General Mathematics Vol. 15, Nr. 2–3(2007), 98–107

First order linear strong differential subordinations

Georgia Irina Oros and Gheorghe Oros

Abstract

The concept of differential subordination was introduced in [6] by S.S. Miller and P.T. Mocanu and the concept of strong differential subordination was introduced in [1] by J.A. Antonino and S. Romaguera. This last concept was applied in the special case of Briot-Bouquet strong differential subordination. In [8] we study the strong differential subordinations in the general case. In this paper we study the first order linear strong differential subordinations.

2000 Mathematical Subject Classification: 30C45, 34A30.

Key words and phrases: analytic function, differential subordination, strong subordination.

1 Introduction

Let $\mathcal{H} = \mathcal{H}(U)$ denote the class of functions analytic in U. For n a positive integer and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H};\ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots,\ z \in U\}.$$

Let A be the class of functions f of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots, \quad z \in U,$$

which are analytic in the unit disk.

In addition, we need the class of convex (univalent) and starlike (univalent) functions given respectively by

$$K = \{f \in A;\ \text{Re } zf''(z)/f'(z) + 1 > 0\}$$

and

$$S^* = \{f \in A,\ \text{Re } zf'(z)/f(z) > 0\}.$$

In order to prove our main results, we use the following definitions and lemma.

Definition 1. [1], [2], [3] *Let $H(z, \xi)$ be analytic in $U \times \overline{U}$ and let $f(z)$ analytic and univalent in U. The function $H(z, \xi)$ is strongly subordinate to $f(z)$, written $H(z, \xi) \prec\prec f(z)$ if for each $\xi \in \overline{U}$, the function of z, $H(z, \xi)$ is subordinate to $f(z)$.*

Remark 1. Since $f(z)$ is analytic and univalent Definition 1 is equivalent to:

$$H(0, \xi) = f(0) \text{ and } H(U \times \overline{U}) \subset f(U).$$

Definition 2. [7, p.24] *We denote by Q the set of functions f that are analytic and injective in $\overline{U}\setminus E(f)$, where*

$$E(f)=\left\{\zeta\in\partial U;\ \lim_{z\to\zeta}f(z)=\infty\right\}$$

and are such that $f'(\zeta)\neq 0$ for $\zeta\in\partial U\setminus E(f)$. The subclass of Q for which $f(0)=a$ is denoted by $Q(a)$.

Definition 3. [8, Definition 4] *Let Ω be a set in $\mathbb{C}$, $q\in Q$ and n be a positive integer. The class of admissible functions $\psi_n[\Omega,q]$ consists of those functions $\psi:\mathbb{C}^2\times U\times\overline{U}\to\mathbb{C}$ that satisfy the admissibility condition:*

$$\psi(r,s;z,\xi)\notin\Omega$$

whenever $r=q(\zeta)$, $s=m\zeta q'(\zeta)$, $z\in U$, $\xi\in\overline{U}$, $\zeta\in\partial U\setminus E(f)$ and $m\geq n$.

Lemma A. [7, Lemma 2.2.d, p.24] *Let $q\in Q(a)$, with $q(0)=a$ and let $p(z)=a+a_nz^n+\dots$ be analytic in U, with $p(z)\not\equiv a$ and $n\geq 1$. If p is not subordinate to q, then there exist points $z_0=r_0e^{i\theta_0}\in U$ and $\zeta_0\in\partial U\setminus E(q)$, and an $m\geq n\geq 1$ for which $p(U_{r_0})\subset q(U)$*

(i) $p(z_0)=q(\zeta_0)$

(ii) $z_0p'(z_0)=m\zeta_0q'(\zeta_0)$.

