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On an expansion theorem in the finite operator calculus of G-C Rota

Emil C. Popa

Abstract

Using a identity for linear operators we present here the Taylor formula in the umbral calculus.

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1 Introduction

We consider the algebra of all polynomials $p(t)$ in one variable over a field of characteristic zero, to be denoted Π.

We denote by Π^* the linear space of linear operators on Π to Π. For example $D \in \Pi^*$, $Dp(t) = p'(t)$ (the derivative), $E^a \in \Pi^*$, $(E^a p)(t) = p(t+a)$ (the shift operator), $\mathcal{I}p(t) = p(t)$ (the identity).

We denote by Π_t^* the set of shift invariant operators

$$\Pi_t^* = \{T \mid TE^a = E^a T, (\forall)a\}$$

and by Π_δ^* the set of delta operators

$$\Pi_\delta^* = \{Q \in \Pi_t^* \mid Qx \, \text{is a nonzero constant}\}.$$

Delta operators possess many of the properties of the derivative operator D. For example if Q is a delta operator, then $Qa = 0$ for every constant a. Next, if $p(t)$ is a polynomial of degree n and $Q \in \Pi_\delta^*$, then $Qp(t)$ is a polynomial of degree $n-1$.

A polynomial sequence $(p_n(t))$ $(\deg p_n = n,\ n = 0,1,2,\dots)$ is called the sequence of basic polynomials for $Q \in \Pi_\delta^*$ if $p_0(t) = 1$, $p_n(0) = 0$ for $n \geq 1$ and $Qp_n(t) = np_{n-1}(t)$ for $n \geq 1$.

It is known the following theorem

Theorem 1. *i) Every delta operator has a unique sequence of basic polynomials.*

ii) If $(p_n(t))$ is a basic sequence for some delta operator Q then it is a sequence of polynomials of binomial type.

iii) If $(p_n(t))$ is a sequence of polynomials of binomial type, then it is a basic sequence for some delta operator.

The following theorem generalizes the Taylor expansion theorem to delta operators and their basic polynomials.

Theorem 2. *For $T \in \Pi_t^*$ and $Q \in \Pi_\delta^*$ with basic set (p_n), we have*

$$T = \sum_{k \geq 0} \frac{(Tp_k)(0)}{k!} Q^k. \tag{1}$$

We consider now a operator $X \notin \Pi_t^*$, defined by $Xp(t) = tp(t)$ and for any operator T defined on Π, the operator

$$T' = TX - XT$$

will be called the Pincherle derivative of the operator T.

We observe that $D' = \mathcal{I}$, $(E^a)' = aE^a$, $\mathcal{I}' = O$ (the null operator).

2 The Bernoulli identity

Let T, S be two linear operators such that

$$TS - ST = \mathcal{I}. \tag{2}$$

For example $DX - XD = \mathcal{I}$. From (2) we obtain

$$TS^2 = S^2T + 2S$$

and by induction

$$TS^n = S^nT + nS^{n-1},\ n \geq 1. \tag{3}$$

Starting with the identity

$$\sum_{k=0}^{n}(\alpha_k - \alpha_{k+1}) = \alpha_0 - \alpha_{n+1} \tag{4}$$

for

$$\alpha_0 = 0,\ \alpha_k = \frac{(-1)^k}{(k-1)!}S^{k-1}T^k,\ k \geq 1 \tag{5}$$

and using (3) we get

$$\alpha_k - \alpha_{k+1} = \frac{(-1)^k}{k!}TS^kT^k$$

and hence

$$T\sum_{k=0}^{n}\frac{(-1)^k}{k!}S^kT^k = \frac{(-1)^n}{n!}S^nT^{n+1}. \tag{6}$$

This is the Bernoulli identity obtained by O.V. Viskov (see [1], [3]).

3 The main result

Let Q be a delta operator with the basic set $(p_n(t))$. Hence $p_0(x) = 1$, $p_n(0) = 0$ for $n \geq 1$ and $Qp_n = np_{n-1}$ for $n \geq 1$.

