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A convexity property for an integral operator on the class $S_p(\beta)$

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Abstract

In this paper we consider an integral operator $F_n(z)$ for analytic functions $f_i(z)$ in the open unit disk U. The object of this paper is to prove the convexity properties for the integral operator $F_n(z)$ on the class $S_p(\beta)$.

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1 Introduction

Let $U = \{z \in C, |z| < 1\}$ be the unit disc of the complex plane and denote by $H(U)$, the class of the olomorphic functions in U. Consider $A = \{f \in H(U), f(z) = z + a_2z^2 + a_3z^3 + ..., z \in U\}$ be the class of analytic functions in U and $S = \{f \in A : f \text{ is univalent in } U\}$.

Denote with K the class of convex functions in U, defined by

$$K=\left\{f\in H\left(U\right):f\left(0\right)=f'\left(0\right)-1=0,\mathbf{Re}\left\{\frac{zf''\left(z\right)}{f'\left(z\right)}+1\right\}>0,z\in U\right\}.$$

A function $f\in S$ is the convex function by the order $\alpha, 0\leq\alpha<1$ and denote this class by $K\left(\alpha\right)$ if f verify the inequality

$$\mathbf{Re}\left\{\frac{zf''\left(z\right)}{f'\left(z\right)}+1\right\}>\alpha, z\in U.$$

Consider the class $S_p\left(\beta\right)$, was is introduced by F. Ronning in the paper [3] and is defined by:

$$f\in S_p\left(\beta\right)\Leftrightarrow\left|\frac{zf'\left(z\right)}{f\left(z\right)}-1\right|\leq\mathbf{Re}\left\{\frac{zf'\left(z\right)}{f\left(z\right)}-\beta\right\}\tag{1}$$

where β is the real number with the property $-1\leq\beta<1$.

For $f_i\left(z\right)\in A$ and $\alpha_i>0, i\in\{1,...,n\}$, we define the integral operator $F_n\left(z\right)$ given by

$$F_n\left(z\right)=\int_0^z\left(\frac{f_1\left(t\right)}{t}\right)^{\alpha_1}\cdot\ldots\cdot\left(\frac{f_n\left(t\right)}{t}\right)^{\alpha_n}dt.\tag{2}$$

This integral operator was first defined by Breaz and Breaz in [1]. It is easy to see that $F_n\left(z\right)\in A$.

2 Main results

Theorem 1. *Let $\alpha_i>0$, for $i\in\{1,...,n\}$, β_i is the real numbers with the property $-1\leq\beta_i<1$ and $f_i\in S_p\left(\beta_i\right)$ for $i\in\{1,...,n\}$.*

If

$$0 < \sum_{i=1}^{n} \alpha_i (1 - \beta_i) \leq 1, \tag{3}$$

the integral operator F_n is convex by the order $1 + \sum_{i=1}^{n} \alpha_i (\beta_i - 1)$.

Proof. We calculate for F_n the derivatives of the first and second order.

From (2) we obtain:

$$F_n'(z) = \left(\frac{f_1(z)}{z}\right)^{\alpha_1} \cdot \ldots \cdot \left(\frac{f_n(z)}{z}\right)^{\alpha_n}$$

and

$$F_n''(z) = \sum_{i=1}^{n} \alpha_i \left(\frac{f_i(z)}{z}\right)^{\alpha_i - 1} \left(\frac{z f_i'(z) - f_i(z)}{z f_i(z)}\right) \prod_{\substack{j=1 \\ j \neq i}}^{n} \left(\frac{f_j(z)}{z}\right)^{\alpha_j}.$$

After the calculus we obtain that:

$$\frac{F_n''(z)}{F_n'(z)} = \alpha_1 \left(\frac{z f_1'(z) - f_1(z)}{z f_1(z)}\right) + \ldots + \alpha_n \left(\frac{z f_n'(z) - f_n(z)}{z f_n(z)}\right).$$

These relation is equivalent with:

$$\frac{F_n''(z)}{F_n'(z)} = \alpha_1 \left(\frac{f_1'(z)}{f_1(z)} - \frac{1}{z}\right) + \ldots + \alpha_n \left(\frac{f_n'(z)}{f_n(z)} - \frac{1}{z}\right). \tag{4}$$

Multiply the relation (4) with z we obtain:

$$\frac{z F_n''(z)}{F_n'(z)} = \sum_{i=1}^{n} \alpha_i \left(\frac{z f_i'(z)}{f_i(z)} - 1\right) = \sum_{i=1}^{n} \alpha_i \frac{z f_i'(z)}{f_i(z)} - \sum_{i=1}^{n} \alpha_i. \tag{5}$$

The relation (5) is equivalent with

$$\frac{zF_n''(z)}{F_n'(z)}+1=\sum_{i=1}^{n}\alpha_i\frac{zf_i'(z)}{f_i(z)}-\sum_{i=1}^{n}\alpha_i+1.$$

This relation is equivalent with:

$$\frac{zF_n''(z)}{F_n'(z)}+1=\sum_{i=1}^{n}\alpha_i\left(\frac{zf_i'(z)}{f_i(z)}-\beta_i\right)+\sum_{i=1}^{n}\alpha_i\beta_i-\sum_{i=1}^{n}\alpha_i+1.$$