2 Main results

Taking Definition 1 as starting point, we define the first order linear strong differential subordination as follows:

Definition 4. *A strong differential subordination of the form*

$$A(z,\xi)zp'(z)+B(z,\xi)p(z)\prec\prec h(z),\quad z\in U,\ \xi\in\overline{U} \tag{1}$$

where $A(z,\xi)zp'(z)+B(z,\xi)p(z)$ *is analytic in* U *for all* $\xi \in \overline{U}$ *and* $h(z)$ *is analytic in* U *is called first order linear strong differential subordination.*

Remark 1. If $A(z,\xi)=1$ and $B(z,\xi)=0$ and $h(z)$ is a convex function then (1) becomes

$$zp'(z) \prec h(z), \quad z \in U.$$

This subordination was studied by G.M. Goluzin in 1935 in [4]. Goluzin proved that if h is a convex function then

$$p(z) \prec q(z) = \int_0^z h(t)t^{-1}dt$$

and q is the best dominant of the differential subordination.

In 1970, T.J. Suffridge [10] showed that Goluzin's result remains true even if function h is only starlike.

Remark 2. If $A(z,\xi)=B(z,\xi)=1$ then (1) becomes

$$zp'(z)+p(z) \prec h(z).$$

This subordination was studied by R.M. Robinson in 1947 in [9]. Robinson proved that if h and $q(z)=z^{-1}\int_0^z h(t)dt$ are univalent then q is the best dominant at least in the disc $|z|<\frac{1}{5}$.

Remark 3. If $A(z,\xi)=\frac{1}{\gamma}$ for $\gamma \neq 0$ and $\operatorname{Re}\gamma \geq 0$ and $B(z,\xi)=1$ then (1) becomes

$$p(z)+\frac{zp'(z)}{\gamma} \prec h(z).$$

This subordination was studied in 1975 by D.J. Hallenbeck and S. Ruscheweyh [5]. They have proved that if h is convex then the function

$$q(z)=\frac{\gamma}{z^\gamma}\int_0^z h(t)t^{-1}dt$$

is the best dominant of the subordination.

Theorem 1. *Let* $p \in \mathcal{H}[0,n]$, $A : U \times \overline{U} \to \mathbb{C}$, $B : U \times \overline{U} \to \mathbb{C}$ *with* $A(z,\xi)zp'(z) + B(z,\xi)p(z)$ *analytic in* U *for all* $\xi \in \overline{U}$ *and*

$$\text{Re }[nA(z,\xi) + B(z,\xi)] \geq 1, \quad \text{Re } A(z,\xi) \geq 0.$$

If

$$A(z,\xi)zp'(z) + B(z,\xi)p(z) \prec\prec Mz, \quad z \in U, \ \xi \in \overline{U} \tag{6}$$

then

$$p(z) \prec Mz, \quad z \in U, \ M > 0.$$

Proof. Let $\psi : \mathbb{C}^2 \times U \times \overline{U} \to \mathbb{C}$, $r = p(z)$, $s = zp'(z)$. We have

$$\psi(r,s;z,\xi) = A(z,\xi)zp'(z) + B(z,\xi)p(z)$$

and (6) becomes

$$\psi(r,s;z,\xi) \prec\prec Mz, \quad z \in U, \ \xi \in \overline{U}. \tag{7}$$

Since $h(z) = Mz$, it results that $h(U) = U(0,M)$. In this case, (7) is equivalent to

$$\psi(r,s;z,\xi) \in U(0,M). \tag{8}$$

Suppose that p is not subordinated to $h(z) = Mz$. Then, by using Lemma A, we have that there exist $z_0 \in U$ and $\zeta_0 \in \partial U$ such that $p(z_0) = h(\zeta_0) = Me^{i\theta_0}$, $\theta_0 \in \mathbb{R}$ when $|\zeta_0| = 1$ and

$$z_0p'(z_0) = m\zeta_0h'(\zeta_0) = Ke^{i\theta_0}, \quad K \geq nM,$$

hence we obtain

$$|\psi(Me^{i\theta_0}, Ke^{i\theta_0}; z_0, \xi)| = |A(z_0,\xi)z_0p'(z_0) + B(z_0,\xi)p(z_0)|$$

$$= |A(z_0,\xi)Ke^{i\theta_0} + B(z_0,\xi)Me^{i\theta_0}| = |A(z_0,\xi)K + B(z_0,\xi)M|$$

$$\geq \text{Re }[KMA(z_0,\xi) + MB(z_0,\xi)| \geq K\text{Re }A(z_0,\xi) + M\text{Re }B(z,\xi)$$

$$\geq M[n\text{Re }A(z,\xi) + \text{Re }B(z,\xi)] \geq M.$$

Since this result contradicts (8), we conclude that the assumption made concerning the subordination relation between p and h is false, hence $p(z) \prec Mz$, $z \in U$.

