On the confluent hypergeometric function coming from the Pareto distribution

Katsuo Takano

Abstract

Making use of the confluent hypergeometric function we can obtain the Laplace-Stieltje transform of the Pareto distribution in the following form

$$\zeta(s) = hU(1; 1 - h; s)$$
$$= {}_1F_1(1; 1 - h; s) - \Gamma(1 - h)s^h {}_1F_1(1 + h; 1 + h; s).$$

About this transform, we obtain an identity,

$$\Gamma(1 + h)|U(1, 1 - h, s)|^2 = \int_0^\infty \int_0^\infty \frac{\lambda^h e^{-\lambda - y}}{|\lambda + s|^2 + \lambda y} dy d\lambda.$$

1   Introduction

If $h = l + h_1$, $0 < h_1 < 1$, where l is a positive integer, let us denote

$$u(t) := \frac{\Gamma(1 - h_1) \sin \pi h_1}{(h - 1) \cdots (h - l)} t^h e^{-t},$$

and

$$v(t) := 1 + \Sigma_{j=1}^{l} \frac{t^j}{(h - 1) \cdots (h - j)}$$

$$+ \frac{t^l}{(h - 1) \cdots (h - l)} \Sigma_{j=1}^{\infty} \frac{t^j}{(h_1 - 1) \cdots (h_1 - j)}$$

$$- \frac{\Gamma(1 - h_1)}{(h - 1) \cdots (h - l)} t^h e^{-t} \cos \pi h_1.$$

Let us denote

$$\delta_j := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j}, \; j = 1, 2, ...$$

and let γ be the Euler constant. If $h = l$, which is the case $h_1 = 0$ in the above, let us denote

$$u(t) := \frac{\pi}{(l - 1)!} t^l e^{-t},$$

and

$$v(t) := 1 + \Sigma_{j=1}^{l-1} \frac{t^j}{(l - 1) \cdots (l - i)}$$

$$+ \frac{t^l}{(l - 1)!} \{\Sigma_{j=1}^{\infty} (-1)^j \delta_j \frac{t^j}{j!} + \gamma e^{-t} - e^{-t} \log \; t\}.$$

Let us denote

$$c_1 := \frac{\Gamma(1 - h_1) \sin \pi h_1}{(h - 1) \cdots (h - l)},$$

and

$$c_2 := \frac{\pi}{(l - 1)!}.$$

It is the purpose of this paper to show that the following identity holds:

(2) $$\frac{\pi}{c(1,h)h}\{u^2(t) + v^2(t)\} = \int_0^\infty \int_0^\infty \frac{\lambda^h e^{-\lambda - y}}{(t-\lambda)^2 + y\lambda} dy d\lambda$$

for positive t, where $c(1,h) = c_1$ if h is non-integer, ; $= c_2$ if $h = l$ is a positive integer.

2   Pareto distribution

The Pareto distribution is usually defined by

$$\begin{aligned} F(x) &= 1 - (\frac{c}{x})^h, \quad x \geq c, \\ &= 0, \quad x < c, \end{aligned}$$

where $c > 0$, $h > 0$ are the determing parameters. In this paper we take $c = 1$ and let the Pareto distribution be

$$\begin{aligned} F(x) &= 1 - \frac{1}{(1+x)^h}, \quad x \geq 0, \\ (3) &= 0, \quad x < 0. \end{aligned}$$

Then the Pareto density is

$$\begin{aligned} F'(x) &= \frac{h}{(1+x)^{1+h}}, \quad x > 0, \\ (4) &= 0, \quad x < 0. \end{aligned}$$

Suppose that $h = 1 + h_1$, $0 < h_1 < 1$. Integrating by parts, we see that

$$\begin{aligned} \zeta(s) &= \int_0^\infty e^{-sx} \frac{h}{(1+x)^{1+h}} dx = he^s \int_1^\infty e^{-st} t^{-1-h} dx \\ &= 1 + \frac{-s}{h-1} + e^s \frac{(-s)^2}{h-1} \int_1^\infty e^{-st} t^{1-h} dt \end{aligned}$$

