General Mathematics Vol. 17, No. 3 (2009), 81–90

Convergence of an iterative algorithm to a fixed point of uniformly L-Lipschitzian mapping [1]

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Abstract

Let K be a non-empty closed convex subset of an arbitrary Banach space E. Let $T : K \to K$ be uniformly L Lipschitzian with $F(T) \neq \emptyset$. Let $\{k_n\} \subseteq [1, \infty)$ be a sequence with $\lim_{n\to\infty} k_n = 1$.For any $x_0 \in K$ and fixed $u \in K$, define the sequence $\{x_n\}$ by

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T^n x_n,$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are real sequences in $(0, 1)$ satisfying some conditions.Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Asymptotically nonexpansive, Asymptotically pseudocontractive, uniformly L-Lipschitzian, Banach space.

[1] *Received 26 August, 2008*

Accepted for publication (in revised form) 22 January, 2009

1 Introduction and preliminaries

Let K be the closed convex subset of a real Banach space E with the dual space E^*. The normalized duality mapping $J : E \to 2^{E^*}$is defined as

$$J(x) = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2, \|f\| = \|x\| \right\} \quad \forall x \in E,$$

where $\langle ., . \rangle$ denotes the duality pairing between E and E^*. The single valued normalized duality mapping is denoted by j. Let $F(T)$ denotes the set of fixed points of the mapping $T : K \to K$.

Definition 1 *The mapping $T : K \to K$ is said to be non-expansive if*

$$\|Tx - Ty\| \leq \|x - y\| \quad \textit{for all } x, y \in K.$$

Definition 2 *The mapping $T : K \to K$ is said to be uniformly L- Lipschitzian if there exists $L > 0$ such that $\forall n \geq 1$*

$$\|T^n x - T^n y\| \leq L \|x - y\| \quad \textit{for all } x, y \in K.$$

Definition 3 *The mapping $T : K \to K$ is said to be asymptotically non expansive if there exists a sequence $\{k_n\} \subseteq [1, \infty)$ with $k_n \to 1$ such that $\forall n \geq 1$*

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \quad \textit{for all } x, y \in K.$$

Definition 4 *The mapping $T : K \to K$ is said to be asymptotically pseudocontractive if there exists a sequence $\{k_n\} \subseteq [1, \infty)$ with $k_n \to 1$ and $j(x - y) \in J(x - y)$ such that $\forall n \geq 1$*

$$\langle T^n x - T^n y, j(x - y) \rangle \leq k_n \|x - y\|^2 \quad \textit{for all } x, y \in K.$$

It is easy to see that every asymptotically nonexpansive mapping is uniformly L- Lipschitzian and every asymptotically non-expansive mapping is asymptotically pseudocontractive but the converse is not true. The class of asymptotically pseudocontractive mappings was introduced by Schu[6] who proved the strong convergence theorem for the iterative approximation of fixed points of asymptotically pseudocontractive mappings in Hilbert space. Chang [2] extended the result of Schu to real uniformly smooth Banach space. In[8], there has been introduced an iteration scheme as

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T x_n, \quad n \geq 0,$$

for given $x_0 \in K$ and arbitrary but fixed $u \in K$. The result proved in [8] is as under:

Theorem 1 *[8] Let C be a nonempty closed convex subset of a uniformly smooth Banach space E. Let $T : C \to C$ be a non expansive mapping such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be three real sequences in $(0,1)$ satisfying the following control conditions*

(*i*) $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \geq 0$;

(*ii*) $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(*iii*) $\lim\limits_{n\to\infty} \gamma_n = 0$, *for given $x_0 \in C$ arbitrarily and fixed $u \in C$, let the sequence $\{x_n\}$ be generated iteratively by*

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T x_n, \quad n \geq 0. \tag{1}$$

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Our main motive here is to generalize the result of [8] to arbitrary Banach space and for uniformly L-Lipschitzian mappings. Our iteration scheme is a modification of the iteration scheme given in (1). Our result improves and generalizes the results proved in [2, 4, 6, 9, 5, 1, 3, 7].

