

Some Subclasses of Close-to-Convex and Quasi-Convex Functions with Respect to k -Symmetric Points¹

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Abstract

In the present paper, the authors introduce two new subclasses $\mathcal{C}^{(k)}(\lambda, \alpha)$ of close-to-convex functions and $\mathcal{QC}^{(k)}(\lambda, \alpha)$ of quasi-convex functions with respect to k -symmetric points. The integral representations and convolution conditions for these classes are provided. Some coefficient inequalities for functions belonging to these classes and their subclasses with negative coefficients are also provided.

2000 Mathematical Subject Classification: 30C45.

Key words: Close-to-convex functions, quasi-convex functions, k -symmetric points, Hadamard product.

1 Introduction

Let \mathcal{A} denote the class of functions of the form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

¹Received 17 July, 2007

Accepted for publication (in revised form) 20 December, 2007

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbf{C} : |z| < 1\}$. Let \mathcal{S} denotes the subclass of \mathcal{A} consisting of all functions which are univalent in \mathcal{U} . Also let $\mathcal{T}(n, p)$ denote the class of functions of the form

$$f(z) = z^p - \sum_{l=n}^{\infty} a_{l+p} z^{l+p} \quad (a_{l+p} \geq 0; \quad p, n \in \mathbf{N} = \{1, 2, 3, \dots\}),$$

which are analytic in \mathcal{U} . Write $\mathcal{T}(1, 1)$ simple as \mathcal{T} .

We denote by \mathcal{S}^* , \mathcal{K} , \mathcal{C} and \mathcal{QC} the familiar subclasses of \mathcal{A} consisting of functions which are, respectively, starlike, convex, close-to-convex and quasi-convex in \mathcal{U} . Thus, by definition, we have (see, for details, [4, 6, 7, 9])

$$\mathcal{S}^* = \left\{ f : f \in \mathcal{A} \text{ and } \Re \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad (z \in \mathcal{U}) \right\},$$

$$\mathcal{K} = \left\{ f : f \in \mathcal{A} \text{ and } \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0 \quad (z \in \mathcal{U}) \right\},$$

$$\mathcal{C} = \left\{ f : f \in \mathcal{A}, \exists g \in \mathcal{S}^*, \text{ such that } \Re \left\{ \frac{zf'(z)}{g(z)} \right\} > 0 \quad (z \in \mathcal{U}) \right\},$$

and

$$\mathcal{QC} = \left\{ f : f \in \mathcal{A}, \exists g \in \mathcal{K}, \text{ such that } \Re \left\{ \frac{(zf'(z))'}{g'(z)} \right\} > 0 \quad (z \in \mathcal{U}) \right\}.$$

Let $\mathcal{T}(n, p, \lambda, \alpha)$ be the subclass of $\mathcal{T}(n, p)$ consisting of functions $f(z)$ which satisfy the inequality

$$\Re \left\{ \frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} \right\} > \alpha \quad (z \in \mathcal{U})$$

for some α ($0 \leq \alpha < 1$) and λ ($0 \leq \lambda \leq 1$). Altintas [1] once introduced and investigated the class $\mathcal{T}(n, 1, \lambda, \alpha)$. In a later paper, Altintas, Irmak and Srivastava [2] derived some other interesting properties of the class $\mathcal{T}(n, p, \lambda, \alpha)$. Write $\mathcal{T}(1, 1, \lambda, \alpha)$ simple as $\mathcal{T}(\lambda, \alpha)$.

Let $\mathcal{C}(n, \lambda, \alpha)$ be the subclass of $\mathcal{T}(n, 1)$ consisting of functions $f(z)$ which satisfy the inequality

$$\Re \left\{ z \frac{\lambda z^2 f'''(z) + (2\lambda + 1)z f''(z) + f'(z)}{\lambda z^2 f''(z) + z f'(z)} \right\} > \alpha \quad (z \in \mathcal{U})$$

for some α ($0 \leq \alpha < 1$) and λ ($0 \leq \lambda \leq 1$). The class $\mathcal{C}(n, \lambda, \alpha)$ was introduced and investigated recently by Kamali and Akbulut [5]. Write $\mathcal{C}(1, \lambda, \alpha)$ simple as $\mathcal{C}(\lambda, \alpha)$.

