General Mathematics Vol. 15, No. 4 (2007), 17-23

On the charateristic equation of Chebyshev matrices[1]

Amelia Bucur, S.Alvarez-Ballesteros, J.L. López-Bonilla

Abstract

We show that the characteristic equation of Chebyshev matrix reveals the existence of "Associated Polynomials of Chebyshev", and we give an explicit expression from them.

2000 Mathematical Subject Classification:15A18.
Key words: Chebyshev polynomials; characteristic equation.

Here we consider the first-kind Chebyshev polynomials $T_n(x)$ given by the recurrence relation $[1-7](|x| \leq 1)$:

$$(1) \qquad T_0 = 0, \qquad T_1 = x, \qquad T_{k+1} = 2xT_k - T_{k-1}, \qquad k = 1, 2, \ldots$$

that is:

$$(2) \qquad T_0 = 1, \qquad T_1 = x, \qquad T_2 = 2x^2 - 1, \qquad T_3 = 4x^3 - 3x,$$

$$T_4 = 8x^4 - 8x^2 + 1, \qquad T_5 = 16x^5 - 20x^3 + 5x, \ldots,$$

which also can be generated as the determinant of Chebyshev matrices $\underline{T_n}(x)[6]$, in fact:

[1] *Received 10 December, 2007*
Accepted for publication (in revised form) 14 December, 2007

(3) $$T_2(x) = Det\underline{T_2}(x) = Det \begin{pmatrix} x & 1 \\ 1 & 2x \end{pmatrix}$$

$$T_3(x) = Det\underline{T_3}(x) = det \begin{pmatrix} x & 1 & 0 \\ 1 & 2x & 1 \\ 0 & 1 & 2x \end{pmatrix}$$

$$T_4(x) = Det\underline{T_4}(x) = det \begin{pmatrix} x & 1 & 0 & 0 \\ 1 & 2x & 1 & 0 \\ 0 & 1 & 2x & 1 \\ 0 & 0 & 1 & 2x \end{pmatrix}, etc.$$

However, we do not know publications studying the characteristic equation (CE)[9-17] of $\underline{T_n}(x)$, by these reason this work is dedicated to analyze the polynomial coefficients in the mentioned CE.

In the literature there are several methods[10,11,13] to obtain the CE of a matrix, here we employ the algorithm of Leverrier-Takeno [9,12,14-17] and the Maple symbolic program, therefore.

1 Characteristic Equation

(4)

1. $\lambda - T_1 = 0$
2. $\lambda^2 - 3x\lambda + T_2 = 0$
3. $\lambda^3 - 5x\lambda^2 + (8x^2 - 2)\lambda - T_3 = 0$
4. $\lambda^4 - 7x\lambda^3 + (18x^2 - 3)\lambda^2 - (20x^3 - 10x)\lambda + T_4 = 0$
5. $\lambda^5 - 9x\lambda^4 + (32x^2 - 4)\lambda^3 - (56x^3 - 21x)\lambda^2 + (48x^4 -$

$-36x^2 + 3)\lambda - T_5 = 0$

6. $\lambda^6 - 11x\lambda^5 + (50x^2 - 5)\lambda^4 - (120x^3 - 36x)\lambda^3 + (160x^4 - 96x^2 + 6)\lambda^2 - -(112x^5 - 112x^3 + 21x)\lambda + T_6 = 0$

7. $\lambda^7 - 13x\lambda^6 + (72x^2 - 6)\lambda^5 - (220x^3 - 55x)\lambda^4 + (400x^4 - 200x^2 + 10)\lambda^3 - -(432x^5 - 360x^3 + 54x)\lambda^2 + (256x^6 - 320x^4 + 96x^4 - 4)\lambda - T_7 = 0$

8. $\lambda^8 - 15x\lambda^7 + (98x^2 - 7)\lambda^6 - (364x^3 - 78x)\lambda^5 + (840x^4 - 360x^2 + +15)\lambda^4 - (1232x^5 - 880x^3 + 110x)\lambda^3 + (1120x^6 - 1200x^4 + 300x^2 - -10)\lambda^2 - (576x^7 - 864x^5 + 360x^3 - 36)\lambda + T_8 = 0$

etc.

