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Multivalued starlike functions of complex order[1]

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Abstract

Let $\mathcal{S}^*_\lambda(1-b)$ $(b \neq 0, \text{complex})$ denote the class of functions $f(z) = z+a_2z^2+\cdots$ analytic in the open unit disc $\mathbb{D} = \{z \in \mathbb{C}||z| < 1\}$ which satisfy for $z = re^{i\theta} \in \mathbb{D}$, $\frac{f(z)}{z} \neq 0$ and $\mathrm{Re}\left[1+\frac{1}{b}\left(z\frac{(\mathrm{D}^\lambda f(z))'}{\mathrm{D}^\lambda f(z)}-1\right)\right] > 0$ $(0 \leq \lambda < 1)$, where $\mathrm{D}^\lambda f(z) = \Gamma(2-\lambda)z^\lambda \mathrm{D}^\lambda_z f(z)$ and $\mathrm{D}^\lambda_z f(z)$ is the fractional derivative of $f(z)$.

The aim of this paper is to investigate certain properties of the mentioned above class.

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1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z) = z + a_2z^2 + \cdots$ which are analytic in $\mathbb{D}$. Let Ω be the family of functions $w(z)$ regular in $\mathbb{D}$, and satisfying the conditions $w(0) = 0$, $|w(z)| < 1$ for all $z \in \mathbb{D}$. We use $\mathcal{P}$ to denote the class of functions $p(z) = 1 + p_1z + p_2z^2 + \cdots$ which are analytic in $\mathbb{D}$ and have a

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positive real part in $\mathbb{D}$. Let $h(z) = z + b_2z^2 + \cdots$ and $s(z) = z + c_2z^2 + \cdots$ be analytic functions in $\mathbb{D}$. We say that $h(z)$ is subordinate to $s(z)$, written $h \prec s$, if

$$h(z) = s(w(z)), \text{ for some } w(z) \in \Omega, \text{ and for all } z \in \mathbb{D}. \tag{1.1}$$

In particular, if $s(z)$ is univalent in $\mathbb{D}$, then $h \prec s$ if and only if $h(0) = s(0)$, and $h(\mathbb{D}) \subset s(\mathbb{D})$ (see [2]).

The fractional integral of order λ is defined for a function $f(z)$, by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta \ (\lambda > 0), \tag{1.2}$$

where the function $f(z)$ is analytic in a simply connected region of the complex plane containing the origin and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$ ([7], [8], [10]).

The fractional derivative of order λ is defined for a function $f(z)$, by

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta \ (0 \le \lambda < 1), \tag{1.3}$$

where the function $f(z)$ is analytic in a simply connected region of the complex plane containing the origin and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed as in definition of the fractional integral ([7], [8], [10]).

Under the hypotheses of the fractional derivative of order λ, the fractional derivative of $(n+\lambda)$ is defined, for a function $f(z)$, by

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z) \ (0 \le \lambda < 1, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \tag{1.4}$$

By means of the definitions above, we see that

$$D_z^{-\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k+1+\lambda)} z^{k+\lambda} \ (\lambda > 0); D_z^{\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\lambda)} z^{k-\lambda} \ (0 \le \lambda < 1)$$

and

$$D_z^{n+\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k+1-n-\lambda)} z^{k-n-\lambda} \ (0 \le \lambda < 1, n \in \mathbb{N}_0).$$

Therefore, we conclude that, for any real λ

$$D_z^{\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\lambda)} z^{k-\lambda}.$$

In this paper we will study some of the properties of the class $\mathcal{S}_{\lambda}^{*}(1-b)$ $(b \neq 0, \text{complex})$ defined as follows.

Definition 1.1. *A function $f \in \mathcal{A}$ is said to be multivalued starlike of complex order $(1-b)$ $(b \neq 0$, complex), that is $f \in \mathcal{S}_{\lambda}^{*}(1-b)$ if and only if*

$$1 + \frac{1}{b}\left(z\frac{(D^{\lambda} f(z))'}{D^{\lambda} f(z)} - 1\right) = p(z) \qquad (z \in \mathbb{D}) \tag{1.5}$$

for some $p(z) \in \mathcal{P}$ and $0 \leq \lambda < 1$.