We see that

$$\int_1^\infty e^{-st}\, t^{-h_1} dt \;=\; \int_0^\infty e^{-st}\, t^{-h_1} dt - \int_0^1 e^{-st}\, t^{-h_1} dt$$

$$=\; \Gamma(1-h_1)s^{h_1-1} - \int_0^1 e^{-st}\, t^{-h_1} dt$$

Repeating integration by parts j times, we obtain

$$\int_0^1 e^{-st}\, t^{-h_1} dt \;=\; -e^{-s}\{\frac{1}{h_1-1} + \frac{(-s)}{(h_1-1)(h_1-2)} + \cdots$$

$$+\; \frac{(-s)^{j-1}}{(h_1-1)\cdots(h_1-j)}\}$$

$$+\; \frac{(-s)^j}{(h_1-1)\cdots(h_1-j)} \int_0^1 e^{-st}\, t^{j-h_1} dt.$$

It is seen that

$$\left|\frac{(-s)^j}{(h_1-1)\cdots(h_1-j)} \int_0^1 e^{-st}\, t^{j-h_1} dt\right| \le \left|\frac{(-s)^j}{(h_1-1)\cdots(h_1-j)}\right|\frac{1}{j+1-h} \to 0$$

as $j \to \infty$ for $|s| < \infty$. Hence we obtain

$$\zeta(s) = 1 + \frac{(-s)}{h-1}$$

$$+\frac{(-s)^2}{h-1}\{\frac{1}{h_1-1} + \frac{(-s)}{(h_1-1)(h_1-2)} + \cdots + \frac{(-s)^{j-1}}{(h_1-1)\cdots(h_1-j)} + \cdots\}$$

$$(5) \quad -\frac{\Gamma(1-h_1)}{h-1}(-s)e^s\, s^{h_1}$$

for $h = 1 + h_1$, $0 < h_1 < 1$ and $|s| < \infty$. In general, if l is a positive integer and $h = l + h_1$, $0 < h_1 < 1$, we obtain

$$\zeta(s) = 1 + \Sigma_{j=1}^l \frac{(-s)^j}{(h-1)\cdots(h-i)}$$

$$+\frac{(-s)^l}{(h-1)\cdots(h-l)}\Sigma_{j=1}^\infty \frac{(-s)^j}{(h_1-1)\cdots(h_1-j)}$$

$$(6) \qquad\qquad -\frac{\Gamma(1-h_1)}{(h-1)\cdots(h-l)}(-s)^l e^s\, s^{h_1}.$$

If $0 < h < 1$, we have

$$\zeta(s) = 1 + \Sigma_{j=1}^{\infty} \frac{(-s)^j}{(h-1)\cdots(h-j)} - \Gamma(1-h)s^h e^s.$$

In the above $\zeta(s)$, in order to have a single-valued function, we take the branch of $s^{h_1} = e^{h_1 \log s}$ as $s^{h_1} > 0$ for $s > 0$. The function $\zeta(s)$ can be defined by analytic continuation on the whole complex plane with the cut along the negtive real axis including zero. If $s = -t + i\,\rho$, $t > 0$, $\rho > 0$, we see that

$$Re\,\zeta^+(-t) = \lim_{\rho \to +0} Re\,\zeta(-t+i\rho)$$

$$= 1 + \Sigma_{j=1}^{l} \frac{t^j}{(h-1)\cdots(h-i)} + \frac{t^l}{(h-1)\cdots(h-l)} \Sigma_{j=1}^{\infty} \frac{t^j}{(h_1-1)\cdots(h_1-j)}$$

$$- \frac{\Gamma(1-h_1)}{(h-1)\cdots(h-l)} t^l e^{-t} \lim_{\rho \to +0} Re\, e^{h_1 \log(-t+i\rho)}$$

$$= 1 + \Sigma_{j=1}^{l} \frac{t^j}{(h-1)\cdots(h-i)} + \frac{t^l}{(h-1)\cdots(h-l)} \Sigma_{j=1}^{\infty} \frac{t^j}{(h_1-1)\cdots(h_1-j)}$$

$$(7) \quad - \frac{\Gamma(1-h_1)}{(h-1)\cdots(h-l)} t^h e^{-t} \cos \pi h_1 \ .$$

In the same manner, we have

$$Re\,\zeta^+(-t) = Re\,\zeta^-(-t).$$

We also see that

$$Im\,\zeta^+(-t) = \lim_{\rho \to +0} Im\,\zeta(-t+i\rho)$$

$$(8) \qquad\qquad = -\frac{\Gamma(1-h_1)}{(h-1)\cdots(h-l)} t^h e^{-t} \sin \pi h_1$$

and $Im\ \zeta^+(-t) = -Im\ \zeta^-(-t)$ for $t > 0$. We have $-Im\ \zeta^+(-t) = u(t)$ and $Re\ \zeta^+(-t) = v(t)$. From (7) we have

$$\zeta(s) = 1 + \Sigma_{j=1}^{l} \frac{s^j}{(1-h)\cdots(i-h)}$$

$$+ \frac{s^l}{(1-h)\cdots(l-h)} \Sigma_{j=1}^{\infty} \frac{s^j}{(1-h_1)\cdots(j-h_1)}$$