Or in compact form:

$$\sum_{m=0}^{n} T_n^n \lambda^{n-m} = 0, T_n^0 = 1, T_n^n = (-1)^n T_n \tag{5}$$

thus, for example:

$$T_3^1 = -5x, T_3^2 = 8x^2 - 2, \tag{6}$$

$$T_5^1 = -9x, T_5^2 = 32x^2 - 4, T_5^3 = -56x^3 + 21x,$$

$$T_5^4 = 48x^4 - 36x^2 + 3$$

etc.

Then it is clear that $T_n^m(x)$, $m = 0, \ldots, n$ is a polynomial in x of degree m, and they may be named as "associated polynomials of Chebyshev", which are not explicitly in the literature and can be generated in terms of Gauss hypergeometric function:

$$T_n^m = (-1)^m \left(\begin{array}{c} 2n-m \\ m \end{array} \right) {}_2F_1(-m, 2n-m; n-m+\frac{1}{2}; \frac{1-x}{2}), \tag{7}$$

where $m = 0, \ldots, n$; with (7) is immediate to reproduce the characteristic equation (4).
As an additional case, from(5) and (7) for $n = 10$ we obtain the CE:

$$\lambda^{10} - 19x\lambda^9 + (162x^2 - 9)\lambda^8 + (-816x^3 + 136x)\lambda^7 + \tag{8}$$

$$+(2688x^4 - 896x^2 + 28)\lambda^6 + (-6048x^5 + 3360x^3 - 315x)\lambda^5 +$$

$$+(9408x^6 - 7840x^4 + 1470x^2 - 35)\lambda^4 + (-9984x^7 + 11648x^5 - 3640x^3 + 260x)\lambda^3 +$$

$$+(6912x^8 - 10752x^6 + 5040x^4 - 720x^2 + 15)\lambda^2 + (-2816x^9 + 5632x^7 - 3696x^5 +$$

$$+880x^3 - 55x)\lambda + T_4 = 0$$

in harmony with the corresponding expression given by the Maple program.
Now we indicate some properties of our associated polynomials of Chebyshev:
a) All roots x_j of $T_n^m(x) = 0$ are real and $|x_j| < 1$, which can be verified directly, employing the Matlab program, for each polynomial coefficient in (4) and(8).
b) $T_n^m(x)$ are generators for the well-known [5,6,8,12-20] four kinds of Chebyshev polynomials:

$$T_n(x) = (-1)^n T_n^n(x), \qquad U_n(x) = (-1)^n \frac{2}{n+2} T_{n+1}^n(x) \tag{9}$$

$$V_n(x) = \frac{(-1)^n}{n+1} T_{2n+1}^{2n}(\sqrt{\frac{1-x}{2}}), W_n(x) = \frac{1}{n+1} T_{2n+1}^{2n}\left(\sqrt{\frac{1+x}{2}}\right).$$

c)$T_n^m(x)$ are solution of differential equation:

$$(1-x^2)\frac{d^2y}{dx^2} - (2n-2m+1)x\frac{dy}{dx} + m(2n-m)y = 0, \tag{10}$$

which for $m = n$ coincides with the Chebyshev equations of the first-kind:

$$(1-x^2)\frac{d^2Tn}{dx^2} - x\frac{dTn}{dx} + n^2Tn = 0, \tag{11}$$

and for $m = N, n = N + 1$ we deduce the coresponding equation of the second-kind:

$$(1 - x^2)\frac{d^2U_N}{dx^2} - 3x\frac{dU_N}{dx} + N(N + 2)U_N = 0, etc. \tag{12}$$

The literature only mentions to $Tn(x)$ as the determinant of Chebyshev matrix, however, our work shows that this matrix has an important hidden information which can be uncovered studying its characteristic equation, revealing thus the existence of the associated polynomials of Chebyshev:

In other paper we will analyse more properties (recurrence, orthogonality, interpolation, etc.) of this interesting $T_n^m(x)$.