Theorem 2. *Let $p \in \mathcal{H}[1,n]$, $A, B : U \times \overline{U} \to \mathbb{C}$ with $A(z,\xi)zp'(z) + B(z,\xi)p(z)$ a function of z, analytic in U for any $\xi \in \overline{U}$ and*

$$\text{Re }A(z,\xi) \geq 0, \quad \text{Im }B(z,\xi) \leq n\text{Re }A(z,\xi).$$

If

$$\text{Re }[A(z,\xi)zp'(z) + B(z,\xi)p(z)] > 0 \tag{9}$$

then

$$\text{Re }p(z) > 0, \quad z \in U.$$

Proof. Let $\psi : \mathbb{C}^2 \times U \times \overline{U} \to \mathbb{C}$,

$$\psi(r,s;z,\xi) = A(z,\xi)s + B(z,\xi)r$$

for $r = p(z)$, $s = zp'(z)$. In this case, (9) becomes

$$\text{Re }\psi(r,s;z,\xi) > 0, \quad z \in U,\ \xi \in \overline{U}. \tag{10}$$

Since

$$h(z)=\frac{1+z}{1-z},\quad h(U)=\{w\in\mathbb{C}:\ \text{Re }w(z)>0\}$$

from which we have that (10) becomes

$$\psi(r,s;z,\xi)\prec\frac{1+z}{1-z}.$$

Suppose that Re $p(z)<0$, meaning p is not subordinated to $h(z)=\frac{1+z}{1-z}$. Using Lemma A, we have that there exist $z_0\in U$ and $\zeta_0\in\partial U$ with $|\zeta_0|=1$ such that

$$p(z_0)=h(\zeta_0)=\rho i,\quad z_0p'(z_0)=m\zeta_0h'(\zeta_0)=\sigma$$

where $\rho,\sigma\in\mathbb{R}$ and $\sigma\le-\frac{n}{2}(1+\rho^2)$, $n\ge1$.

Then we obtain:

$$\text{Re }\psi(p(z_0),z_0p'(z_0);z,\xi)=\text{Re }\psi(h(\zeta_0),m\zeta_0h'(\zeta_0),z;\xi)$$

$$=\text{Re }\psi(\rho i,\sigma;z,\xi)=\text{Re }[A(z,\xi)\sigma+B(z,\xi)\rho i]$$

$$=\text{Re }\{A(z,\xi)\sigma+[B_1(z,\xi)+iB_2(z,\xi)]\rho i\}$$

$$=\sigma\text{Re }A(z,\xi)-\rho\text{Im }B(z,\xi)\le-\frac{n}{2}(1+\rho^2)\text{Re }(z,\xi)-\rho\text{Im }B(z,\xi)$$

$$\le-\frac{n}{2}\rho^2\text{Re }A(z,\xi)-\rho\text{Im }B(z,\xi)-\frac{n}{2}\le0.$$

Hence Re $\psi(p(z_0),z_0p'(z_0);z,\xi)\le0$ which contradicts (10) and we conclude that Re $p(z)>0$, $z\in U$.