(9)
$$- \frac{\Gamma(1-h_1)}{(1-h)\cdots(l-h)} s^l e^s s^{h_1},$$

and using the confluent hypergeometric function,

$$_1F_1(a;b;s) = 1 + \frac{a}{b}s + \frac{a(a+1)}{b(b+1)}\frac{s^2}{2!} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)}\frac{s^3}{3!} + \cdots ,$$

where $b \neq -1, -2, \cdots$, we can write as follows;

$$\zeta(s) = {_1F_1}(1; 1-h; s) - \Gamma(1-h)s^h {_1F_1}(1+h; 1+h; s)$$

(10)
$$= hU(1; 1-h; s),$$

(cf. [5. (1.3.1)]). Until here, we supposed that $h = l + h_1$, $0 < h_1 < 1$, l is a positive integer. Next, we suppose that h is a positive integer, and now we consider the limit case as $h_1 \to +0$. We show that as $h_1 \to 0$,

$$\frac{1}{h_1}\{1 + \Sigma_{j=1}^{\infty} \frac{(-1)^j t^j}{(1-h_1)\cdots(j-h_1)} - e^{-t}\}$$

(11)
$$\to \Sigma_{j=1}^{\infty}(-1)^j(1 + \frac{1}{2} + \cdots + \frac{1}{j})\frac{1}{j!}t^j$$

for $|t| < \infty$. As $x \to +0$, it holds that

$$|\frac{1}{x}\{1 + \Sigma_{j=1}^{\infty} \frac{(-1)^j t^j}{(1-x)\cdots(j-x)} - e^{-t}\}|$$

$$= |\frac{1}{x}\Sigma_{j=1}^{\infty}\{\frac{1}{(1-x)\cdots(j-x)} - \frac{1}{j!}\}(-1)^j t^j|$$

(12)
$$\leq \Sigma_{j=1}^{\infty}\frac{1}{j!}|t|^j\frac{1}{x}\{\frac{1}{(1-x)(1-\frac{x}{2})\cdots(1-\frac{x}{j})} - 1\}|.$$

By the mean value theorem we see that

$$|\frac{1}{x}\{\frac{1}{(1-x)(1-\frac{x}{2})\cdots(1-\frac{x}{j})}-1\}| \le \frac{1}{x}\{\frac{1}{(1-x)^j}-1\} = j(1-\theta x)^{-j-1} \le 2^{j+1}j,$$

$(0 < \theta < 1)$ for $0 < x \le \frac{1}{2}$. Hence if $0 < x \le 1/2$, (11) converges uniformly with respect to t in each finite interval including the origin. Since

$$\lim_{x\to+0} \frac{1}{x}\{\frac{1}{(1-x)(1-\frac{x}{2})\cdots(1-\frac{x}{j})} - 1\} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j},$$

differentiating by term by term, we obtain (10). As $h_1 \to +0$, we have

$$u(t) = \frac{\Gamma(1-h_1)\sin \pi h_1}{(h-1)\cdots(h-l)}t^h e^{-t} \to u(l;t) = \frac{\pi}{(l-1)!}t^l e^{-t}.$$

We obtain that

$$v(t) = 1 + \Sigma_{j=1}^l \frac{t^j}{(h-1)\cdots(h-i)}$$

$$+\frac{t^l}{(h-1)\cdots(h-l)}\Sigma_{j=1}^\infty \frac{t^j}{(h_1-1)\cdots(h_1-j)}$$

$$-\frac{\Gamma(1-h_1)}{(h-1)\cdots(h-l)}t^h e^{-t}\cos\pi h_1$$

$$\to v(l;t) = 1 + \Sigma_{j=1}^{l-1}\frac{t^j}{(l-1)\cdots(l-i)}$$

(13) $$+\frac{t^l}{(l-1)!}\{\Sigma_{j=1}^\infty (-1)^j\delta_j\frac{t^j}{j!} + \gamma e^{-t} - e^{-t}\log t\}$$