Definition 1. *We define the Q-integral operator as a linear operator $\mathcal{I}_Q = \oint dt$ by*

$$\left(\mathcal{I}_Q p_n\right)(t) = \oint p_n(t)dt = \frac{1}{n+1}p_{n+1}(t),$$

for $n \geq 0$. We denote

$$\oint_\alpha^x (Qp)(t)dt = p(x) - p(\alpha). \tag{7}$$

Definition 2. *We define next the pseudo Q-integral operator*

$$T_Q \in \Pi^*, (T_Q p_n)(t) = p_{n+1}(t).$$

Remark 1.*For $Q = D$ we have $p_n(t) = t^n$, $n = 0, 1, 2, \ldots$ and $T_Q = T_D = X$, $(Xp)(t) = tp(t)$.*

Theorem 3. *We have the following Taylor expansion formula*

$$\sum_{k=0}^{n} \frac{\left((x\mathcal{I} - T_Q)^k Q^k f\right)(x)}{k!} = \sum_{k=0}^{n} \frac{\left((x\mathcal{I} - T_Q)^k Q^k f\right)(\alpha)}{k!} + \oint_\alpha^x \frac{\left((x\mathcal{I} - T_Q)^n Q^{n+1} f\right)(t)}{n!}dt \tag{8}$$

with the rest term in the Cauchy form.

Proof. Let T, S be as below

$$T = Q,\ S = T_Q - x\mathcal{I}.$$

We have $(TS-ST)p_n(t)=p_n(t)$ and hence $TS-ST=\mathcal{I}$. After submition into (6) we get

$$Q\sum_{k=0}^{n}\frac{(-1)^k}{k!}(T_Q-x\mathcal{I})^kQ^kp(t)=\frac{(-1)^n}{n!}(T_Q-x\mathcal{I})^nQ^{n+1}p(t),\ p(t)\in\Pi.$$

Apply $\oint_{\alpha}^{x}dt$ to both sides where, of course, t is the variable, and using (7) to obtain

$$\sum_{k=0}^{n}\frac{\left((x\mathcal{I}-T_Q)^kQ^kp\right)(x)}{k!}=\sum_{k=0}^{n}\frac{\left((x\mathcal{I}-T_Q)^kQ^kp\right)(\alpha)}{k!}+\mathcal{R}_{n+1}(x) \tag{9}$$

where the rest term $\mathcal{R}_{n+1}$ is in the Cauchy form

$$\mathcal{R}_{n+1}(x)=\oint_{\alpha}^{x}\frac{((x\mathcal{I}-T_Q)^nQ^{n+1}p)(t)}{n!}dt. \tag{10}$$

Remark 2. *For $Q=D$ we observe that $T_D=X$ and hence*

$$(T_Dp)(t)=tp(t).$$

Next $((x\mathcal{I}-T_D\ )\,p)\,(x)=(xp(t)-tp(t))\big|_{t=x}=0$ and of (9) we obtain

$$p(x)=\sum_{k=0}^{n}\frac{(x-\alpha)^k}{k!}p^{(k)}(\alpha)+\mathcal{R}_{n+1}(x) \tag{11}$$

with $\mathcal{R}_{n+1}(x)=\int_{\alpha}^{x}\frac{(x-t)^n}{n!}p^{(n+1)}(t)dt$.

Remark 3. *We have $DX-XD=\mathcal{I}$ and for $T=D$ and $S=X$ in the Bernoulli identity we obtain*

$$D\sum_{k=0}^{n}\frac{(-1)^k}{k!}X^kD^k=\frac{(-1)^n}{n!}X^nD^{n+1}$$

and finally a McLaurin expansion formula in the following form

$$p(0)=\sum_{k=0}^{n}\frac{(-\alpha)^k}{k!}p^{(k)}(\alpha)+\int_{0}^{\alpha}\frac{(-t)^{n+1}}{n!}p^{(n+1)}(t)dt. \tag{12}$$

References

[1] A. K. Kwasniewski, *More on the Bernouli-Taylor formula for extended umbral clculus*, Advances in Appl. Clifford Algebras, 16 (2006), 29-39.

[2] G-C. Rota, D. Kahaner, A. Odlyzko, Finite operator calculus, *J. Math. Analysis and Appl.*, Vol. 42 (3), June, 1973.

[3] O.V. Viskov, *Noncommutative approach to classical problems of analysis*, Trudy Matematiceskovo Instituta AN SSSR, 177 (1986)(in russian).

Emil C. Popa
"Lucian Blaga" University
Department of Mathematics
Sibiu, Romania
e-mail: ecpopa2002@yahoo.com