Using (1.3), Owa and Srivastava [9] introduced the operator

$$D^{\lambda}\colon \mathcal{A} \longrightarrow \mathcal{A} \text{ defined by } D^{\lambda} f(z) = \gamma(2-\lambda) z^{\lambda} D_z^{\lambda} f(z) \ (0 \leq \lambda < 1).$$

We note that $\mathcal{S}_0^{*}(1-b) = \mathcal{S}(1-b)$, where $\mathcal{S}(1-b)$ is the class of starlike functions of complex order introduced and studied by Nasr and Aouf [5].

The following lemma due to Jack [3] (also [4]) plays an important role in our proofs.

Lemma 1.2. *Let $w(z)$ be a non-constant and analytic function in the open unit disc $\mathbb{D}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r$ at the point z_1, then $z_1 w'(z_1) = k w(z_1)$ for some real $k \geq 1$.*

2 Main Results

Lemma 2.1. *Let $f(z)$ be an element of $\mathcal{A}$. Then the fractional $\lambda-$operator*

$$\mathrm{D}^{\lambda} f(z) = \Gamma(2-\lambda) z^{\lambda} \mathrm{D}_z^{\lambda} f(z) \tag{2.1}$$

satisfies the following equalities

(i) $\mathrm{D}^{\lambda} f(z) = \Gamma(2-\lambda) z^{\lambda} \mathrm{D}_z^{\lambda} f(z) = z + \sum_{n=2}^{\infty} a_n \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} z^n$

(ii) *For* $\lambda = 1$, $\mathrm{D}f(z) = \lim_{\lambda\to 1} \mathrm{D}^\lambda f(z) = zf'(z)$

(iii) *For* $\lambda < 1$, $\alpha < 1$,

$$\mathrm{D}^\alpha(D^\lambda f(z)) = \mathrm{D}^\lambda(\mathrm{D}^\alpha f(z)) = z + \sum_{n=2}^{\infty} a_n \frac{\Gamma(2-\alpha)\Gamma(2-\lambda)(\Gamma(n+1))^2}{\Gamma(n+1-\lambda)\Gamma(n+1-\alpha)} z^n$$

(iv) $$\mathrm{D}(\mathrm{D}^\lambda f(z)) = z + \sum_{n=2}^{\infty} n a_n \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} z^n = z(\mathrm{D}^\lambda f(z))'$$
$$= \Gamma(2-\lambda) z^\lambda (\lambda \mathrm{D}_z^\lambda f(z) + z\mathrm{D}_z^{\lambda+1} f(z)).$$

Proof. The proof of this lemma follows from the definition of $\mathrm{D}^\lambda f(z)$.

Lemma 2.2. *Let* $f(z) = z + a_2 z^2 + \cdots$ *and* $g(z) = z + b_2 z^2 + \cdots$ *be analytic functions in the open unit disc* $\mathbb{D}$. *Then the solution of the fractional differential equation*

(2.2) $$\mathrm{D}_z^\lambda f(z) = \frac{1}{\Gamma(2-\lambda)} z^{-\lambda} g(z) \text{ is },$$

(2.3) $$f(z) = z + \sum_{n=2}^{\infty} b_n \frac{\Gamma(n+1-\lambda)}{\Gamma(2-\lambda)\Gamma(n+1)} z^n.$$

Proof. Using the definitions of fractional integral, fractional derivative and fractional derivative of order $(n+\lambda)$, we get

(2.4)
$$\mathrm{D}_z^\lambda f(z) = \frac{\Gamma(2)}{\Gamma(2-\lambda)} z^{1-\lambda} + a_2 \frac{\Gamma(3)}{\Gamma(3-\lambda)} z^{2-\lambda} + \cdots + a_n \frac{\Gamma(n+1)}{\Gamma(n+1-\lambda)} z^{n-\lambda} + \cdots$$

(2.5) $$\frac{1}{\Gamma(2-\lambda)} z^{-\lambda} g(z) = \frac{1}{\Gamma(2-\lambda)} z^{-\lambda}(z + b_2 z^2 + \cdots) = \frac{1}{\Gamma(2-\lambda)} z^{1-\lambda} + b_2 \frac{1}{\Gamma(2-\lambda)} z^{2-\lambda} + \cdots + b_n \frac{1}{\Gamma(2-\lambda)} z^{n-\lambda} + \cdots.$$