In fact, it is seen that

$$c_1 = \frac{\Gamma(1-h_1)\sin \pi h_1}{(h-1)\cdots(1+h_1)h_1} \to \frac{\pi}{(l-1)!} = c_2 \quad as \quad h_1 \to +0,$$

and from (6) we have

$$v(t) = 1 + \Sigma_{j=1}^{l-1}\frac{t^j}{(h-1)\cdots(h-i)}$$

$$+\frac{t^l}{(h-1)\cdots(h-l+1)h_1}\{1 + \Sigma_{j=1}^\infty \frac{t^j}{(h_1-1)\cdots(h_1-j)}$$

(14) $$-\Gamma(1-h_1)t^{h_1} e^{-t}\cos\pi h_1\},$$

and we obtain that

$$\frac{1}{h_1}\{1 + \Sigma_{j=1}^{\infty} \frac{t^j}{(h_1 - 1)\cdots(h_1 - j)} - \Gamma(1 - h_1)t^{h_1} \; e^{-t} \cos \pi h_1\}$$

$$\text{(15)} \qquad \to \Sigma_{j=1}^{\infty} (-1)^j \delta_j \frac{t^j}{j!} + \gamma e^{-t} - e^{-t} \log \; t$$

as $h_1 \to +0$.

3   An identity on the confluent hypergeometric function

Let us denote the Pareto density by

$$p_h(x) = \frac{h}{(1 + x)^{1+h}}, \quad x > 0.$$

Theorem 1. *It holds that*

$$\text{(16)} \qquad \Gamma(1 + h)|U(1, 1 - h, s)|^2 = \int_0^{\infty} \int_0^{\infty} \frac{\lambda^h e^{-\lambda - y}}{|\lambda + s|^2 + \lambda y} dy d\lambda$$

for $-\pi < \arg s < +\pi$. *Hence the confluent hypergeometric function* $U(1, 1 - h, s)$ *does not have zeros outside except cut along nonpositive real line. If* $s = -t < 0$ *it holds that*

$$\Gamma(1 + h)|U(1, 1 - h, -t)|^2 = \frac{\pi}{c(1, h)h}\{u(t)^2 + v(t)^2\}$$

$$\text{(17)} \qquad = \int_0^{\infty} \int_0^{\infty} \frac{\lambda^h e^{-\lambda - y}}{|\lambda - t|^2 + \lambda y} dy d\lambda$$

and the double integral is convergent for $t > 0$.

Proof. The Laplace transform of the Pareto density $\zeta(s)$ is defined for Re $s \geq 0$ and $\zeta(+0) = 1$. Let us denote the extension of $\zeta(s)$ by analytic continuation by $\phi(s)$. The density function $p_h(x)$ can be extended to the complex plane by analytic continuation and we have

$$\phi(s) = e^{i\theta} \int_0^\infty e^{-se^{i\theta}x} p_h(e^{i\theta}x) dx$$

for $-\pi < \arg s < \pi$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, such that Re $se^{i\theta} > 0$. (cf. G. Sansone and J.C.H. Gerretsen, [4], section 8.13) By Schwartz's reflection principle it holds that

$$\overline{\phi}(s) = e^{-i\theta} \int_0^\infty e^{-\overline{s}e^{-i\theta}x} p_h(e^{-i\theta}x) dx.$$

Hence we have

$$|\phi(s)|^2 = \int_0^\infty \int_0^\infty \exp\{-se^{i\theta}x - \overline{s}e^{-i\theta}y\} p_h(e^{i\theta}x) p_h(e^{-i\theta}y) dxdy,$$

and by change of variables, $x = uv$, $y = \frac{u}{v}$, we have

$$|\phi(s)|^2 = \int_0^\infty \int_0^\infty \frac{2u}{v} \exp\{-se^{i\theta}uv - \overline{s}e^{-i\theta}\frac{u}{v}\} p_h(e^{i\theta}uv) p_h(e^{-i\theta}\frac{u}{v}) dvdu,$$

From the following relation,

$$p_h(uv)p_h(\frac{u}{v}) = h^2 \frac{1}{\{1 + u(v + 1/v) + u^2\}^{1+h}}$$

$$(18) \qquad = \int_0^\infty \exp\{-\lambda u(v + \frac{1}{v})\} \frac{h^2}{\Gamma(1+h\}} \exp\{-\lambda(1 + u^2)\} \lambda^h d\lambda$$

we see that

$$|\phi(s)|^2 = \int_0^\infty \int_0^\infty \frac{2u}{v} \exp\{-se^{i\theta}uv - \bar{s}e^{-i\theta}\frac{u}{v}\}dudv$$

$$\cdot \int_0^\infty \exp\{-\lambda u(e^{i\theta}v + \frac{e^{-i\theta}}{v})\} \cdot \frac{h^2}{\Gamma(1+h)} \exp\{-\lambda(1+u^2)\} \lambda^h d\lambda$$

$$= \int_0^\infty \int_0^\infty 2u\lambda^h \frac{h^2}{\Gamma(1+h)} \exp\{-\lambda(1+u^2)\}d\lambda du$$

