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On some integral classes of integral operators[1]

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Abstract

Let A be the class of the functions f which are analytic in the unit disk $U = \{z \in C;\ |z| < 1\}$ and $f(0) = f'(0) - 1 = 0$. The object of the present paper is to derive univalence conditions of certain integral operators for $f(z) \in A$ and $f(z)$ has the form: $f(z) = z + \sum_{k=3}^{\infty} a_k z^k$.

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1 Introduction

Let A be the class of the functions $f(z)$ which are analytic in the unit disk $U = \{z \in C : |z| < 1\}$ and $f(0) = f'(0) - 1 = 0$.

We denote by S the class of the functions $f(z) \in A$ which are univalent in U.

In this paper we consider the integral operators

$$F_\alpha(z) = \int_0^z [f'(u)]^\alpha \ du \tag{1.1}$$

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$$H_{\beta,\gamma}(z) = \left\{ \beta \int_0^z u^{\beta-1} \left[f'(u) \right]^{\gamma} \ du \right\}^{\frac{1}{\beta}} \tag{1.2}$$

$$L_{\beta}(z) = \left[\beta \int_0^z u^{\beta-1} \left[f'(u) \right] \ du \right]^{\frac{1}{\beta}} \tag{1.3}$$

2 Preliminary Results

We need the following theorems.

Lemma 2.1. [1]. *If* $f(z) \in A$ *satisfies*

$$\left(1 - |z|^2\right) \left| \frac{z\ f''(z)}{f'(z)} \right| \leq 1, \ \ z \in U \tag{2.1}$$

then $f(z) \in S$.

Theorem 2.2. [3]. *Let* α *be a complex number,* $Re\,\alpha > 0$ *and* $f(z) \in A$. *If*

$$\frac{1 - |z|^{2Re\,\alpha}}{Re\,\alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1, \tag{2.2}$$

for all $z \in U$, *then for any complex number* β, $Re\,\beta \geq Re\,\alpha$ *the function*

$$F_{\beta}(z) = \left[\beta \int_0^z u^{\beta-1} \, f'(u) du \right]^{\frac{1}{\beta}} \tag{2.3}$$

is in the class S.

Theorem 2.3. [2]. *If the function* $g(z)$ *is regular in* U *and* $|g(z)| < 1$ *in* U, *then for all* $\xi \in U$ *and* $z \in U$ *the following inequalities hold:*

$$\left| \frac{g(\xi) - g(z)}{1 - \overline{g(z)} g(\xi)} \right| \leq \left| \frac{\xi - z}{1 - \overline{z}\xi} \right|, \tag{2.4}$$

$$|g'(z)| \leq \frac{1 - |g(z)|^2}{1 - |z|^2}, \tag{2.5}$$

the equalities hold only in the case $g(z) = \frac{\epsilon(z+u)}{1+\overline{u}z}$, *where* $|\epsilon| = 1$ *and* $|u| < 1$.

Remark 2.4. [2] *For* $z = 0$, *from inequality* (2.4). *We have*

$$\left| \frac{g(\xi) - g(0)}{1 - \overline{g(0)}g(\xi)} \right| \leq |\xi| \tag{2.6}$$

and, hence

$$|g(\xi)| \leq \frac{|\xi| + |g(0)|}{1 + |g(0)|\,|\xi|}. \tag{2.7}$$

Considering $g(0) = a$ *and* $\xi = z$,

$$|g(z)| \leq \frac{|z| + |a|}{1 + |a||z|} \tag{2.8}$$

for all $z \in U$.

3 Main Results

Theorem 3.1. *Let* α *be a complex number and* $f(z) \in A$,
$f(z) = z + \sum_{k=3}^{\infty} a_k z^k$. *If*

$$\left| \frac{f''(z)}{f'(z)} \right| < 1, \;\; z \in U \tag{3.1}$$

and

$$|\alpha| \leq 4 \tag{3.2}$$

then the function

$$F_\alpha(z) = \int_0^z [f'(u)]^\alpha \; du \tag{3.3}$$

is in the class S.

Proof. The function $F_\alpha(z)$ is regular in U. Let us consider the function

$$p(z) = \frac{1}{|\alpha|} \frac{F''_\alpha(z)}{F'_\alpha(z)} \tag{3.4}$$

where the constant $|\alpha|$ satisfies the inequality (3.2).

The function $p(z)$ is regular in U. From (3.4) and (3.3) we obtain

$$p(z) = \frac{\alpha}{|\alpha|}\frac{f''(z)}{f'(z)}. \tag{3.5}$$

Using (3.1) and (3.5) we get

$$|p(z)| \leq 1, \ \ z \in U \tag{3.6}$$

and we have $p(0) = 0$.