Therefore we have

$$D_z^\lambda f(z) = \frac{1}{\Gamma(2-\lambda)} z^{-\lambda} g(z) \text{ which implies}$$

$$\frac{\Gamma(2)}{\Gamma(2-\lambda)}z^{1-\lambda}+a_2\frac{\Gamma(3)}{\Gamma(3-\lambda)}z^{2-\lambda}+\cdots+a_n\frac{\Gamma(n+1)}{\Gamma(n+1-\lambda)}z^{n-\lambda}+\cdots \tag{2.6}$$

$$=\frac{1}{\Gamma(2-\lambda)}z^{1-\lambda}+b_2\frac{1}{\Gamma(2-\lambda)}z^{2-\lambda}+\cdots+b_n\frac{1}{\Gamma(2-\lambda)}z^{n-\lambda}+\cdots.$$

Comparing the coefficient of $z^{n-\lambda}$ in both sides of (2.5) we obtain

$$a_n=\frac{\Gamma(n+1-\lambda)}{\Gamma(2-\lambda)\Gamma(n+1)}b_n \tag{2.7}$$

Theorem 2.3. *Let $f(z)\in\mathcal{A}$ and satisfies the condition*

$$\left(z\frac{(\mathrm{D}^\lambda f(z))'}{\mathrm{D}^\lambda f(z)}-1\right)\prec\frac{2bz}{1-z}=F(z), \tag{2.8}$$

then $f(z)\in S^*_\lambda(1-b)$. This result is sharp since the function $f(z)$ satisfies the fractional differential equation $\mathrm{D}^\lambda_z f(z)=\dfrac{1}{\Gamma(2-\lambda)}z^{1-\lambda}(1-z)^{-2b}$.

Proof. We define the function $w(z)$ by

$$\frac{\mathrm{D}^\lambda f(z)}{z}=(1-w(z))^{-2b},$$

where the value of $(1-w(z))^{-2b}$ is 1 at $z=0$ (i.e, we consider the corresponding Riemann branch), then $w(z)$ is analytic in $\mathbb{D}$, $w(0)=0$, and

$$\left(z\frac{(\mathrm{D}^\lambda f(z))'}{\mathrm{D}^\lambda f(z)}-1\right)=\frac{2bzw'(z)}{1-w(z)}. \tag{2.9}$$

Now, it is easy to realize that the subordination (2.9) is equivalent to $|w(z)|<1$ for all $z\in\mathbb{D}$. Indeed, assume the contrary: then, there exists $z_1\in\mathbb{D}$ such that $|w(z)|$ attains its maximum value on the circle $|z|=r$ at the point z_1, that is $|w(z_1)|=1$. Then, by I.S. Jack's lemma, $z_1w'(z_1)=kw(z_1)$ for some real $k\geq 1$. For such z_1 we have

$$\left(z_1\frac{(\mathrm{D}^\lambda f(z_1))'}{\mathrm{D}^\lambda f(z_1)}-1\right)=\frac{2kbw(z_1)}{1-w(z_1)}=F(w(z_1))\notin F(\mathbb{D}),$$

because $|w(z_1)| = 1$ and $k \geq 1$. But this contradicts (2.9), so the assumption is wrong, i.e, $|w(z)| < 1$ for all $z \in \mathbb{D}$.

The sharpness of this result follows from the fact that

$$\mathrm{D}_z^{\lambda} f(z) = \frac{1}{\Gamma(2-\lambda)} z^{1-\lambda}(1-z)^{-2b} = \frac{1}{\Gamma(2-\lambda)} z^{-\lambda} z(1-z)^{-2b} \Rightarrow$$

$$\mathrm{D}^{\lambda} f(z) = \Gamma(2-\lambda) z^{\lambda} \mathrm{D}_z^{\lambda} f(z) = z(1-z)^{-2b} \Rightarrow \left(z\frac{(\mathrm{D}^{\lambda} f(z))'}{\mathrm{D}^{\lambda} f(z)} - 1 \right) = \frac{2bz}{1-z}.$$