Lemma 1 *Let E be real Banach space and J be the normalized duality mapping. Then for any given $x, y \in E$, we have*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall j(x+y) \in J(x+y).$$

Lemma 2 *Let $\{a_n\}$ and $\{b_n\}$ be two nonnegative real sequences satisfying the following condition*

$$a_{n+1} \leq (1+\lambda_n) a_n + b_n, \quad \forall n \geq n_0,$$

where $\{\lambda_n\}$ is a sequence in $(0,1)$ with $\sum_{n=0}^{\infty} \lambda_n < \infty$. If $\sum_{n=0}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Lemma 3 *Let $\{\theta_n\}$ be a sequence of nonnegative real numbers and $\{\lambda_n\} \subseteq [0,1]$ be the real sequence satisfying the following condition*

$$\sum_{n=0}^{\infty} \lambda_n = \infty.$$

If there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ such that

$$\theta_{n+1}^2 \leq \theta_n^2 - \lambda_n \phi(\theta_{n+1}) + \sigma_n, \ \forall n \geq n_0,$$

where n_0 is some non negative integer and $\{\sigma_n\}$ is a sequence of nonnegative numbers such that $\sigma_n = o(\lambda_n)$, then $\theta_n \to 0$ as $n \to \infty$.

2 Main results

Theorem 2 *Let K be a non empty closed convex subset of an arbitrary Banach space E. Let $T : K \to K$ be a uniformly L- Lipschitzian mapping with $F(T) \neq \emptyset$. Let $\{k_n\} \subseteq [1, \infty)$ be a sequence with $\lim_{n\to\infty} k_n = 1$.For any $x_0 \in K$, and fixed $u \in K$ define the sequence $\{x_n\}$ by*

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T^n x_n, \tag{2}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are real sequences in $(0, 1)$ satisfying the following conditions
(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$; *(ii)* $\sum_{n=0}^{\infty} \gamma_n^2 < \infty$; *(iii)* $\sum_{n=0}^{\infty} \alpha_n < \infty$; *(iv)* $\sum_{n=0}^{\infty} \gamma_n k_n < \infty$;
(v) $\alpha_n = o(\gamma_n)$; *(vi)* $\lim_{n\to\infty} \gamma_n = 0$.

If there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle T^n x - p, j(x - p)\rangle \leq k_n \|x - p\|^2 - \phi(\|x - p\|), \quad \forall x \in K,$$

then the iterative scheme $\{x_n\}$ defined by (2) converges strongly to a fixed point of T.

Proof. Let $p \in F(T)$. First we prove that $\{x_n\}$ is bounded. By using Lemma 6 we have:

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\alpha_n (u - p) + \beta_n (x_n - p) + \gamma_n (T^n x_n - p)\|^2 \\ &\leq \beta_n^2 \|x_n - p\|^2 + 2\langle \alpha_n (u - p) + \gamma_n (T^n x_n - p), j(x_{n+1} - p)\rangle\end{aligned}$$

$$
\begin{aligned}
&\leq \beta_n^2 \|x_n - p\|^2 + 2\left\langle \alpha_n (u-p), j(x_{n+1}-p)\right\rangle \\
&\quad +2\left\langle \gamma_n (T^n x_n - p), j(x_{n+1}-p)\right\rangle \\
&\leq \beta_n^2\|x_n-p\|^2+2\alpha_n\|u-p\|\|x_{n+1}-p\|+2\gamma_n\|T^n x_{n+1}-T^n x_n\|\|x_{n+1}-p\| \\
&\quad +2\gamma_n\left\langle T^n x_{n+1}-p, j(x_{n+1}-p)\right\rangle \\
&\leq \beta_n^2\|x_n-p\|^2+2\alpha_n\|u-p\|\|x_{n+1}-p\|+2\gamma_n\|T^n x_{n+1}-T^n x_n\|\|x_{n+1}-p\| \\
&\quad +2\gamma_n\left(k_n\|x_{n+1}-p\|^2-\phi(\|x_{n+1}-p\|)\right) \\
&\leq \beta_n^2\|x_n-p\|^2+2\alpha_n\|u-p\|\|x_{n+1}-p\|+2\gamma_n L\|x_{n+1}-x_n\|\|x_{n+1}-p\| \\
&\quad +2\gamma_n\left(k_n\|x_{n+1}-p\|^2-\phi(\|x_{n+1}-p\|)\right) \qquad (3)
\end{aligned}
$$