(19) $$\cdot \int_0^\infty \frac{1}{v} \exp\{-(\lambda+s)e^{i\theta}uv - (\lambda+\bar{s})e^{-i\theta}\frac{u}{v}\}dv.$$

In fact, by the assumption that $-\pi < \arg\ s\ < \pi$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, such that Re $se^{i\theta} > 0$, the three fold integral is written as follows,

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{2u}{v} |\exp\{-se^{i\theta}uv - \bar{s}e^{-i\theta}\frac{u}{v}\}|$$

$$\cdot |\exp\{-\lambda u(e^{i\theta}v + \frac{e^{-i\theta}}{v})\}| \cdot \frac{h^2}{\Gamma(1+h)} \exp\{-\lambda(1+u^2)\} \lambda^h d\lambda dudv$$

(20) $$= \int_0^\infty \int_0^\infty \exp[-(\text{Re } se^{i\theta})(x+y)]\frac{h^2}{[1+(x+y)\cos\theta+xy]^{1+h}}dxdy$$

and the double integral is convergent and so we can change the order of integration in the three fold integral by the Fubini theorem. Let

$$\Theta = \arg u(\lambda+\bar{s})e^{-i\theta}$$

and suppose $\Theta > 0$. From $\text{Re}u(\lambda+\bar{s})e^{-i\theta} > 0$ it holds that $\Theta < \frac{\pi}{2}$. On the following integral

$$\int_0^\infty \frac{1}{v} \exp\{-(\lambda+s)e^{i\theta}uv - (\lambda+\bar{s}e^{-i\theta})\frac{u}{v}\}dv$$

consider the contour integral along the curve C_{aABb}, which is compose of a real interval $[a, A]$ and a large arc $A \frown B$ with radius R and a segment

starting from $b = re^{i\Theta}$ to $B = Re^{i\Theta}$ and a small arc $a \frown b$ with radius r. We can see that the contour integral along the arc $A \frown B$ tends to 0 as R tends to ∞ and the contour integral along the small arc $a \frown b$ tends to 0 as r tends 0. By the Cauchy theorem it is seen that the variable can be changed such as $v = u(\lambda + \overline{s})e^{-i\theta}\rho$. Hence we obtain

$$
\begin{aligned}
|\phi(s)|^2 &= \int_0^\infty \int_0^\infty d\lambda du \int_0^\infty 2u\lambda^h \frac{h^2}{\Gamma(1+h)} \exp\{-\lambda(1+u^2)\} \\
&\quad \cdot \exp\{|\lambda + s|^2 u^2 \rho - \frac{1}{\rho}\} \frac{d\rho}{\rho} \\
&= \int_0^\infty \int_0^\infty d\lambda d\rho u \frac{h^2}{\Gamma(1+h)} \lambda^h \exp\{-\lambda - \frac{1}{\rho}\} \cdot \frac{1}{\rho} \\
&\quad \cdot \int_0^\infty 2u \exp\{-(|\lambda + s|^2 \rho + \lambda)u^2\} du \\
&= \frac{h^2}{\Gamma(1+h)} \int_0^\infty \int_0^\infty \frac{\lambda^h e^{-\lambda - y}}{|\lambda + s|^2 + \lambda y} d\lambda du.
\end{aligned}
$$

(21)

From this and the relation

$$
\frac{\pi}{c(1,h)h} = \frac{\Gamma(h)^2}{\Gamma(1+h)},
$$

we obtain the identity (15) and the right hand side of (15) is always positive and finite if Im $s \neq 0$. Considering the limit of the left hand side of (15) as $s = -t + i\rho \to -t < 0$ then the integrand of right hand side is positive, and by using the Lebesgue monotone convergence theorem and by the fact that the left hand side of the series is convergent we see that the right hand side of (16), i.e. the double integral, is convergent for $t > 0$. q.e.d.

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Department of Mathematics, Faculty of Science, Ibaraki University

Mito-city, Ibaraki 310 JAPAN

ktaka@mx.ibaraki.ac.jp