On the other hand we have

$$\left(z\frac{(\mathrm{D}^{\lambda} f(z))'}{\mathrm{D}^{\lambda} f(z)} - 1 \right) \prec \frac{2bz}{1-z} \Rightarrow 1 + \frac{1}{b}\left(\frac{(\mathrm{D}^{\lambda} f(z))'}{\mathrm{D}^{\lambda} f(z)} - 1 \right) = \frac{1+w(z)}{1-w(z)}.$$

Corollary 2.4. *If $f(z) \in \mathcal{S}_{\lambda}^*(1-b)$, then*

$$\left| \left(\frac{z^{1-\lambda}}{\Gamma(2-\lambda)\mathrm{D}_z^{\lambda} f(z)} \right)^{1/2b} - 1 \right| < 1.$$

This corollary is a simple consequence of Theorem 2.3.

Remark 1. Putting $\lambda = 0$, we obtain the result obtained in [6].

Theorem 2.5. *If $f(z) \in \mathcal{S}_{\lambda}^*(1-b)$, then*

$$\frac{r^{1-\lambda}(1-r)^{|b|-\mathrm{Re}b}}{\Gamma(2-\lambda)(1+r)^{|b|+\mathrm{Re}b}} \leq \left| \mathrm{D}_z^{\lambda} f(z) \right| \leq \frac{r^{1-\lambda}(1+r)^{|b|-\mathrm{Re}b}}{\Gamma(2-\lambda)(1-r)^{|b|+\mathrm{Re}b}}. \tag{2.11}$$

This result is sharp since the function satisfies the fractional differential equation $\mathrm{D}_z^{\lambda} f(z) = \dfrac{1}{\Gamma(2-\lambda)} z^{1-\lambda}(1-z)^{-2b}$.

Proof. If $p(z) \in \mathcal{P}$, then we have ([2])

$$\left| p(z) - \frac{1+r^2}{1-r^2} \right| \leq \frac{2r}{1-r^2}. \tag{2.12}$$

Using the definition of the class $\mathcal{S}_{\lambda}^*(1-b)$, then we can write

$$\left| \left[1 + \frac{1}{b}\left(z\frac{(\mathrm{D}^{\lambda} f(z))'}{\mathrm{D}^{\lambda} f(z)} - 1 \right) \right] - \frac{1+r^2}{1-r^2} \right| \leq \frac{2r}{1-r^2}. \tag{2.13}$$

It follows from (2.16) that

$$\left|\left(z\frac{(\mathrm{D}^\lambda f(z))'}{\mathrm{D}^\lambda f(z)}-1\right)-\frac{2br^2}{1-r^2}\right|\le\frac{2|b|r}{1-r^2}$$

which implies that

$$\frac{2r^2\mathrm{Re}b-2r|b|}{1-r^2}\le\mathrm{Re}\left(z\frac{(\mathrm{D}^\lambda f(z))'}{\mathrm{D}^\lambda f(z)}-1\right)\le\frac{2r^2\mathrm{Re}b+2r|b|}{1-r^2}.$$

Thus we obtain that

(2.16)
$$\frac{1-2|b|r-(1-2\mathrm{Re}b)r^2}{1-r^2}\le\mathrm{Re}\left(z\frac{(\mathrm{D}^\lambda f(z))'}{\mathrm{D}^\lambda f(z)}\right)\le\frac{1+2|b|r-(1-2\mathrm{Re}b)r^2}{1-r^2}.$$

On the other hand, since

$$\mathrm{Re}\left(z\frac{(\mathrm{D}^\lambda f(z))'}{\mathrm{D}^\lambda f(z)}\right)=r\frac{\partial}{\partial r}\log|\mathrm{D}^\lambda f(z)|,$$

then, the inequality (2.16) can be written in the form

(2.17)
$$\frac{1-2|b|r-(1-2\mathrm{Re}b)r^2}{r(1-r)(1+r)}\le\frac{\partial}{\partial r}\log|\mathrm{D}^\lambda f(z)|\le\frac{1+2|b|r-(1-2\mathrm{Re}b)r^2}{r(1-r)(1+r)}.$$

Integrating both sides of the inequality (2.17) from 0 to r we obtain (2.11).
Remark 2. Putting $\lambda=0$, we obtain the result obtained in [1].