Now consider

$$
\begin{aligned}
\|x_{n+1}-x_n\| &= \|\alpha_n(u-x_n)+\gamma_n(T^n x_n - x_n)\| \\
&\leq \alpha_n\|u-x_n\|+\gamma_n\|T^n x_n-x_n\| \\
&\leq \alpha_n(\|u-p\|+\|x_n-p\|)+\gamma_n(\|T^n x_n-p\|+\|x_n-p\|) \\
&\leq [\alpha_n+\gamma_n(L+1)]\|x_n-p\|+\alpha_n\|u-p\|. \qquad (4)
\end{aligned}
$$

Substituting (4) into (3)

$$
\begin{aligned}
\|x_{n+1}-p\|^2 \leq\ & \beta_n^2\|x_n-p\|^2+2\alpha_n\|u-p\|\|x_{n+1}-p\| \\
& +2\gamma_n L\left((\alpha_n+\gamma_n(L+1))\|x_n-p\|+\alpha_n\|u-p\|\right)\|x_{n+1}-p\| \\
& +2\gamma_n\left(k_n\|x_{n+1}-p\|^2-\phi(\|x_{n+1}-p\|)\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \beta_n^2 \|x_n - p\|^2 + \alpha_n \|u - p\|^2 + \alpha_n \|x_{n+1} - p\|^2 \\
&\quad + \gamma_n L (\alpha_n + \gamma_n (L+1)) \|x_n - p\|^2 \\
&\quad + \gamma_n L (\alpha_n + \gamma_n (L+1)) \|x_{n+1} - p\|^2 \\
&\quad + \gamma_n L \alpha_n \|u - p\|^2 + \gamma_n L \alpha_n \|x_{n+1} - p\|^2 \\
&\quad + 2\gamma_n \left(k_n \|x_{n+1} - p\|^2 - \phi(\|x_{n+1} - p\|)\right) \\
&= \left(\beta_n^2 + \gamma_n \alpha_n L + \gamma_n^2 L (L+1)\right) \|x_n - p\|^2 + (\alpha_n + \gamma_n \alpha_n L) \|u - p\|^2 \\
&\quad + \left(\alpha_n + 2\gamma_n \alpha_n L + \gamma_n^2 L (L+1) + 2\gamma_n k_n\right) \|x_{n+1} - p\|^2 - 2\gamma_n \phi(\|x_{n+1} - p\|) \\
&\leq \left(\beta_n^2 + \gamma_n \alpha_n L + \gamma_n^2 L (L+1)\right) \|x_n - p\|^2 + \alpha_n (1+L) \|u - p\|^2 \\
&\quad + \left(\alpha_n + 2\gamma_n \alpha_n L + \gamma_n^2 L (L+1) + 2\gamma_n k_n\right) \|x_{n+1} - p\|^2 - 2\gamma_n \phi(\|x_{n+1} - p\|). \qquad (5)
\end{aligned}
$$