Sakaguchi [8] once introduced a class \mathcal{S}_s^* of functions starlike with respect to symmetric points, it consists of functions $f(z) \in \mathcal{S}$ satisfying

$$\Re \left\{ \frac{zf'(z)}{f(z) - f(-z)} \right\} > 0 \quad (z \in \mathcal{U}).$$

Following him, many authors discussed this class and its subclasses. And a function $f(z) \in \mathcal{A}$ is in the class \mathcal{C}_s if and only if $zf'(z) \in \mathcal{S}_s^*$.

Let $\mathcal{S}_s^{(k)}(\alpha)$ denote the class of functions in \mathcal{S} satisfying the following inequality

$$\Re \left\{ \frac{zf'(z)}{f_k(z)} \right\} > \alpha \quad (z \in \mathcal{U}),$$

where $0 \leq \alpha < 1$, $k \geq 2$ is a fixed positive integer and $f_k(z)$ is defined by the following equality

$$(1.2) \quad f_k(z) = \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{-\nu} f(\varepsilon^\nu z) \quad (\varepsilon = \exp(2\pi i/k); z \in \mathcal{U}).$$

And a function $f(z) \in \mathcal{A}$ is in the class $\mathcal{C}_s^{(k)}(\alpha)$ if and only if $zf'(z) \in \mathcal{S}_s^{(k)}(\alpha)$. The class $\mathcal{S}_s^{(k)}(\alpha)$ of functions starlike with respect to k -symmetric points of order α was studied by Chand and Singh [3], and the class $\mathcal{C}_s^{(k)}(\alpha)$ of functions convex with respect to k -symmetric points of order α is a corresponding special class defined in [10].

Motivated by the classes $\mathcal{T}(\lambda, \alpha)$, $\mathcal{C}(\lambda, \alpha)$, $\mathcal{S}_s^{(k)}(\alpha)$ and $\mathcal{C}_s^{(k)}(\alpha)$, we now introduce and investigate the following subclasses of \mathcal{A} with respect to k -symmetric points, and obtain some interesting results.

Definition 1. Let $\mathcal{C}^{(k)}(\lambda, \alpha)$ denote the class of functions in \mathcal{A} satisfying the following inequality

$$(1.3) \quad \Re \left\{ \frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f_k(z) + \lambda z f'_k(z)} \right\} > \alpha \quad (z \in \mathcal{U}),$$

where $0 \leq \alpha < 1$, $0 \leq \lambda \leq 1$, $k \geq 2$ is a fixed positive integer and $f_k(z)$ is defined by equality (1.2).

Definition 2. Let $\mathcal{QC}^{(k)}(\lambda, \alpha)$ denote the class of functions in \mathcal{A} satisfying the following inequality

$$\Re \left\{ z \frac{\lambda z^2 f_k'''(z) + (2\lambda + 1)z f_k''(z) + f_k'(z)}{\lambda z^2 f_k''(z) + z f_k'(z)} \right\} > \alpha \quad (z \in \mathcal{U}),$$

where $0 \leq \alpha < 1$, $0 \leq \lambda \leq 1$, $k \geq 2$ is a fixed positive integer and $f_k(z)$ is defined by equality (1.2).

For convenience, we write $\mathcal{C}^{(k)}(\lambda, \alpha) \cap \mathcal{T}$ simple as $\mathcal{C}_{\mathcal{T}}^{(k)}(\lambda, \alpha)$, and $\mathcal{QC}^{(k)}(\lambda, \alpha) \cap \mathcal{T}$ simple as $\mathcal{QC}_{\mathcal{T}}^{(k)}(\lambda, \alpha)$.

In our proposed investigation of functions in the classes $\mathcal{C}^{(k)}(\lambda, \alpha)$ and $\mathcal{QC}^{(k)}(\lambda, \alpha)$, we shall also make use of the following lemmas.

