** *Assume that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on $(a, b)$. Then we have*

$$|R_\Delta(f, \xi_i)| \leq \frac{1}{2h} \|f + lf'\|_\infty \cdot \sum_{i=0}^{n-1} h_i^2 \tag{7}$$

*where $h = \min\{|a - \alpha|, |b - \alpha|\}$.*

**Proof.** Applying the theorem 2.2 on the interval $[x_i, x_{i+1}]$ for the intermediate points $\xi_i$, to obtain

$$\left| \int\limits_{x_i}^{x_{i+1}} f(t)\mathrm{d}t - \frac{f(\xi_i)}{\xi_i - \alpha} \left( \frac{x_{i+1} + x_i}{2} - \alpha \right) h_i \right| \leq$$

$$\leq \frac{h_i^2}{|\xi_i - \alpha|} \left[ \frac{1}{4} + \left( \frac{\xi_i - \frac{x_{i+1} + x_i}{2}}{h_i} \right)^2 \right] \cdot \|f + lf'\|_\infty.$$

But $\left| \xi_i - \frac{x_{i+1} + x_i}{2} \right| \leq \frac{x_{i+1} - x_i}{2}$ and

$$|\xi_i - \alpha| \geq \min\{|a - \alpha|, |b - \alpha|\}.$$

Hence

$$\left|\int\limits_{x_i}^{x_{i+1}} f(t)\mathrm{d}t - \frac{f(\xi_i)}{\xi_i-\alpha}\left(\frac{x_{i+1}+x_i}{2}-\alpha\right)h_i\right| \le$$

$$\le \frac{1}{2h}\|f+lf'\|_\infty \cdot h_i^2.$$

Summer over $i$ from 0 to $n-1$ we deduce

$$\left|\int\limits_a^b f(t)\mathrm{d}t - \sum_{i=0}^{n-1}\frac{f(\xi_i)}{\xi_i-\alpha}\left(\frac{x_{i+1}+x_i}{2}-\alpha\right)h_i\right| \le$$

$$\le \frac{1}{2h}\cdot\|f+lf'\|_\infty \cdot \sum_{i=0}^{n-1} h_i^2$$

so

$$|R_\Delta(f,\xi_i)| \le \frac{1}{2h}\|f+lf'\|_\infty \cdot \sum_{i=0}^{n-1} h_i^2.$$

**Remark 3.1** *For* $h_i = \dfrac{b-a}{n}$, $i=\overline{0,n-1}$ *we get*

$$|R_\Delta(f,\xi_i)| \le \frac{(b-a)^2}{2hn}\cdot\|f+lf'\|_\infty.$$

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