By (5) we get

$$
\begin{aligned}
(1-\sigma_n) \|x_{n+1} - p\|^2 &\leq \left(\beta_n^2 + \gamma_n \alpha_n L + \gamma_n^2 L(L+1)\right) \|x_n - p\|^2 + \alpha_n (1+L) \|u - p\|^2 \\
&\quad - 2\gamma_n \phi(\|x_{n+1} - p\|) \\
&\leq \left(\beta_n + \gamma_n \alpha_n L + \gamma_n^2 L (L+1)\right) \|x_n - p\|^2 + \alpha_n (1+L) \|u - p\|^2 \qquad (6)
\end{aligned}
$$

where $\sigma_n = \alpha_n + 2\gamma_n \alpha_n L + \gamma_n^2 L (L+1) + 2\gamma_n k_n$. By conditions (i) and (ii) it is obvious that $\lim_{n\to\infty} \sigma_n = 0$, therefore there exists a positive integer N such that $0 < \sigma_n < \frac{1}{2}$ for all $n \geq N$, which implies $1 - \sigma_n > 0$, hence $\frac{\gamma_n}{1-\sigma_n} > 0$. By (6) we have

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \left(1 + \frac{3L\gamma_n \alpha_n + 2\gamma_n k_n + 2\gamma_n^2 L (L+1)}{1-\sigma_n} - \frac{\gamma_n}{1-\sigma_n}\right) \|x_n - p\|^2 \\
&\quad + \frac{\alpha_n (1+L)}{1-\sigma_n} \|u - p\|^2 - \frac{2\gamma_n}{1-\sigma_n} \phi(\|x_{n+1} - p\|)
\end{aligned}
$$

$$\begin{aligned} &\leq \left(1+\frac{3L\gamma_n\alpha_n+2\gamma_n k_n+2\gamma_n^2 L(L+1)}{1-\sigma_n}\right)\|x_n-p\|^2 \\ &\quad +\frac{\alpha_n(1+L)}{1-\sigma_n}\|u-p\|^2-\frac{2\gamma_n}{1-\sigma_n}\phi(\|x_{n+1}-p\|) \\ &\leq \left(1+6L\gamma_n\alpha_n+4\gamma_n k_n+4\gamma_n^2 L(L+1)\right)\|x_n-p\|^2 \\ &\quad +2\alpha_n(1+L)\|u-p\|^2-\frac{2\gamma_n}{1-\sigma_n}\phi(\|x_{n+1}-p\|) \\ &\leq \left(1+6L\alpha_n+4\gamma_n k_n+4\gamma_n^2 L(L+1)\right)\|x_n-p\|^2 \\ &\quad +2\alpha_n(1+L)\|u-p\|^2. \end{aligned} \tag{7}$$

By conditions $(ii),(iii),(iv)$ we deduce that

$$\sum_{n=0}^{\infty}\left(6L\alpha_n+4\gamma_n k_n+4\gamma_n^2 L(L+1)\right)<\infty,$$

and

$$\sum_{n=0}^{\infty}2\alpha_n(1+L)\|u-p\|^2<\infty.$$

Hence by Lemma 7 and inequality (7), we deduce that $\lim_{n\to\infty}\|x_n-p\|$ exists. Therefore the sequence $\{x_n\}$is bounded. Let $M=\min\left\{\sup_n\|x_n-p\|,\|u-p\|\right\}$, for a positive constant M. Secondly we prove that $x_n\to p$ as $n\to\infty$.Considering inequality (7) as $n\to\infty$, we have:

$$\begin{aligned} \|x_{n+1}-p\|^2 &\leq \|x_n-p\|^2-2\gamma_n\phi(\|x_{n+1}-p\|) \\ &\quad +2\gamma_n\left(3L\frac{\alpha_n}{\gamma_n}+2k_n+2\gamma_n L(L+1)+\frac{\alpha_n}{\gamma_n}(L+1)\right)M^2. \end{aligned}$$

Let $\theta_n:=\|x_n-p\|$, $\lambda_n:=\gamma_n$, $\delta_n\coloneqq\left(3L\frac{\alpha_n}{\gamma_n}+2k_n+2\gamma_n L(L+1)+\frac{\alpha_n}{\gamma_n}(L+1)\right)M^2$, then by conditions (i) to (vi) $\delta_n=o(\lambda_n)$. Therefore using Lemma 3 it follows that $\|x_n-p\|\to 0$. Hence $x_n\to p$ as $n\to\infty$. This completes the proof.

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