Lemma 1. Let $\gamma \geq 0$ and $f \in \mathcal{C}$, then

$$F(z) = \frac{1 + \gamma}{z^\gamma} \int_0^z f(t) t^{\gamma-1} dt \in \mathcal{C}.$$

This lemma is a special case of Theorem 4 in [11].

Lemma 2 [6]. Let $0 < \lambda \leq 1$ and $f \in \mathcal{QC}$, then

$$F(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z f(t) t^{\frac{1}{\lambda}-2} dt \in \mathcal{QC} \subset \mathcal{C}.$$

Lemma 3. $\mathcal{C}^{(k)}(\lambda, \alpha) \subset \mathcal{C} \subset \mathcal{S}$.

Proof. Let $F(z) = (1 - \lambda)f(z) + \lambda z f'(z)$, $F_k(z) = (1 - \lambda)f_k(z) + \lambda z f_k'(z)$ with $f(z) \in \mathcal{C}^{(k)}(\lambda, \alpha)$, substituting z by $\varepsilon^\mu z$ in (1.1) ($\mu = 0, 1, 2, \dots, k-1$), we get

$$(1.4) \quad \Re \left\{ \frac{\varepsilon^\mu z f'(\varepsilon^\mu z) + \lambda (\varepsilon^\mu z)^2 f''(\varepsilon^\mu z)}{(1 - \lambda) f_k(\varepsilon^\mu z) + \lambda \varepsilon^\mu z f_k'(\varepsilon^\mu z)} \right\} > \alpha \quad (z \in \mathcal{U}).$$

Note that $f_k(\varepsilon^\mu z) = \varepsilon^\mu f_k(z)$ and $f_k'(\varepsilon^\mu z) = f_k'(z)$, thus, inequality (1.4) can be written as

$$(1.5) \quad \Re \left\{ \frac{z f'(\varepsilon^\mu z) + \lambda z^2 \varepsilon^\mu f''(\varepsilon^\mu z)}{(1 - \lambda) f_k(z) + \lambda z f_k'(z)} \right\} > \alpha \quad (z \in \mathcal{U}).$$

Letting $\mu = 0, 1, 2, \dots, k-1$ in (1.5), respectively, and summing them we can obtain

$$\Re \left\{ \frac{1}{k} \sum_{\mu=0}^{k-1} \frac{z f'(\varepsilon^\mu z) + \lambda z^2 \varepsilon^\mu f''(\varepsilon^\mu z)}{(1-\lambda) f_k(z) + \lambda z f'_k(z)} \right\} > \alpha \quad (z \in \mathcal{U}),$$

or equivalently,

$$\Re \left\{ \frac{z f'_k(z) + \lambda z^2 f''_k(z)}{(1-\lambda) f_k(z) + \lambda z f'_k(z)} \right\} = \Re \left\{ \frac{z F'_k(z)}{F_k(z)} \right\} > \alpha \quad (z \in \mathcal{U}),$$

that is $F_k(z) \in \mathcal{S}^*(\alpha)$, which is the usual class of starlike functions of order α in \mathcal{U} . Note that $\mathcal{S}^*(0) = \mathcal{S}^*$, this implies that $F(z) = (1-\lambda)f(z) + \lambda z f'(z) \in \mathcal{C}$. We now split it into two cases to prove.

Case 1. When $\lambda = 0$. It is obvious that $f(z) = F(z) \in \mathcal{C}$.

Case 2. When $0 < \lambda \leq 1$. From $F(z) = (1-\lambda)f(z) + \lambda z f'(z)$ and $0 < \lambda \leq 1$, we have

$$f(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z F(t) t^{\frac{1}{\lambda}-2} dt.$$

Since $\gamma = \frac{1}{\lambda} - 1 \geq 0$, by Lemma 1, we obtain that $f(z) \in \mathcal{C}$. Hence $\mathcal{C}^{(k)}(\lambda, \alpha) \subset \mathcal{C} \subset \mathcal{S}$, and the proof of Lemma 3 is complete.

By means of Lemma 2, using the similar method as in Lemma 3, we may prove the following Lemma.