By Remark 2.4 we have

$$|p(z)| \leq |z|, \ \ z \in U. \tag{3.7}$$

From (3.4) and (3.7) we obtain

$$\frac{1}{|\alpha|}\left|\frac{F''_\alpha(z)}{F'_\alpha(z)}\right| \leq |z|, \ \ z \in U \tag{3.8}$$

and

$$\left(1-|z|^2\right)\left|\frac{zF''_\alpha(z)}{F'_\alpha(z)}\right| \leq |\alpha| \max_{|z|<1}\left(1-|z|^2\right)|z|^2. \tag{3.9}$$

Because $\max\limits_{|z|<1}\left(1-|z|^2\right)|z|^2 = \frac{1}{4}$, from (3.9) and (3.2) we get

$$\left(1-|z|^2\right)\left|\frac{zF''_\alpha(z)}{F'_\alpha(z)}\right| \leq 1, \ \ z \in U. \tag{3.10}$$

By Lemma 2.1 it results that the function $F_\alpha(z) \in S$.

Theorem 3.2. *Let γ be a complex number and the function $f(z) \in A$, $f(z) = z + \sum_{k=3}^{\infty} a_k z^k$. If*

$$\left|\frac{f''(z)}{f'(z)}\right| < 1, \ \ z \in U \tag{3.11}$$

and

$$|\gamma| \le 4 \tag{3.12}$$

then for any complex number β, $Re\,\beta \ge 1$ the function

$$H_{\beta,\gamma}(z) = \left\{ \beta \int_0^z u^{\beta-1} \left[f'(u)\right]^{\gamma} \, du \right\}^{\frac{1}{\beta}} \tag{3.13}$$

is in the class S.

Proof. Let us consider the function

$$g(z) = \int_0^z \left[f'(u)\right]^{\gamma} \, du. \tag{3.14}$$

The function

$$p(z) = \frac{1}{|\gamma|} \frac{g''(z)}{g'(z)}, \tag{3.15}$$

where the constant $|\gamma|$ satisfies the inequality (3.12), is regular in U.

From (3.15) and (3.14) we obtain

$$p(z) = \frac{\gamma}{|\gamma|} \frac{f''(z)}{f'(z)}. \tag{3.16}$$

and using (3.11) we have

$$|p(z)| \le 1, \;\; z \in U$$

Remark 2.4 applied to the function $p(z)$ give

$$\frac{1}{|\gamma|} \left| \frac{g''(z)}{g'(z)} \right| \le |z|, \;\; z \in U \tag{3.17}$$

and, hence

$$\left(1 - |z|^2\right) \left| \frac{z g''(z)}{g'(z)} \right| \le |\gamma| \max_{|z|<1} \left(1 - |z|^2\right) |z|^2. \tag{3.18}$$

From (3.18) and (3.12) we obtain

$$(1-|z|^2)\left|\frac{zg''(z)}{g'(z)}\right| \leq 1, \quad z \in U. \tag{3.19}$$

By Theorem 2.2 for $Re\,\alpha = 1$, it results that $H_{\beta,\gamma}(z) \in S$.

Theorem 3.3. *Let β a complex number, $Re\,\beta \geq 1$ and $f(z) \in A$, $f(z) = z + a_3z^3 + \ldots$, $\frac{f(z)}{z} \neq 0$, $z \in U$. If*

$$\left|\frac{f''(z)}{f'(z)}\right| \leq 4, \quad z \in U \tag{3.20}$$

then the function

$$L_\beta(z) = \left[\beta \int_0^z u^{\beta-1}\left[f'(u)\right]\, du\right]^{\frac{1}{\beta}} \tag{3.21}$$

is in the class S.

Proof. Let us consider the function

$$g(z) = \frac{1}{4}\frac{f''(z)}{f'(z)}$$

which is regular in U. Remark 2.4 applied to the function $g(z)$ give

$$\frac{1}{4}\left|\frac{f''(z)}{f'(z)}\right| \leq |z|, \quad z \in U \tag{3.22}$$

and, hence, we obtain

$$(1-|z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \leq 4\max_{|z|<1}(1-|z|^2)\,|z|^2, \quad z \in U \tag{3.23}$$

Since $\max_{|z|<1}(1-|z|^2)\,|z|^2 = \dfrac{1}{4}$, from (3.23) we have

$$(1-|z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \quad z \in U. \tag{3.24}$$

From (3.24) and Theorem 2.2 for $Re\,\alpha = 1$, we obtain $F_\beta(z) \in S$.

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