Theorem 3. *Let* $p\in\mathcal{H}[1,n]$, $A,B:U\times\overline{U}\to\mathbb{C}$ *with* $A(z,\xi)zp'(z)+B(z,\xi)p(z)$ *a function of* z*, analytic in* U *for all* $\xi\in\overline{U}$ *and*

$$\text{Re }A(z,\xi)\ge0,\quad\text{Im }B(z,\xi)\le n\text{Re }A(z,\xi)[-n\text{Re }A(z,\xi)+z].$$

If

(11) $$A(z,\xi)zp'(z)+B(z,\xi)p(z) \prec\prec z, \quad z\in U,\ \xi\in\overline{U}$$

then

$$p(z)\prec \frac{1+z}{1-z}, \quad z\in U.$$

Proof. Let $\psi:\mathbb{C}^2\times U\times\overline{U}\to\mathbb{C}$,

$$\psi(r,s;z,\xi)=A(z,\xi)s+B(z,\xi)r,$$

for $r=p(z)$, $s=zp'(z)$. Then (11) becomes

(12) $$\psi(r,s;z,\xi)\prec\prec z, \quad z\in U,\ \xi\in\overline{U}.$$

Since $h(z)=z$, $h(U)=U$ we obtain

(13) $$\psi(r,s;z,\xi)\subset U, \quad z\in U,\ \xi\in\overline{U}.$$

Suppose that p is not subordinated to $q(z)=\dfrac{1+z}{1-z}$. Using Lemma A, we have that there exist $z_0\in U$, $\zeta_0\in\partial U$ such that

$$p(z_0)=q(\zeta_0)=\rho i, \quad z_0p'(z_0)=m\zeta_0q'(\zeta_0)=\sigma$$

where $\rho,\sigma\in\mathbb{R}$ and

$$\sigma\le -\frac{n}{2}(1+\rho^2), \quad n\ge 1.$$

Then we obtain

$$\text{Re }\psi(p(z_0),z_0p'(z_0);z_0,\xi)=\text{Re }\psi(\rho i,\sigma;z_0,\xi)$$

$$=\text{Re }[A(z,\xi)\sigma+B(z,\xi)\rho i]=\sigma\text{Re }A(z,\xi)-\rho\text{Im }B(z,\xi)$$

$$\leq -\frac{n}{2}(1+\rho^2)\text{Re } A(z,\xi) - \rho\text{Im } B(z,\xi)$$
$$\leq -\frac{n}{2}\rho^2\text{Re } A(z,\xi) - \rho\text{Im } B(z,\xi) - \frac{n}{2}\text{Re } A(z,\xi) \leq -1.$$

Hence, we have

$$\text{Re } \psi(p(z_0), zp'(z_0); z_0, \xi) \leq -1$$

which contradicts (13) and we conclude that

$$p(z) \prec \frac{1+z}{1-z}, \quad z \in U.$$

References

[1] José A. Antonino and Salvador Romaguera, *Strong differential subordination to Briot-Bouquet differential equations*, Journal of Differential Equations 114(1994), 101-105.

[2] José A. Antonino, *Strong differential subordination and applications to univalency conditions*, J. Korean Math. Soc., **43**(2006), no.2, 311-322.

[3] José A. Antonino, *Strong differential subordination to a class of first order differential equations* (to appear).

[4] G.M. Goluzin, *On the majorization principle in function theory*, Dokl. Akad. SSSR, 42(1935), 647-649.

[5] D.J. Hallenbeck, S. Ruscheweyh, *Subordination by convex functions*, Proc. Amer. Math. Soc., 52(1975), 191-195, p.192.

[6] S.S. Miller and P.T. Mocanu, *Differential subordinations and univalent functions*, Michigan Math. J., **28**(1981), no.2, 157-172.

[7] S.S. Miller and P.T. Mocanu, *Differential subordinations. Theory and applications*, Pure and Applied Mathematics, Marcel Dekker, Inc., New York, 2000.

[8] Georgia Irina Oros and Gheorghe Oros, *Strong differential subordination* (to appear).

[9] R.M. Robinson, *Univalent majorants*, Trans. Amer. Math. Soc., 61(1947)(1-35), p.22.

[10] K. Sakaguchi, *A note on p-valent functions*, J. Math. Soc. Japan, 14(1962), 312-321.

[11] T.J. Suffridge, *Some remarks on convex maps of the unit disk*, Duke Math. J., 37(1970), 775-777, p.777.

Department of Mathematics
University of Oradea
Str. Universităţii, No.1
410087 Oradea, Romania
E-mail: georgia_oros_ro@yahoo.co.uk