Lemma 4. $\mathcal{QC}^{(k)}(\lambda, \alpha) \subset \mathcal{QC} \subset \mathcal{C}$.

In the present paper, we shall provide the integral representations and convolution conditions for the classes $\mathcal{C}^{(k)}(\lambda, \alpha)$ and $\mathcal{QC}^{(k)}(\lambda, \alpha)$, we shall also provide some coefficient inequalities for functions belonging to these classes and their subclasses with negative coefficients.

2 Integral Representations

At first, we give the integral representations of functions belonging to the classes $\mathcal{C}^{(k)}(\lambda, \alpha)$ and $\mathcal{QC}^{(k)}(\lambda, \alpha)$.

Theorem 1. Let $f(z) \in \mathcal{C}^{(k)}(\lambda, \alpha)$ with $0 < \lambda \leq 1$, then we have

$$(2.1) \quad f_k(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z \exp \left\{ \frac{1}{k} \sum_{\mu=0}^{k-1} \int_0^{\varepsilon^\mu u} \frac{2(1-\alpha)\omega(t)}{t(1-\omega(t))} dt \right\} u^{\frac{1}{\lambda}-1} du,$$

where $f_k(z)$ is defined by equality (1.2), $\omega(z)$ is analytic in \mathcal{U} and $\omega(0) = 0$, $|\omega(z)| < 1$.

Proof. Suppose that $f(z) \in \mathcal{C}^{(k)}(\lambda, \alpha)$, it is easy to know that the condition (1.3) can be written as

$$\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f_k(z) + \lambda z f'_k(z)} \prec \frac{1 + (1-2\alpha)z}{1-z},$$

where " \prec " stands for the usual subordination, it follows that

$$(2.2) \quad \frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f_k(z) + \lambda z f'_k(z)} = \frac{1 + (1-2\alpha)\omega(z)}{1-\omega(z)},$$

where $\omega(z)$ is analytic in \mathcal{U} and $\omega(0) = 0$, $|\omega(z)| < 1$. By applying the similar method as in Lemma 3 to equality (2.2), we can obtain

$$(2.3) \quad \frac{(1-\lambda)zf'_k(z) + \lambda z(zf'_k(z))'}{(1-\lambda)f_k(z) + \lambda z f'_k(z)} = \frac{1}{k} \sum_{\mu=0}^{k-1} \frac{1 + (1-2\alpha)\omega(\varepsilon^\mu z)}{1-\omega(\varepsilon^\mu z)},$$

from equality (2.3), we get

$$(2.4) \quad \frac{(1-\lambda)f'_k(z) + \lambda(zf'_k(z))'}{(1-\lambda)f_k(z) + \lambda z f'_k(z)} - \frac{1}{z} = \frac{1}{k} \sum_{\mu=0}^{k-1} \frac{2(1-\alpha)\omega(\varepsilon^\mu z)}{z(1-\omega(\varepsilon^\mu z))}.$$

Integrating equality (2.4), we have

$$\begin{aligned} \log \left\{ \frac{(1-\lambda)f_k(z) + \lambda z f'_k(z)}{z} \right\} &= \frac{1}{k} \sum_{\mu=0}^{k-1} \int_0^z \frac{2(1-\alpha)\omega(\varepsilon^\mu \zeta)}{\zeta(1-\omega(\varepsilon^\mu \zeta))} d\zeta \\ &= \frac{1}{k} \sum_{\mu=0}^{k-1} \int_0^{\varepsilon^\mu z} \frac{2(1-\alpha)\omega(t)}{t(1-\omega(t))} dt, \end{aligned}$$

that is,

$$(2.5) \quad (1 - \lambda)f_k(z) + \lambda z f'_k(z) = z \cdot \exp \left\{ \frac{1}{k} \sum_{\mu=0}^{k-1} \int_0^{\varepsilon^\mu z} \frac{2(1 - \alpha)\omega(t)}{t(1 - \omega(t))} dt \right\}.$$

From equality (2.5), we can get equality (2.1) easily. Hence the proof of Theorem 1 is complete.