References

[1] P.L. Chebyshev, *Théorie des mecanismes connus sous le nom de parallélogrammes*, Mém. Acad. Sci. Pétersb.7(1854)539-568.

[2] C. Lanczos, *Tables of Chebyshev polynomials*, Nat. Bur. Std. Appl. Math. Series No. 9 (1952).

[3] B.K.P. Scaife, *Studies in numerical analysis*, Academic press(1974), p. 186.

[4] T.J Rivilin, *The Chebyshev polynomials*, Wiley-Intersciences, New York (1974).

[5] C. Lanczos, *Applied analysis*, Dover, New York(1988) Chap. 3..

[6] J.C. Mason and D. Handscomb, *Chebyshev polynomials*, Chapman & Hall-CRC Press, London(2002)Chap. 1.

[7] B. Sinwell, *The Chebyshev polynomials:Patterns and derivation*,Mats. Teacher 98, No 1(2004) 20-25.

[8] M.Galaz L., R. Garcia O. and J.López-Bonilla, *A new class of chebyshev-like polynomials*, Bol. Soc. Cub. Mat. Comp. 5 No.1 (2007) 33-39.

[9] U.J.J. Leverrier, *Sur les variation séculaires des éléments des orbites*, J. de Math. Pures Appl. Série 1,5(1840) 220-254.

[10] P. Horst, *A method of determinig the coefficients of a characteristic equation*, Ann. Math. Stat.6 (1935) 83-84.

[11] H. Wayland, *Expansion of determinantal equation into polynomials form*, Quart. Appl. Math. 2(1945) 277-306.

[12] H. Takeno, *A theorem concerning the chatracteristic equation of the matrix of a tensor of the second order*, Tensor N.S. 3(1945) 119-122.

[13] A.S. Householder and F.L. Bauer, *On certain methods for expanding the characteristic polynomial*, Numerische Mathematik 1 (1959) 29-37.

[14] E.B. Wilson, J.C. Decius and P.C. Cross, *Molecular vibrations*, Dover, New York (1980) 216-217.

[15] J.C. Gower, *A modified Leverrier-Faddeev algorithm for matrices with multiple eigenvalues*, Linear Algebra Appls. 31 (1980) 61-70.

[16] J.H. Caltenco,J. López-Bonilla and R. Peña R., *Characteristic polynomial of A and Faddeev's method for* A^{-1}, Aligarh Bull. Math. 19(2000) 55-59.

[17] J. López-Bonilla, J. Morales, G. Ovando and E. Ramirez G.,*Leverrier-Faddeev's algorithm applied to space-time of class one*, Proc. Pakistan Acad. Sci,43,No.1 (2006) 47-50.

[18] W. Gautschi, *On mean convergense of extended Lagrange interpolation*, J. Comp. Appl. Math. 43 (1992) 19-35.

[19] L. Boyadjiev and R. Scherer, *On the Chebyshev polynomials*, Kuwait J. Sci. Eng. 28, No.2 (2001) 227-240.

[20] J. Garcia-Revelo, R. Cuevas, A. Queijeiro, J.J. Peña and J. Morales, *Chebyshev function of half-integer order*, Integral Transforms and Special Functions 18, No. 10 (2007) 743-749.

Amelia Bucur
Department of Mathematics,
Faculty of Sciences,
"Lucian Blaga" University of Sibiu,
Sibiu, Romania
E-mail: amelia.bucur@ulbsibiu.ro

J.L. López-Bonilla, S.Alvarez-Ballesteros
SEPI-ESIME-Zacatenco
Instituto Politécnico Nacional
Edif. Z-4, 3er Piso, Col. Lindavista CP07738 México DF
E-mail: jlopezb@ipn.mx