Theorem 2.6. *If $f(z)\in\mathcal{S}^*_\lambda(1-b)$, then*

$$|a_n|\le\frac{\Gamma(n+1-\lambda)}{(n-1)!\Gamma(2-\lambda)\Gamma(n+1)}\prod_{k=0}^{n-2}(k+2|b|)\ n=2,3,\cdots. \tag{2.18}$$

This inequality is sharp because the extremal function satisfies the fractional differential equation $\mathrm{D}^\lambda_z f(z)=\dfrac{1}{\Gamma(2-\lambda)}z^{1-\lambda}(1-z)^{-2b}$.

Proof. Using the definition of the class $\mathcal{S}^*_\lambda(1-b)$, we can write

$$z(\mathrm{D}^\lambda f(z))'=(\mathrm{D}^\lambda f(z))(1+bp_1z+bp_2z^2+\cdots+bp_nz^n+\cdots). \tag{2.19}$$

Comparison of the coefficient of z^n in both sides (2.19) gives

$$a_n = \frac{1}{(n-1)}\frac{\Gamma(n+1-\lambda)}{\Gamma(n+1)}\sum_{k=1}^{n-1}\frac{\Gamma(k+1)}{\Gamma(k+1-\lambda)}a_k b p_{n-k},\ a_1 \equiv 1. \tag{2.20}$$

Therefore we have

$$|a_n| \le \frac{2|b|}{(n-1)}\frac{\Gamma(n+1-\lambda)}{\Gamma(n+1)}\sum_{k=1}^{n-1}\frac{\Gamma(k+1)}{\Gamma(k+1-\lambda)}|a_k|,\ |a_1| \equiv 1 \tag{2.21}$$

because $|p_n| \le 2$ for all $n \ge 1$ whenever $p(z) \in \mathcal{P}$ ([2]).

It follows that

$$|a_2| \le \frac{\gamma(3-\lambda)\gamma(2)}{\gamma(3)\gamma(2-\lambda)}2|b| = (2-\lambda)|b|,$$

$$\begin{aligned}|a_3| &\le \frac{2|b|\gamma(4-\lambda)}{2\gamma(4)}\left(\frac{\gamma(2)}{\gamma(2-\lambda)}|a_1| + \frac{\gamma(3)}{\gamma(3-\lambda)}|a_2|\right)\\ &\le \frac{\gamma(4-\lambda)}{2!\gamma(4)\gamma(2-\lambda)}2|b|\left(1+2|b|\right),\end{aligned}$$

and

$$\begin{aligned}|a_4| &\le \frac{2|b|\gamma(5-\lambda)}{3\gamma(5)}\left(\frac{\gamma(2)}{\gamma(2-\lambda)}|a_1| + \frac{\gamma(3)}{\gamma(3-\lambda)}|a_2| + \frac{\gamma(4)}{\gamma(4-\lambda)}|a_3|\right)\\ &\le \frac{\gamma(5-\lambda)}{3!\gamma(5)\gamma(2-\lambda)}2|b|\left(1+2|b|\right)\left(2+2|b|\right).\end{aligned}$$

Therefore, applying the mathematical induction, we complete the proof of the theorem.

Now we consider the fractional differential equation

$$\mathrm{D}_z^\lambda f(z) = \frac{1}{\Gamma(2-\lambda)}z^{1-\lambda}(1-z)^{-2b}.$$

Using by Lemma 2.2 we can write

$$\begin{aligned}\mathrm{D}_z^\lambda f(z) &= \frac{\Gamma(2)}{\Gamma(2-\lambda)}z^{1-\lambda} + a_2\frac{\Gamma(3)}{\Gamma(3-\lambda)}z^{2-\lambda} + \cdots + a_n\frac{\Gamma(n+1)}{\Gamma(n+1-\lambda)}z^{n-\lambda}\\ &= \frac{1}{\Gamma(2-\lambda)}z^{-\lambda}\left[z + \sum_{n=2}^{\infty}\left(\prod_{k=0}^{n-2}\frac{2b+k}{k+1}\right)z^n\right],\ \text{which implies}\end{aligned} \tag{2.22}$$

$$f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1-\lambda)}{\Gamma(2-\lambda)\Gamma(n+1)} \left(\prod_{k=0}^{n-2} \frac{2b+k}{k+1} \right) z^n.$$

Remark 3. Putting $\lambda = 0$, we obtain the result obtained by Nasr and Aouf [5].

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