Theorem 2. Let $f(z) \in \mathcal{C}^{(k)}(\lambda, \alpha)$ with $0 < \lambda \leq 1$, then we have

$$(2.6) \quad f(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z \int_0^u \exp \left\{ \frac{1}{k} \sum_{\mu=0}^{k-1} \int_0^{\varepsilon^\mu \zeta} \frac{2(1 - \alpha)\omega(t)}{t(1 - \omega(t))} dt \right\} \cdot \frac{1 + (1 - 2\alpha)\omega(\zeta)}{1 - \omega(\zeta)} d\zeta u^{\frac{1}{\lambda}-2} du,$$

where $\omega(z)$ is analytic in \mathcal{U} and $\omega(0) = 0$, $|\omega(z)| < 1$.

Proof. Suppose that $f(z) \in \mathcal{C}^{(k)}(\lambda, \alpha)$, from equalities (2.2) and (2.5), we can get

$$\begin{aligned} (1 - \lambda)f'(z) + \lambda(zf'(z))' &= \frac{(1 - \lambda)f_k(z) + \lambda z f'_k(z)}{z} \cdot \frac{1 + (1 - 2\alpha)\omega(z)}{1 - \omega(z)} \\ &= \exp \left\{ \frac{1}{k} \sum_{\mu=0}^{k-1} \int_0^{\varepsilon^\mu z} \frac{2(1 - \alpha)\omega(t)}{t(1 - \omega(t))} dt \right\} \cdot \frac{1 + (1 - 2\alpha)\omega(z)}{1 - \omega(z)}. \end{aligned}$$

Integrating the above equality, we can get equality (2.6) easily.

Similarly, for the class $\mathcal{QC}^{(k)}(\lambda, \alpha)$, we have

Corollary 1. Let $f(z) \in \mathcal{QC}^{(k)}(\lambda, \alpha)$ with $0 < \lambda \leq 1$, then we have

$$f_k(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z \int_0^u \exp \left\{ \frac{1}{k} \sum_{\mu=0}^{k-1} \int_0^{\varepsilon^\mu \zeta} \frac{2(1 - \alpha)\omega(t)}{t(1 - \omega(t))} dt \right\} d\zeta u^{\frac{1}{\lambda}-2} du,$$

where $f_k(z)$ is defined by equality (1.2), $\omega(z)$ is analytic in \mathcal{U} and $\omega(0) = 0$, $|\omega(z)| < 1$.

Corollary 2. Let $f(z) \in \mathcal{QC}^{(k)}(\lambda, \alpha)$ with $0 < \lambda \leq 1$, then we have

$$f(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z \int_0^u \frac{1}{\xi} \int_0^\xi \exp \left\{ \frac{1}{k} \sum_{\mu=0}^{k-1} \int_0^{\varepsilon^\mu \zeta} \frac{2(1-\alpha)\omega(t)}{t(1-\omega(t))} dt \right\} \cdot \frac{1 + (1-2\alpha)\omega(\zeta)}{1-\omega(\zeta)} d\zeta d\xi u^{\frac{1}{\lambda}-2} du,$$

where $\omega(z)$ is analytic in \mathcal{U} and $\omega(0) = 0$, $|\omega(z)| < 1$.

3 Convolution Conditions

In this section, we give the convolution conditions for the classes $\mathcal{C}^{(k)}(\lambda, \alpha)$ and $\mathcal{QC}^{(k)}(\lambda, \alpha)$. Let $f, g \in \mathcal{A}$, where $f(z)$ is given by (1.1) and $g(z)$ is defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

then the Hadamard product (or convolution) $f * g$ is defined (as usual) by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z).$$

Theorem 3. A function $f(z) \in \mathcal{C}^{(k)}(\lambda, \alpha)$ if and only if

$$(3.1) \quad \frac{1}{z} \left\{ f * \left\{ (1-\lambda) \left\{ \frac{z}{(1-z)^2} (1-e^{i\theta}) - [1 + (1-2\alpha)e^{i\theta}] h \right\} + \lambda z \left\{ \frac{z}{(1-z)^2} (1-e^{i\theta}) - [1 + (1-2\alpha)e^{i\theta}] h \right\}' \right\} (z) \right\} \neq 0$$

for all $z \in \mathcal{U}$ and $0 \leq \theta < 2\pi$, where $h(z)$ is given by (3.6).