We calculate the real part from both terms of the above equality and obtain:

$$\mathbf{Re}\left(\frac{zF_n''(z)}{F_n'(z)}+1\right)=\sum_{i=1}^{n}\alpha_i\mathbf{Re}\left(\frac{zf_i'(z)}{f_i(z)}-\beta_i\right)+\sum_{i=1}^{n}\alpha_i\beta_i-\sum_{i=1}^{n}\alpha_i+1.$$

Because $f_i\in S_p(\beta_i)$ for $i=\{1,...,n\}$, we apply in the above relation the inequality (1) and obtain:

$$\mathbf{Re}\left(\frac{zF_n''(z)}{F_n'(z)}+1\right)>\sum_{i=1}^{n}\alpha_i\left|\frac{zf_i'(z)}{f_i(z)}-1\right|+\sum_{i=1}^{n}\alpha_i(\beta_i-1)+1.$$

Because $\alpha_i\left|\frac{zf_i'(z)}{f_i(z)}-1\right|>0$ for all $i\in\{1,...,n\}$, obtain that

$$\mathbf{Re}\left(\frac{zF_n''(z)}{F_n'(z)}+1\right)>\sum_{i=1}^{n}\alpha_i(\beta_i-1)+1.$$

So, F_n is convex by the order $\sum_{i=1}^{n}\alpha_i(\beta_i-1)+1$.

Theorem 2. *Let $\alpha_i, i\in\{1,...,n\}$ the real positive numbers and $f_i\in S_p(\beta)$ for $i\in\{1,...,n\}$.*

If

$$0 < \sum_{i=1}^{n} \alpha_i \leq \frac{1}{1-\beta},$$

the integral operator F_n *is convex by the order* $(\beta - 1)\sum\limits_{i=1}^{n} \alpha_i + 1$.

Proof. Since

$$F_n'(z) = \left(\frac{f_1(z)}{z}\right)^{\alpha_1} \cdot \ldots \cdot \left(\frac{f_n(z)}{z}\right)^{\alpha_n}$$

and

$$F_n''(z) = \sum_{i=1}^{n} \alpha_i \left(\frac{f_i(z)}{z}\right)^{\alpha_i - 1} \left(\frac{zf_i'(z) - f_i(z)}{zf_i(z)}\right) \prod_{\substack{j=1 \\ j \neq i}}^{n} \left(\frac{f_j(z)}{z}\right)^{\alpha_j}$$

we obtain

$$\frac{zF_n''(z)}{F_n'(z)} + 1 = \sum_{i=1}^{n} \alpha_i \frac{zf_i'(z)}{f_i(z)} - \sum_{i=1}^{n} \alpha_i + 1.$$

Thus we see, for $f_i(z) \in S_p(\beta)$, for all $i \in \{1, ..., n\}$

$$\mathbf{Re}\left(\frac{zF_n''(z)}{F_n'(z)} + 1\right) > \sum_{i=1}^{n} \alpha_i \left|\frac{zf_i'(z)}{f_i(z)} - 1\right| - (\beta - 1)\sum_{i=1}^{n} \alpha_i + 1.$$

Because $\alpha_i \left|\frac{zf_i'(z)}{f_i(z)} - 1\right| > 0$ for all $i \in \{1, ..., n\}$, obtain that

$$\mathbf{Re}\left(\frac{zF_n''(z)}{F_n'(z)} + 1\right) > (\beta - 1)\sum_{i=1}^{n} \alpha_i + 1.$$

Because $-1 \leq \beta < 1$ and $0 < \sum\limits_{i=1}^{n} \alpha_i \leq \dfrac{1}{1-\beta}$ obtain that $0 \leq (\beta - 1)\sum\limits_{i=1}^{n} \alpha_i$ $+1 < 1$. So F_n is convex by the order $(\beta - 1)\sum\limits_{i=1}^{n} \alpha_i + 1$.

Remark 1.*If* $\beta = 0$ *and* $\sum\limits_{i=1}^{n} \alpha_i = 1$ *then*

$$\mathbf{Re}\left(\frac{zF_n''(z)}{F_n'(z)} + 1\right) > 0$$

so, F_n *is the convex function.*

Corollary 1. *Let* γ *the real number,* $\gamma > 0$. *We suppose that the functions* $f \in S_p(\beta)$ *and* $0 < \gamma \leq \dfrac{1}{1-\beta}$. *In this conditions the integral operator* $F_1(z) = \int\limits_0^z \left(\frac{f(t)}{t}\right)^\gamma dt$ *is convex by the order* $(\beta - 1)\gamma + 1$.

Proof. In the Theorem 2, we consider $n = 1$.

Corollary 2. *Let* $f \in S_p(\beta)$ *and consider the integral operator of Alexander,* $F(z) = \int\limits_0^z \frac{f(t)}{t} dt$. *In this condition* F *is convex by the order* β.

Proof. We have:

$$\frac{zF''(z)}{F'(z)} = \frac{zf'(z)}{f(z)} - 1. \tag{6}$$

From (6) we have:

$$\mathbf{Re}\left(\frac{zF''(z)}{F'(z)} + 1\right) = \mathbf{Re}\left(\frac{zf'(z)}{f(z)} - \beta\right) + \beta > k\left|\frac{zf'(z)}{f(z)} - 1\right| + \beta > \beta. \tag{7}$$

So, the relation (7) imply that the Alexander operator is convex.

References

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