Proof. Suppose that $f(z) \in \mathcal{C}^{(k)}(\lambda, \alpha)$, since (1.3) is equivalent to

$$(3.2) \quad \frac{z f'(z) + \lambda z^2 f''(z)}{(1-\lambda) f_k(z) + \lambda z f'_k(z)} \neq \frac{1 + (1-2\alpha)e^{i\theta}}{1 - e^{i\theta}}$$

for all $z \in \mathcal{U}$ and $0 \leq \theta < 2\pi$. And the condition (3.2) can be written as

$$(3.3) \quad \frac{1}{z} \{ [(1-\lambda)zf'(z) + \lambda z(zf'(z))'] (1 - e^{i\theta}) - [(1-\lambda)f_k(z) + \lambda z f'_k(z)] [1 + (1-2\alpha)e^{i\theta}] \} \neq 0.$$

On the other hand, it is well known that

$$(3.4) \quad zf'(z) = f(z) * \frac{z}{(1-z)^2}.$$

And from the definition of $f_k(z)$, we know

$$(3.5) \quad f_k(z) = (f * h)(z),$$

where

$$(3.6) \quad h(z) = \frac{1}{k} \sum_{v=0}^{k-1} \frac{z}{1 - \varepsilon^v z}.$$

Substituting (3.4) and (3.5) into (3.3), we can get (3.1) easily. This completes the proof of Theorem 3.

Similarly, for the class $\mathcal{QC}^{(k)}(\lambda, \alpha)$, we have

Corollary 3. *A function $f(z) \in \mathcal{QC}^{(k)}(\lambda, \alpha)$ if and only if*

$$\frac{1}{z} \left\{ f * \left\{ z \left\{ (1-\lambda) \left\{ \frac{z}{(1-z)^2} (1 - e^{i\theta}) - [1 + (1-2\alpha)e^{i\theta}] h \right\} + \lambda z \left\{ \frac{z}{(1-z)^2} (1 - e^{i\theta}) - [1 + (1-2\alpha)e^{i\theta}] h \right\}' \right\}' \right\} (z) \right\} \neq 0$$

for all $z \in \mathcal{U}$ and $0 \leq \theta < 2\pi$, where $h(z)$ is given by (3.6).

4 Coefficient Inequalities

In this section, we first provide the sufficient conditions for functions belonging to the classes $\mathcal{C}^{(k)}(\lambda, \alpha)$ and $\mathcal{QC}^{(k)}(\lambda, \alpha)$.

Theorem 4. Let $0 \leq \alpha < 1$ and $0 \leq \lambda < 1$. If

$$(4.1) \quad \sum_{n=1}^{\infty} (1 + \lambda nk)(nk + 1 - \alpha)|a_{nk+1}| + \sum_{\substack{n=2 \\ n \neq lk+1}}^{\infty} [1 + \lambda(n-1)]n|a_n| \leq 1 - \alpha,$$

then $f(z) \in \mathcal{C}^{(k)}(\lambda, \alpha)$.

Proof. It suffices to show that

$$\left| \frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f_k(z) + \lambda z f'_k(z)} - 1 \right| < 1 - \alpha.$$

Note that for $|z| = r < 1$, we have

$$\begin{aligned} \left| \frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f_k(z) + \lambda z f'_k(z)} - 1 \right| &= \left| \frac{\sum_{n=2}^{\infty} [1 + \lambda(n-1)](n - b_n)a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} [\lambda n + (1-\lambda)]b_n a_n z^{n-1}} \right| \\ &\leq \frac{\sum_{n=2}^{\infty} [1 + \lambda(n-1)](n - b_n)|a_n|}{1 - \sum_{n=2}^{\infty} [\lambda n + (1-\lambda)]b_n |a_n|}. \end{aligned}$$

where

$$(4.2) \quad b_n = \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{(n-1)\nu} = \begin{cases} 1, & n = lk + 1, \\ 0, & n \neq lk + 1. \end{cases}$$

This last expression is bounded above by $1 - \alpha$ if

$$(4.3) \quad \sum_{n=2}^{\infty} [1 + \lambda(n-1)](n - \alpha b_n)|a_n| \leq 1 - \alpha.$$

Since inequality (4.3) can be written as inequality (4.1), hence $f(z)$ satisfies the condition (1.3). This completes the proof of Theorem 4.

Similarly, for the class $\mathcal{QC}^{(k)}(\lambda, \alpha)$, we have

Corollary 4. Let $0 \leq \alpha < 1$ and $0 \leq \lambda < 1$. If

$$\sum_{n=1}^{\infty} (nk+1)(1 + \lambda nk)(nk + 1 - \alpha)|a_{nk+1}| + \sum_{\substack{n=2 \\ n \neq lk+1}}^{\infty} [1 + \lambda(n-1)]n^2|a_n| \leq 1 - \alpha,$$

then $f(z) \in \mathcal{QC}^{(k)}(\lambda, \alpha)$.

We now provide the necessary and sufficient coefficient conditions for functions belonging to the classes $\mathcal{C}_{\mathcal{T}}^{(k)}(\lambda, \alpha)$ and $\mathcal{QC}_{\mathcal{T}}^{(k)}(\lambda, \alpha)$.

Theorem 5. *Let $0 \leq \alpha < 1$, $0 \leq \lambda < 1$ and $f(z) \in \mathcal{T}$, then $f(z) \in \mathcal{C}_{\mathcal{T}}^{(k)}(\lambda, \alpha)$ if and only if*

$$(4.4) \quad \sum_{n=1}^{\infty} (1 + \lambda nk)(nk + 1 - \alpha)a_{nk+1} + \sum_{\substack{n=2 \\ n \neq lk+1}}^{\infty} [1 + \lambda(n-1)]na_n \leq 1 - \alpha.$$

Proof. In view of Theorem 4, we need only to prove the necessity. Suppose that $f(z) \in \mathcal{C}_{\mathcal{T}}^{(k)}(\lambda, \alpha)$, then from (1.3), we can get

$$(4.5) \quad \Re \left\{ \frac{1 - \sum_{n=2}^{\infty} na_n z^{n-1} - \lambda \sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} [\lambda n + (1-\lambda)]b_n a_n z^{n-1}} \right\} > \alpha,$$

where b_n is given by (4.2). By letting $|z| = r \rightarrow 1^-$ through real values in (4.5), we can get

$$\frac{1 - \sum_{n=2}^{\infty} na_n - \lambda \sum_{n=2}^{\infty} n(n-1)a_n}{1 - \sum_{n=2}^{\infty} [\lambda n + (1-\lambda)]b_n a_n} \geq \alpha,$$

or equivalently,

$$(4.6) \quad \sum_{n=2}^{\infty} [1 + \lambda(n-1)](n - \alpha b_n)a_n \leq 1 - \alpha.$$

Substituting (4.2) into inequality (4.6), we can get inequality (4.4) easily. This completes the proof of Theorem 5.

Similarly, for the class $\mathcal{QC}_{\mathcal{T}}^{(k)}(\lambda, \alpha)$, we have

Corollary 5. *Let $0 \leq \alpha < 1$, $0 \leq \lambda < 1$ and $f(z) \in \mathcal{T}$, then $f(z) \in \mathcal{QC}_{\mathcal{T}}^{(k)}(\lambda, \alpha)$ if and only if*

$$\sum_{n=1}^{\infty} (nk+1)(1+\lambda nk)(nk+1-\alpha)|a_{nk+1}| + \sum_{\substack{n=2 \\ n \neq tk+1}}^{\infty} [1+\lambda(n-1)]n^2|a_n| \leq 1-\alpha.$$

Acknowledgements. This work was supported by the Scientific Research Fund of Hunan Provincial Education Department and the Hunan Provincial Natural Science Foundation (No. 05JJ30013) of People's Republic of China. The authors would like to thank Professor Ming-Sheng Liu for his helpful suggestions in the preparation of this paper.

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